Opening Possibilities: An Approach for Investigating Students' Transfer of Mathematical Reasoning

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Abstract

How can teachers and researchers engender opportunities for students to engage in mathematical reasoning across a range of situations? We posit an approach—*Opening Possibilities*, in which we link theory and method to investigate students' transfer of mathematical reasoning. The approach affords the interweaving of multiple theoretical perspectives: Lobato's theory of Actor Oriented Transfer, Marton's Variation Theory, and Thompson's theory of Quantitative Reasoning, to theorize students' transfer, discernment, and reasoning. To demonstrate the viability of the Opening Possibilities approach, we report empirical data to provide evidence of a student's transfer of a particular form of mathematical reasoning, covariational reasoning. By interweaving theories, we foreground difference and similarity in an Actor Oriented Transfer perspective. Through this approach, we expand objects, theories, and methods for researchers' investigations of students' transfer, and in turn engender opportunities for students' engagement in mathematical reasoning.

Keywords: Transfer, Theoretical Approach, Methodology, Task Design, Actor Oriented Transfer, Variation Theory, Quantitative Reasoning, Covariational Reasoning

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Opening Possibilities: An Approach for Investigating Students' Transfer of Mathematical Reasoning

How might teachers and researchers engender students' mathematical reasoning across a range of situations? Or, put another way, how might students' transfer of mathematical reasoning be promoted? Yet, what counts as transfer of mathematical reasoning? And what might serve as evidence of such transfer?

Researchers' views of transfer afford what constitutes evidence of transfer (Lobato, 2003, 2008, 2012), as well as the scope of what counts as possible to be transferred. We view transfer as something more than the application of a procedure from one situation to another (Lobato, 2003). Meaning, students can engage in transfer even if they do not accurately apply a procedure across different situations. To weigh what could serve as evidence of transfer, we navigate tensions between our own researcher perspectives and students' perspectives. Hence, we draw on Actor Oriented Transfer (AOT) theory (Lobato, 2003, 2008, 2012), in which Lobato problematizes the perspectives that researchers employ when investigating students' transfer.

To locally integrate theories (Bikner-Ahsbahs & Prediger, 2010), researchers extend beyond combining or coordinating theories to explain empirical phenomena, to build new theories and approaches. We draw on three theories to investigate students' transfer of mathematical reasoning: Lobato's theory of Actor Oriented Transfer (Lobato, 2003, 2008, 2012), Marton's Variation Theory (Kullberg, Runesson Kempe, & Marton, 2017; Marton, 2015), and Thompson's theory of Quantitative Reasoning (Thompson, 1994, 2002, 2011; Thompson & Carlson, 2017). In each of their theories, these scholars distinguish between the perspectives of students, and those of the researchers. Lobato (2003) centered the student perspective when

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expanding the scope of what could count as evidence of transfer. Marton (2015) distinguished between adults' and children's perspectives, explaining that adults cannot expect that by showing and telling children something they as adults discern, that they will necessitate children's discernment. Thompson (1994) argued that a quantity is something more than a label for a unit (e.g., 5 feet), explaining that quantities depend on individuals' conceptions of attributes of objects. By integrating these theories, we center the student perspective in our investigation of students' transfer.

The Opening Possibilities approach stems from Johnson's program of research, consisting of iterative design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), in which Johnson led fine grained investigations of secondary students' reasoning related to rate and function. With this approach, we aim to open possibilities for researchers to investigate students' transfer and for students to engage in mathematical reasoning. By focusing on students' transfer of mathematical reasoning (e.g., Johnson, McClintock, & Hornbein, 2017) researchers can extend the objects of their transfer study. By integrating different theoretical perspectives (Lobato, 2003; Marton, 2015; Thompson, 2011), researchers can expand how they theorize transfer. By linking theory and method, in a way that mutually informs, rather than prescribes, the other (Chan & Clarke, 2019), researchers can broaden methods for transfer study. To demonstrate the viability of this approach, we provide an empirical example of a secondary student's transfer of a particular form of mathematical reasoning—covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2017). We conclude with implications for the design of transfer studies.

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Theoretical Background: Students' Transfer, Discernment, and Reasoning

Integrating theories, we bring together different assumptions. First, researchers' focus on students' perspectives impacts claims of what can constitute evidence of transfer (Lobato 2003, 2008). Second, students' discernment plays a role in their transfer, and students discern both difference and similarity (Marton, 2006). Third, the object of students' transfer can extend beyond knowledge of mathematical concepts to include forms of mathematical reasoning (Johnson, McClintock, et al., 2017).

Transfer and Discernment

From an AOT perspective, transfer is generalization, rather than application (Lobato, 2003, 2008). Meaning, transfer is something other than the accurate application of a solution method across situations. Lobato (2008) defines transfer as "the generalization of learning, which also can be understood as the influence of a learner's prior activities on his or her activity in novel situations" (p. 169). Hence, students transfer their mathematical reasoning when they generalize some form of reasoning from one situation to a novel one. For example, consider two situations: a Cannon Man, flying up into the air, then parachuting back down; and a Toy Car, moving along a curved path, with a stationary object nearby. In each situation students can sketch a Cartesian graph to represent a relationship between attributes: Cannon Man's height from the ground and his total distance traveled, and the Toy Car's distance from the stationary object and its total distance traveled. Even if students do not sketch accurate graphs in either situation, they may still transfer reasoning from the Cannon Man to the Toy Car. To gather evidence of students' transfer, researchers employing an AOT perspective scour data for relationships of sameness that students may construct (Lobato, 2003, 2008). For example, students may recognize that the total distance traveled continues to increase in both situations.

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While there has been a focus on sameness, Lobato (2008) has acknowledged the possibility for researchers' AOT analysis methods to include attention to difference.

For a given graph in a Cartesian coordinate system, some students attend to attributes represented on the axes, while other students attend to only a trace in the plane. Yet, it is important for each and every student to attend to graph attributes. Employing Marton's Variation Theory (Kullberg et al., 2017; Marton, 2015), designers can develop task sequences to provide opportunities for students to *discern* particular aspects of graphs. Discernment involves more than noticing. It implies separation of an object's features from the object itself (Marton, 2015). For example, to discern attributes represented on graph axes, students would separate those attributes from other aspects of a graph.

