VALIDATING AN ASSESSMENT OF STUDENTS' COVARIATIONAL REASONING

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In this mixed methods study, we validated a fully online assessment of students' covariational reasoning. We combined qualitative and quantitative methods to analyze 30 responses from undergraduate college algebra students during individual task based interviews. Our findings were statistically significant; students' total number of items correct could be explained by their evidence of covariational reasoning. We conclude with discussion of our work moving forward.

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Covariational reasoning is a high leverage form of reasoning, which can foster students' understanding of key mathematical concepts, such as rate and function (Carlson, 2002; Thompson & Carlson, 2017). Function is central to the undergraduate college algebra curriculum; hence, promoting students' covariational reasoning dovetails with the goals of the course (Olson & Johnson, 2021). The mixed methods study we report is part of a larger research project designed to promote college algebra students' engagement in covariational reasoning.

We explain how we validated a fully online assessment of students' covariational reasoning. Combining qualitative and quantitative methods, we analyzed 30 responses from undergraduate college algebra students taking part in individual, task-based interviews. Our findings were statistically significant; students' total number of items correct could be explained by evidence of their covariational reasoning. We conclude with discussion of our work moving forward.

Theoretical and Conceptual Background

We ground the construct of covariational reasoning in Thompson's theory of quantitative reasoning (Thompson, 1994), which explains how individuals can conceive of attributes of objects as being possible to measure. For example, consider a toy car moving around a square track. There are a variety of attributes to which students might attend, such as the toy car's total distance traveled around the track or the toy car's distance from a center point. A student engaging in quantitative reasoning could conceive of the attributes as possible to measure. Thompson (1994) calls such a conception a "quantity." For instance, a student may mark off lengths of string to measure one of the distances. To engage in quantitative reasoning, it is sufficient for students to conceive of the possibility of such measurement; they may or may not do the measuring itself.

By variational reasoning, we mean students' conceptions of a single attribute that is not only possible to measure, but also capable of varying (Thompson & Carlson, 2017). For example, a student could conceive of the toy car's distance from a center point as increasing and decreasing, as the car moves around the track. By covariational reasoning, we mean students' conceptions of relationships between attributes that are capable of varying and possible to measure (Carlson et al., 2002; Thompson & Carlson, 2017). For example, a student could conceive of a relationship between the different distances; for instance, the toy car's distance around the track continues to increase while the toy car's distance from the center point increases and decreases.

Students' variational and covariational reasoning intertwine with their conceptions of what Cartesian graphs represent (Johnson et al., 2020). For example, a student engaging in covariational reasoning could interpret a graph as representing a relationship between quantities, such as toy car's total distance traveled and distance from the center. Yet students may conceive of graphs as representing the motion of an object (Kerslake, 1997) or a literal depiction of an object (Leinhardt et al., 1990). For example, students may conceive of a graph of the toy car situation as representing the motion of the toy car around a track or as the literal track itself.

Drawing on Piaget's theory, Moore et al. (2019) distinguished between individuals' figurative and operative thinking when sketching and interpreting Cartesian graphs, as well as graphs with unconventional coordinate systems. Individuals who conceived of graphs as representing literal motion or depictions of objects would engage in figurative thinking. In contrast, individuals who conceived of graphs in terms of quantities and relationships would engage in operative thinking.

Johnson et al. (2020) developed a four-item coding framework in which they made distinctions between students' conceptions of what graphs represent: Covariation (COV), Variation (VAR), Motion (MO), and Iconic (IC). Rather than making fine grained distinctions within covariational and variational reasoning, Johnson et al. (2020) targeted particular levels posited by Thompson and Carlson (2017): gross variation and gross coordination of values. The first marked a student's conception of a quantity as being capable of varying; Johnson and McClintock (2018) called this type of reasoning quantitative variational reasoning. The second marked a student's shift from conceiving of variation in individual quantities (e.g., this one, then that other one) to forming a relationship between quantities (both quantities vary together). Put another way, the COV and VAR codes would evidence operative thinking, while the MO or IC codes would evidence figurative thinking, per the constructs of Moore et al. (2019).

Methods

Assessment Design

To design the covariation assessment, we have integrated different theoretical perspectives and consulted experts in the field (Johnson et al., 2018). The assessment, developed in Qualtrics, is fully online; students can complete it on smartphones, tablets, or computers. There are four items, appearing in random order. Each item incorporates a situation involving changing attributes, with two question groups per item (Table 1). The situations include a turning Ferris wheel (Ferris Wheel item), a person (Nat) walking on a path to and from a tree (Nat + Tree item), a fish bowl filling with water (Fish Bowl item), and a toy car moving around a square track (Toy Car item). We have incorporated innovative elements, including items containing unconventional graphs that do not pass the vertical line test (e.g., Moore et al., 2014).

Table 1: The Covariation Assessment: Item Question Groups

Question Groups	