Through systematic variation, designers can engender opportunities for students' discernment (Kullberg et al., 2017; Marton, 2006, 2015); in the task sequences, difference (contrast) should precede sameness (generalization). Systemic variation necessitates patterns of variation and invariance. For example, suppose researchers intend Cartesian graphs to be an object of learning for students. In the first task, students can encounter different kinds of graphs (contrast), so that students may discern graphs as an object, and Cartesian graphs as a dimension of variation of the broader object of graphs. The relationship between variables would remain invariant, and the type of graph would vary. In a subsequent task, students can encounter different kinds of Cartesian graphs (generalization). Now, the type of graph (Cartesian) would remain invariant, and the relationship between variables would vary. Notably, the object of learning is the first thing varied (the type of graph), then characteristics of the object of learning (relationships between variables), so that students may discern which aspects of Cartesian graphs are necessary, and which aspects are optional.

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Researchers can employ Variation Theory in their study of transfer. Broadly, Marton (2006) defines transfer as being “about how what is learned in one situation affects or influences what the learner is capable of doing in another situation” (p. 499). Summarizing results of different studies, Marton (2006) argues that students’ discernment of both difference and sameness contributes to their transfer. To illustrate, consider Marton’s (2006) example of the Cantonese spoken language, which includes both sound and tone. Suppose a student hears two Cantonese words in succession, both with the same sound, but different tones; this can provide the student an opportunity to discern, or separate, the tone from the sound, not only in the second word, but also in the first. This kind of discernment also can apply to the Cannon Man and Toy Car situations. For example, a student may discern, or separate, the difference in literal movement of each object from the object’s total distance traveled. Hence, it is possible for the discernment of difference (e.g., the difference in tone or literal movement) to be what a learner transfers from one situation to another.

Both Marton and Lobato use the term *generalization*. We interpret their uses of the term to be compatible, yet not synonymous. Lobato uses generalization in a broader sense, whereas Marton uses generalization to address a specific kind of variation. We view Lobato’s explanation of transfer as “generalization of learning” to be consistent with Marton’s definition of transfer; that is, the influence of one situation on a new situation. Marton employs generalization to refer to a pattern of sameness in task sequences, which should follow patterns of difference (contrast). For example, suppose a teacher intends to develop a task sequence for students to discern, or separate, the attribute of “increasing” on a graph. The teacher would begin with contrast, for instance providing students graphs that increase, decrease, and remain constant. Then the teacher would follow with generalization, for instance providing students with graphs having different

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kinds of increases (e.g., linear, quadratic, exponential). Integrating theories, we aim to illustrate how difference can play a role in the generalization of learning, or transfer, from an AOT perspective.

Discernment and Reasoning

In the theory of Quantitative Reasoning (Thompson, 1994, 2011; Thompson & Carlson, 2017), Thompson focuses on students' conceptions of attributes, which may be involved in problem situations or represented in graphs. Whether an attribute is also a quantity depends on the students' perspectives, rather than the observers' perspectives. When a student conceives of some attribute as being possible to measure, then that attribute is a *quantity* for the student. For example, an observer may conceive of how it could be possible to measure a toy car's distance from a stationary object, yet students may wonder where to even look for, let alone measure such a distance. Thompson's theory centers students' conceptions of possibilities for measurement (e.g., using a string to measure the distance between two objects), rather than on their end results of measurements (e.g., exactly how far the toy car is from the stationary object at a given moment). Therefore, students can engage in quantitative reasoning without applying particular procedures or determining certain results. Integrating theories, we explain a particular kind of discernment, a conception of graph attributes as being possible to measure, that we aim to promote in students.

The Opening Possibilities Approach

The Opening Possibilities approach, shown in Fig. 1, links theory and method to investigate students' transfer of mathematical reasoning. Theory and method are positioned across from each other, to represent a complementary, rather than hierarchical, relationship between them. The double headed arrow in the center shows that theory and method mutually

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inform, rather than prescribe the other. Three overarching questions guide the approach: What counts as students' transfer of mathematical reasoning? How can researchers engender students' transfer of mathematical reasoning? What constitutes evidence of students' transfer of mathematical reasoning? In response, researchers may draw on a range of theories and methods, which in turn afford and constrain their design decisions, data collection, and data analysis.

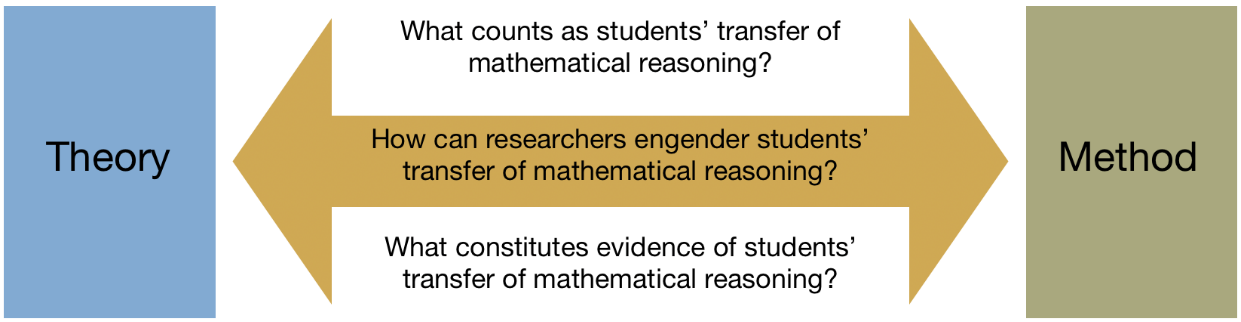

Fig. 1. The Opening Possibilities approach

What Counts as Transfer of Students' Mathematical Reasoning?

How researchers theorize students' mathematical reasoning influences what counts as evidence of students' reasoning. By a student's mathematical reasoning, we mean purposeful thinking in action, occurring in a setting that constitutes mathematics for the student. With Thompson's theory of Quantitative Reasoning, we focus on students' conceptions of what may be possible to measure, rather than on end results obtained from measurement. Bringing together Lobato's AOT theory and Marton's Variation Theory, by transfer of students' mathematical reasoning, we mean how students' mathematical reasoning in prior situations influences their mathematical reasoning in new situations. Through our methods, we aim to infer students' reasoning (and transfer of reasoning) based on their observable behaviors. To gather evidence of students' engagement in the intended mathematical reasoning, we focus on students' conceptions as they are engaging with task sequences, rather than on their end results.

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How Can Researchers Engender Students' Transfer of Mathematical Reasoning?

We aim to promote students' engagement in mathematical reasoning, rather than answer finding. Hence, we design task sequences, in which we work to engineer opportunities for students' reasoning writ large, as well as for students to engage in mathematical reasoning that we intend. Our stance on students' reasoning influences our assumptions about the viability of their reasoning, which in turn influences our methods. First, we assume that students working on a task may have goals for the task that are different from our own (Johnson, Coles, & Clarke, 2017). Second, we acknowledge that the reasoning we intend may be different from the reasoning that students engage in during task sequences. Third, we assume that students' reasoning is viable and productive, regardless of its form. In our methods, we do not seek to "fix" students' reasoning. Rather, we seek to understand and engender students' mathematical reasoning, in its many forms.

What Constitutes Evidence of Students' Transfer of Mathematical Reasoning?

We view students as experts in their own mathematical reasoning, and thereby our role as researchers is to elicit and explain that reasoning. To gather evidence of students' transfer of mathematical reasoning, we build from four criteria, put forth by Lobato (2008). First, students demonstrate a change in their reasoning, from one task to another. Second, prior to the task sequences, students demonstrate limited evidence of the intended reasoning. Third, students' reasoning on earlier tasks influences their reasoning on later tasks. Fourth, students' change in reasoning is something other than a spontaneous occurrence. When analyzing for evidence of influence of students' reasoning, from earlier tasks to later tasks, we consider both contrast and generalization (Marton, 2006). That is, we take as evidence of transfer not only students' perspectives of how tasks are similar, but also how they perceive those tasks to be different.

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Opening Possibilities for Students' Covariational Reasoning

To operationalize the Opening Possibilities approach, we address a particular form of mathematical reasoning, covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). Not confined to a single area of mathematics, covariational reasoning transcends different mathematical concepts, including the gatekeeping concepts of rate and function.

What Counts as Transfer of Students' Covariational Reasoning?

When students engage in covariational reasoning, they can form and interpret relationships between attributes which they conceive to be capable of varying and possible to measure. Meaning, covariational reasoning involves both students' conceptions of attributes, and their conceptions of a relationship between those attributes (Carlson et al., 2002; Thompson & Carlson, 2017). To illustrate, in the Toy Car situation, a student may conceive of varying lengths of a stretchable cord connecting the car to a stationary object, and a trace of the distance traveled as the car moves along its path. Furthermore, that student may conceive of a relationship between the cord length and distance traveled: The cord could start off longer, then shorten, while the toy car's total distance traveled keeps increasing. By transfer of students' covariational reasoning, we mean how that students' covariational reasoning in one situation (e.g., the Cannon Man) influences their covariational reasoning in a new situation (e.g., the Toy Car).

How Can Researchers Engender Students' Transfer of Students' Covariational Reasoning?

We view tasks to be more than a problem statement. Tasks encompass intentions of those designing, implementing, and interacting with the tasks, as well as physical materials (Johnson, Coles, et al., 2017). Our task sequences comprise students' sketching and interpreting Cartesian graphs, which means we address both students' covariational reasoning and their conceptions of graphs themselves. By incorporating patterns of difference and sameness, we intend to provide

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opportunities for students to discern necessary aspects of graphs from optional ones. For example, even though the Cannon Man flies up and down while the Toy Car moves along a path, the total distance traveled for both continues to increase. If students were to only experience one kind of motion, they may not have sufficient opportunities to separate the literal motion of the objects from a measurable attribute of the objects, such as their total distance traveled.

What Constitutes Evidence of Students' Transfer of Covariational Reasoning?

To gather evidence of students' covariational reasoning, we infer students' conceptions based on their observable behavior. We examine students' work when sketching Cartesian graphs, because sketching graphs can provide students opportunities to represent relationships between attributes. We focus on students' process of sketching graphs, rather than on assessing the accuracy of their resulting graphs. While students may engage in covariational reasoning when doing things other than graph sketching, we have found instances of students' graph sketching to offer compelling evidence of their covariational reasoning. Yet, students' difficulties or facilities with graphs can present challenges when analyzing for reasoning. Integrating different theories affords us opportunities to explain students' discernment of graph attributes in conjunction with their transfer of covariational reasoning.

The Promise of Opening Possibilities: An Instantiation of the Approach

To demonstrate the promise of the Opening Possibilities approach, we report data from a larger study, in which Johnson conducted a set of three individual, task based interviews (Goldin, 2000) with each of 13 secondary students, to investigate their covariational reasoning and conceptions of graphs. We report data from one of those students, Aisha, who demonstrated transfer of covariational reasoning. To contextualize the data, we explain the design of our task sequences and our methods for data analysis. With this instantiation of the Opening Possibilities

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approach, we build on Johnson and colleagues' earlier investigation of a secondary student's transfer of covariational reasoning (Johnson, McClintock, et al., 2017).

The Task Sequences

We implemented three task sequences, each with a different background: a Ferris Wheel, a Cannon Man, and a Toy Car, respectively. Across the task sequences students explored different situations, then sketched one or more Cartesian graphs to represent a relationship between attributes in a situation given in an animation. We adapted the Ferris Wheel task sequence from Johnson and colleagues' earlier research (Johnson, McClintock, et al., 2017). We developed the Cannon Man and Toy Car task sequences in Desmos, in collaboration with Meyer, the chief academic officer of Desmos.

The Ferris Wheel task sequence incorporated three key elements. First, students manipulated an online interactive of a turning Ferris wheel. Second, students sketched a single graph representing a relationship between a Ferris wheel cart's height from the ground, and its total distance traveled around the wheel for one revolution of a Ferris wheel. Third, students interpreted a replica of another student's graph, explaining how they thought that student may have been thinking when sketching the graph.

The Cannon Man and Toy Car task sequences each incorporated six key elements (Johnson, McClintock, & Gardner, under review). First, students viewed a video animation, then discussed how it could be possible to measure different attributes in the situation (e.g., Cannon Man's height from the ground and his total distance traveled). Second, students explored variation in each of the individual attributes, by manipulating dynamic segments on the horizontal and vertical axes. Fig 2. shows a dynamic segment in the Cannon Man task sequence. Third, students sketched a graph to represent a relationship between attributes, then viewed a

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computer generated graph. Fourth, students re-explored variation in each of the individual attributes, with the attributes represented on different axes. Fifth, students sketched a new graph to represent the same relationship between attributes, then viewed a computer generated graph. Figs. 3 and 4 show the two different computer generated graphs in the Cannon Man and Toy Car task sequences, respectively. Sixth, students responded to questions about relationships represented by both graphs.

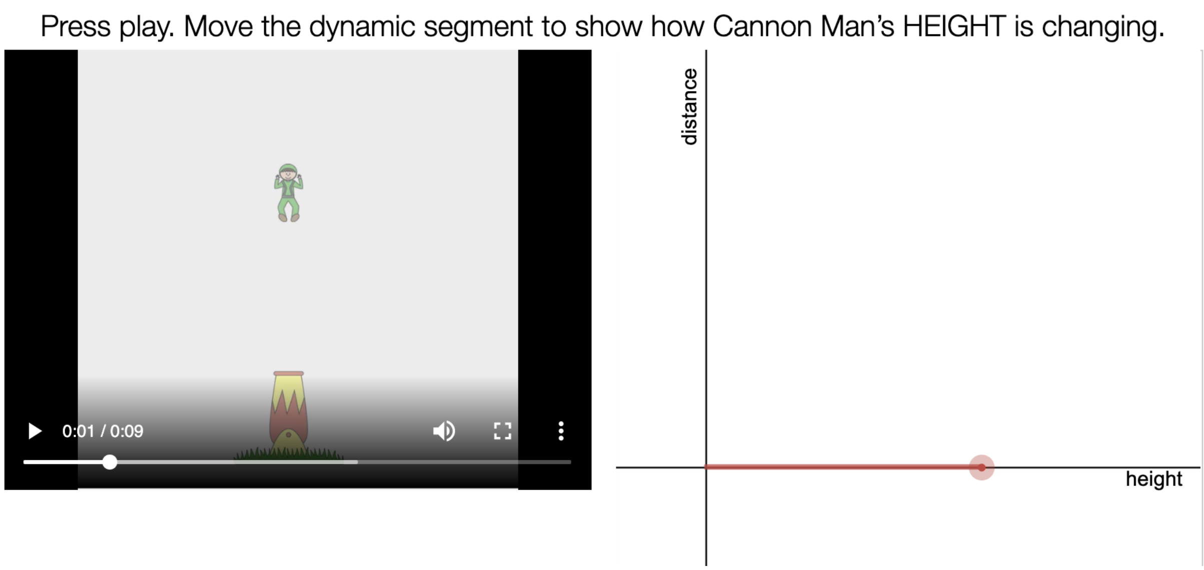

Fig. 2. A dynamic segment in the Cannon Man task sequence

Fig. 3. Two different graphs in the Cannon Man task sequence

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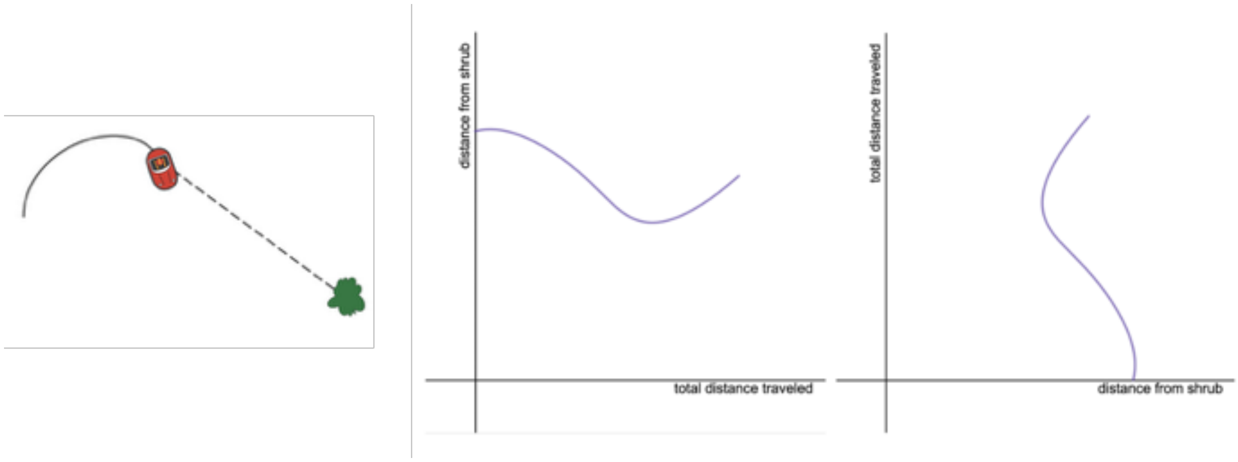

Fig. 4. Two different graphs in the Toy Car task sequence

We integrated Thompson's theory of Quantitative Reasoning and Marton's Variation Theory in our design of the Cannon Man and Toy Car task sequences. First, students could vary each attribute individually, then both attributes together. With the dynamic segments (e.g., Fig. 2), we operationalized Thompson's recommendation that students use their fingers as tools to represent variation in individual attributes (Thompson, 2002). Furthermore, the design provides opportunities for students to discern each graph axis as representing variation in a single attribute (Marton's Variation Theory). After manipulating individual attributes, students sketched a graph to represent a relationship between attributes.

Second, students repeated the process for a new Cartesian plane with the same attributes represented on different axes. This design choice was not a novelty; Moore and colleagues also leveraged this design move (Moore, Silverman, Paoletti, & LaForest, 2014; Moore, Stevens, Paoletti, Hobson, & Liang, 2019). Our theoretical underpinning for this design choice rests in Marton's Variation Theory. With the new graph, we incorporated contrast. The relationship between variables in the Cannon Man task sequence remained invariant; only the graph was different. With this move, we intended to provide opportunities for students to discern a Cartesian plane as separable from a specific instance of a Cartesian graph.

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We designed the first and second patterns of variation and invariance against a single background (the Cannon Man). Next, we engaged in generalization, per Marton's Variation Theory, repeating those patterns against a new background (the Toy Car). In the video animation (the first element of the task sequence), the literal motion of the Toy Car was different from the literal motion of the Cannon Man. For example, the Toy Car moved along a curved path, but Cannon Man moved up and down. We intended this difference to provide students' opportunities to discern what was necessary (e.g., direction of variation in attributes) from what was optional (e.g., literal motion of objects). Across both task sequences, we kept the kind of attributes invariant, because we anticipated it would be less difficult for students to conceive of measuring length attributes (e.g., height, distance) than for other kinds of attributes, such as area or volume (see also Johnson, McClintock, et al., 2017).

Data Analysis Methods

To claim that students transferred their covariational reasoning, first we provide evidence of students' engagement in covariational reasoning within and across tasks (Thompson's theory of Quantitative Reasoning). Second, we identify differences and commonalities that students discerned across tasks (Marton's Variation Theory). Third, we demonstrate that students meet Lobato's (2008) four criteria for evidence of transfer from an AOT perspective.

Covariational reasoning. Our analysis focused on two areas: students' conceptions of attributes as possible to measure and capable of varying, and students' conceptions of relationships between those attributes. The framework put forth by Thompson and Carlson (2017) provided fine grained distinctions regarding different levels of students' covariational reasoning. We gathered evidence of the presence of covariational reasoning, rather than distinguishing between different levels of covariational reasoning. As a litmus test for

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covariational reasoning, we identified the level that Thompson and Carlson (2017) term gross coordination, in which students conceive of a relationship as a loose joining of two attributes. To illustrate, to claim a student engaged in covariational reasoning in the Toy Car situation, we drew on two pieces of evidence. First, the student conceived of both distance attributes as capable of varying and possible to measure; for example, the student could separate a distance attribute from the situation itself (possible to measure) and show or explain how that distance could vary, beyond just describing literal motion of an object (capable of varying). Second, the student conceived of a loose joining of those distances, for example, by showing or explaining how those different distances could vary together (e.g., one distance increased and decrease while the other distance continued to increase).

Transfer of covariational reasoning. Our analysis focused on students' discernment of difference and sameness, and students' evidence of engagement in transfer, from an AOT perspective. Drawing on Marton's theory, we analyzed students' discernment when they encountered what we intended to be instances of contrast and generalization. For example, we examined how students discerned attributes represented on each graph axis (a necessary aspect), or the differences in literal motion between the Cannon Man and the Toy Car (an optional aspect). We specified the four criteria put forth by Lobato (2008) to our task sequence. First, students demonstrated a change in reasoning from the Ferris Wheel task sequence (first interview) to the Toy Car task sequence (third interview). Second, in the Ferris Wheel task sequence, students demonstrated limited evidence of covariational reasoning. Third, students' reasoning during the Cannon Man task sequence (second interview) influenced their reasoning during the Toy Car task sequence (third interview). Fourth, students' change in reasoning resulted from their work on interview tasks, and it was not just a spontaneous occurrence.

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Empirical Evidence: Aisha's Engagement with the Task Sequences

Aisha attended a high performing suburban high school in the metropolitan area of a large US city, with just over half of the student population identifying as students of color. Aisha was near the end of ninth grade (≈15 years old), and enrolled in an Algebra I course, which was typical for students in a general college preparatory track at her school. Aisha's interviews spanned a two week time frame, with at least one day between; interviews occurred during the school day when she had a free period. She engaged with one task sequence in each interview: Ferris Wheel, Cannon Man, and Toy Car, in that order, working on a tablet (an iPad), with paper and pencil available.

We begin with transcripts and description from each of the task sequences, across the three interviews, followed by our analysis within and across tasks. Fig. 5 shows some of the graphs that Aisha drew during the interviews. Aisha's Ferris Wheel graph is shown in Fig. 5, left. The Cannon Man and Toy Car graphs, shown in Fig. 5 (middle, right), are the second Cartesian graphs that Aisha drew in the task sequence (graphs that we intended to provide contrast, per Marton's Variation Theory).

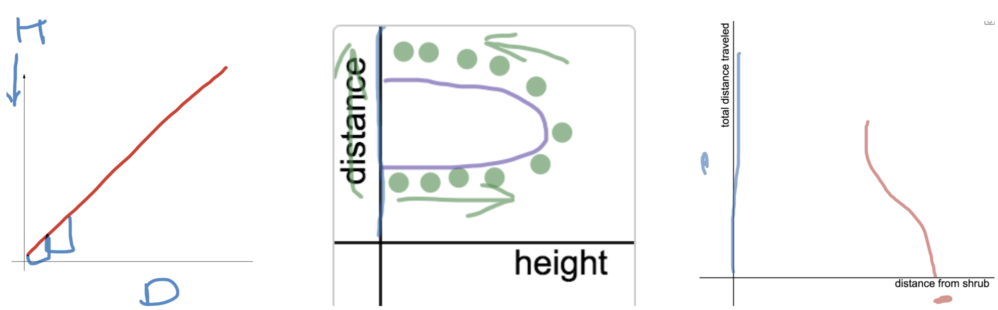

Fig. 5. Aisha's Ferris Wheel, Cannon Man, and Toy Car graphs, respectively

Ferris Wheel. Aisha sketched a graph relating a Ferris wheel cart's height from the ground and total distance traveled, around one revolution of the Ferris wheel. While sketching, Aisha explained why she drew the graph in the manner that she did.

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Aisha: I feel like the height would be more like the line (sketches a line, Fig. 5, left). Distance would be more like the rise and run of the situation (sketches small segments, Fig. 5, left). Cause you're using the rise and run to find the line, and you need to use the distance to find the height.

Cannon Man. Aisha sketched a graph relating Cannon Man's height from the ground and total distance traveled, with the height represented on the horizontal axis and the distance on the vertical axis. Next, Johnson asked Aisha to explain how the graph showed both Cannon Man's height and distance.

Johnson: Can you show me how you see the height increasing and decreasing in this purple graph? (Points to the curved graph Aisha drew, Fig. 5., middle)

Aisha: It's (the height's) increasing here, since it's (the graph's) backwards in my opinion (Sketches green dots, beginning on bottom left near the vertical axis, then moving outward, Fig. 5, middle). Decreasing here (Continues to sketch green dots, until getting close to the vertical axis, adding arrows after sketching dots, Fig. 5, middle).

Johnson: How is the distance changing?

Aisha: (Turns iPad so that vertical axis is horizontal. Draws arrow parallel to vertical axis, Fig. 5, middle.) That way. Continues to get bigger.

Toy Car. Before sketching the graph shown in Fig. 5, right, Aisha spontaneously stated that the Toy Car's distance traveled was the "same as the Cannon Man." Following up, Johnson asked Aisha to explain how those different distances could possibly be the same.

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Johnson: So, you said the total distance traveled is like the Cannon Man. Why is that like the Cannon Man again? Cause Cannon Man goes up and down, and this one moves around. How are those things the same?

Aisha: Just because Cannon Man is coming back down, doesn't mean his distance is going down. His distance is still rising.

To explore change in the Toy Car's total distance traveled and the Toy Car's distance from the shrub, Aisha manipulated dynamic segments located on the vertical and horizontal axes, respectively. For the total distance, Aisha began at the origin, continually moving the segment up, along the vertical axis. She explained: "I moved it up. It continuously went up, because the distance doesn't decrease. The total distance traveled doesn't decrease." For the distance from the shrub, Aisha began to the right of the origin, initially moving the segment to the left, and then to the right, along the vertical axis. She explained: "I moved it (the segment) to the left, because it (the Toy Car) was getting closer to the shrub. Then, when it (the Toy Car) started to turn, I started to move it (the segment) back up to the right, because it (the Toy Car) was getting closer to the shrub." Next, Aisha sketched the graph shown in Fig. 5, right. After viewing the computer-generated graph, Aisha stated what she thought the curved graph represented. Aisha stated: "This (moving her finger from left to right along the horizontal axis) is tracking the distance from the shrub, and this (moving her finger along the curved graph, beginning near the horizontal axis) is also tracking the distance."

Analysis: Aisha's Reasoning Within and Across Tasks

Within tasks: The Ferris Wheel task sequence. Before sketching a graph, Johnson asked Aisha to explain how she might use a string to measure the Ferris wheel cart's height from the ground and total distance traveled. Appealing to a nonstandard unit, such as a string, was a

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typical move by Johnson, to encourage students to do something other than try to find an answer. For the height, Aisha told Johnson that she would tie the string to the Ferris wheel cart, then drop it down to the ground. For the distance, Aisha said that she would start at the base of the Ferris wheel, and then just “go around,” moving her finger counter clockwise around the wheel until she ended up back at the base. Aisha’s actions demonstrated that she could conceive of the height and distance as attributes possible to measure, or as quantities, per Thompson’s theory.

When sketching a graph, Aisha treated height and distance as inputs and outputs, explaining how one might use a formula or rule to determine one amount (height), given another amount (distance). Aisha included both height and distance in a single graph and labeled the axes, but the height and distance were juxtaposed as individual parts of a line graph. A loose joining of attributes would give evidence of covariational reasoning at the gross coordination level. However, Aisha had yet to demonstrate if she could conceive of a relationship between different values of the attributes (e.g., when the cart is this far off the ground, the cart would have traveled this much distance), or even of those attributes as varying together (e.g., the cart’s height increased and decreased while the cart’s distance traveled continued to increase). Per Thompson’s theory, Aisha demonstrated limited evidence of the object of transfer (covariational reasoning). Hence, per Lobato’s (2008) criteria, if Aisha were to demonstrate covariational reasoning during a subsequent task sequence, an argument for transfer could be built.

Within tasks: The Cannon Man task sequence. The interview began with Johnson telling Aisha to view the video animation, then explain what she thought she might be able to measure in the situation. With this question, Johnson intended to investigate what attributes students might discern on their own. Aisha came up with two attributes: the distance from when the parachute deploys, and how high Cannon Man gets in the air, both of which she interpreted

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in relationship to the ground. To encourage Aisha to talk more about how she might measure the attributes, Johnson asked Aisha how the height was changing. Aisha said that she could measure Cannon Man's height using feet, and there would be more feet when Cannon Man was higher in the air. If a student did not spontaneously identify one of the intended attributes, Johnson would introduce that attribute; here, it was total distance traveled. Aisha said that she thought of it the same way as the height—the further Cannon Man is in the air, the more feet he would have. Johnson then suggested that Aisha think of the total distance as a round trip. With such a move, Johnson intended to give students opportunities to extend beyond their initial impressions of attributes. Aisha responded by explaining that the distance would keep getting bigger, and that you could find it by doubling the distance from the ground to Cannon Man's highest point (which she called the "vertex"). Again, in this task sequence, Aisha provided evidence that she conceived of the different attributes as possible to measure (quantities, per Thompson's theory).

Unlike the Ferris Wheel, in the Cannon Man task sequence Aisha demonstrated evidence of covariational reasoning. This happened when Aisha sketched the second graph (Fig. 5, middle). When annotating the graph that she drew in the Cannon Man task sequence (Fig. 5, middle), Aisha explained how she showed the height to be both increasing and decreasing, as well as the distance to be increasing. Taken together with earlier evidence of her conceptions of the attributes as being possible to measure, Aisha's loose joining of the varying attributes demonstrates evidence of her covariational reasoning, at the gross coordination level, per Thompson's theory. Building our case for Aisha's transfer, per Lobato's (2008) criteria, Aisha demonstrated a change in reasoning from the Ferris Wheel to the Cannon Man.

Aisha's engagement in covariational reasoning occurred not with her first graph, but with her second. Per Marton's Variation Theory, we designed the second graph as contrast, so that

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students could have an opportunity to discern the Cartesian plane itself as being separate from the particular graph being sketched. Aisha discerned the representation of the total distance traveled on Cartesian plane in the second graph, stating: "I imagine the distance on the ground, which I can't do." In sketching her second graph (Fig 5., middle), Aisha demonstrated that she discerned necessary aspects of Cartesian graphs (that axes represent measurable attributes) from optional aspects (that the location of an attribute on a graph axis matches the literal orientation of the attribute in a situation). By designing to promote students' discernment of difference in the Cartesian plane, we aimed to engineer opportunities for students to engage in covariational reasoning, and Aisha's actions pointed to the viability of this design move.

Within tasks: The Toy Car task sequence. As did the Cannon Man, the interview began with Aisha identifying "the distance the car drove" as an attribute. Aisha was not sure how she might measure it, so Johnson asked her to sketch the path that she saw the car taking. As in the Cannon Man, Johnson asked Aisha how the attribute was changing. Aisha said that it would keep increasing, if one were thinking about the distance the car was going, and not from the start to the end, because the car's ending point is close to the starting point. Next, Johnson introduced the attribute of the distance from the shrub, and asked Aisha how she saw that attribute changing, to which Aisha responded that the car went "closer to" and then "further from" the shrub, moving her finger along the path of the car. To investigate how Aisha might separate the attribute of the distance from the shrub from the literal motion of the car, Johnson asked Aisha to draw where she saw the distance. Aisha sketched dotted lines from the car's starting point to the shrub, and the car's ending point to the shrub. At this point, Aisha had not seen the dotted line image shown in Fig. 4; she had only seen the video animation of the moving car, which had no annotations for distance. As she did in the Cannon Man task sequence, Aisha provided evidence

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that she conceived of the different attributes as possible to measure (quantities, per Thompson's theory).

Aisha demonstrated covariational reasoning during the Toy Car task sequence, but as happened in the Cannon Man task sequence, it was not until she sketched the second graph. When sketching that graph (Fig. 5., right), Aisha accounted for both the increase in the total distance, and the increase and decrease in the distance from the shrub. As with Cannon Man, Aisha identified the segment along the vertical axis as tracking the total distance traveled, which continually increased, and the trace in the plane as tracking the attribute that both increased and decreased. She found the vertical dynamic segment (Fig 5., right) to be necessary to "show" the total distance traveled. Hence, her representation of the joined attributes entailed two connected inscriptions, the dynamic segment and the trace. Building our case for transfer, per Lobato's (2008) criteria, Aisha's reasoning on the Cannon Man task influenced her reasoning on the Toy Car task. In both tasks, she conceived of the total distance traveled to be continually increasing, and she represented that increase by sketching a segment along the vertical axis, beginning at the origin, and extending upward.

Across tasks: From the Cannon Man to the Toy Car. We draw further evidence of transfer from Aisha's spontaneous utterance of a sameness that she identified across the Toy Car and Cannon Man task sequences. When working on the Toy Car task, without prompting, Aisha spontaneously stated that she thought an attribute—total distance—was "the same" in both the Toy Car and the Cannon Man. We contend that Aisha's discernment of differences across the task situations contributed to her spontaneous identification of this sameness. Per Marton's theory, we incorporated contrast across the Toy Car and Cannon Man task situations, with difference in the literal motion of each object (Cannon Man moved up and down, while the Toy

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Car moved in a curved path). We did not assume that our design alone would be sufficient to ensure students' discernment; we provided conditions under which discernment may occur. Aisha evidenced such discernment, as she separated the direction of the literal motion of each object from the variation in an attribute (total distance) in each situation. For example, Aisha moved the dynamic segment representing the Toy Car's total distance traveled to show that the distance continued to increase, despite the Toy Car moving along a curved path. Consistent with our intent, Aisha distinguished necessary attributes (e.g., continual increase in total distance traveled) from optional aspects (the literal motion of the objects). Drawing on the corpus of evidence, we claim that Aisha transferred her covariational reasoning from the Cannon Man task sequence to the Toy Car task sequence, and her discernment of differences in the literal motion of each object played a role in that transfer.

Discussion

What Is Possible to Transfer?

With the Opening Possibilities approach, we aim to expand objects of transfer study. In Lobato's investigation of transfer from an AOT perspective, the focus has been on students' transfer of mathematical concepts, such as slope (e.g., Lobato, 2003, 2008, 2012). We demonstrate how the object of transfer can be a form of mathematical reasoning, which can transcend different mathematical concepts. In our application of this approach to students' covariational reasoning, we leave open possibilities for concepts that researchers may address. For example, researchers may engender students' covariational reasoning to develop students' understanding of function writ large, or even inverse function more specifically. In our approach, we center students' mathematical reasoning as something that is more than just a process whose

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value rests in its service to students' development of understanding of mathematical concepts. As a result, we expand what can count as mathematics, and in turn, what can be transferred.

Integrating Theories to Open Possibilities: Reasoning, Discernment, and Transfer

We open possibilities for investigating students' covariational reasoning when interpreting and sketching Cartesian graphs, which are ubiquitous in students' math courses. To address both students' covariational reasoning and their conceptions of graphs, we have drawn on theories that explain students' reasoning (Thompson's theory) and discernment (Marton's theory). Researchers have found that Cartesian graphs may mitigate opportunities for covariational reasoning; university students and prospective teachers may not demonstrate covariational reasoning when sketching graphs, despite evidence to suggest their engagement in covariational reasoning in situations not involving graphs (Carlson et al., 2002; Moore et al., 2019). One response to such findings can be to question the potential for researchers and teachers to leverage Cartesian graphs to engender students' covariational reasoning. We take a different stance, provided that students also have opportunities to conceive of graphs as representing relationships between quantities. Integrating theories has afforded our creation of such opportunities, with Marton's Variation Theory being instrumental in this work. By incorporating contrast and generalization in our task sequences, we have made efforts to problematize aspects of Cartesian graphs as dimensions of variation, and empirical evidence points to the viability of such design.

Our empirical work has focused on secondary students' covariational reasoning, yet this design can be applicable to university students, or even younger students. By engineering opportunities for students' reasoning in a familiar setting (a Cartesian graph) without specifying a particular mathematical concept, we create room for students to engage in reasoning that may

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be different from what they have done in previous math courses, or in their work with graphs. Furthermore, we connect graphs to situations, such as the Toy Car, so that students can have opportunities to conceive of graphs as representing measurable attributes of events that could occur in the world. Too often, students experience mathematics as a game with rules determined by people in authority (Gutiérrez, 2013), rather than an opportunity to engage in reasoning and thinking to quantify their world in ways that make sense to them. If students expect that we intend for them to arrive at particular answers or demonstrate their knowledge of certain procedures (even if that was not our intent), the reasoning students demonstrate can be quite different from the reasoning we intend to promote, even if students are capable of demonstrating the intended reasoning. We view our focus on covariational reasoning and Cartesian graphs as one of many avenues for the Opening Possibilities approach. In future studies, researchers may investigate different forms of reasoning in other situations, such as geometric reasoning in dynamic geometry platforms.

Integrating theories has afforded our articulation of a role of difference, as well as sameness, in investigating students' transfer of mathematical reasoning from an AOT perspective. Again, Marton's Variation Theory has been crucial in this work. Designing for contrast and generalization has opened possibilities for us to scour the data for differences and similarities that students construe between situations, as well as for students to distinguish between necessary and optional aspects of the situations. In Aisha's case, we opened opportunities for her to discern physical characteristics of the situation as optional and measurable attributes as necessary (e.g., the total distance of both Cannon Man and Toy Car continuing to increase despite differences in their literal motion), and this discernment played a role in her transfer of covariational reasoning. The objects of students' covariational reasoning

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are something more than things students might notice (e.g., the literal movement of a toy car); they are measurable attributes of situations (e.g., a toy car's distance from a stationary object). Yet, it can be difficult for students to even conceive of situations as having measurable attributes. When integrating theories, we layer different explanations to guide our larger aim. Thompson's theory explains a form of students' reasoning to promote; Marton's theory provides guidance for design choices to engineer opportunities for students to discern measurable attributes of the situations, to foster students' engagement in the intended reasoning. In future studies, researchers can investigate how designing for contrast and generalization, to promote discernment of difference, may afford students' transfer of other forms of reasoning.

Expanding Design Possibilities for Transfer Studies

Through the Opening Possibilities approach, we work to expand design possibilities for investigating students' transfer, to extend beyond pre/post designs. Lobato (2008) has distinguished between tasks implemented during a design experiment study, and tasks implemented in pre or post interviews. To provide evidence of transfer from an AOT perspective, researchers demonstrate that students' conceptions changed from tasks in a pre interview to tasks in a post interview, and that students' work during the design experiment tasks has influenced their changed conceptions. Rather than separating design experiment tasks from post interview tasks, we illustrate how a student can transfer mathematical reasoning from one design experiment task to another, similar to how Marton (2006) describes the possibility for students to transfer their discernment of tone from sound, when hearing Cantonese words in succession.

We concur with Cobb's (2007) appeal for theory expansion, rather than replacement. With the design expansion we propose, we intend to open new possibilities for investigations

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from an AOT perspective, in particular, by foregrounding roles of difference and similarity. Across the Cannon Man and Toy Car task sequences, we have designed for contrast and generalization, and subsequently have analyzed for both difference and similarity. Integrating Marton's Variation Theory with an AOT perspective has afforded us this possibility. In turn, we then have been able to analyze for students' transfer of reasoning within the design experiment tasks themselves, rather than examining students' reasoning on a separate set of transfer tasks, as done in an earlier study (Johnson, McClintock, et al., 2017).

Conclusion

With theory integration comes responsibility, including the consideration of the epistemological roots of different theories (Bikner-Ahsbahs & Prediger, 2010). Such responsibility is both a limitation and an affordance of the approach, as each theory needs to be weighed in light of the other(s). Integrating theories is a purposeful choice, so that researchers can explain phenomena that extend beyond the bounds of a single theory. We have integrated theories specific to reasoning and transfer (Thompson's and Lobato's theories, respectively), with a theory that addresses discernment of different content and extends beyond transfer (Marton's theory). The *grain size* (Watson, 2016) of the theories differ, with two being more domain specific, and one being broader. Yet, we have not imposed a hierarchy of theories onto our analysis, as we have layered analytic techniques from each theory. To guide our choices, we have drawn on scholars' assumptions of distinctions between researchers' and students' perspectives, and have articulated how those assumptions have influenced our work.

With Opening Possibilities, we offer an approach to navigate complexities in researchers' investigations of students' transfer of mathematical reasoning. Although our focus is on transfer, we can conceive of the guiding questions as applicable to the broader work of research.

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Researchers can examine what counts as their object of investigation, how they may engender the study of that object, and what may constitute evidence of the objects of study. Assuming that theory and method mutually inform each other, our approach affords the integration of different theories to embrace, rather than reduce complexities. Through this approach, we expand design possibilities for investigating students' transfer, acknowledging a symbiotic relationship between the theories that we integrate and the contributions that those theories and methods make possible.

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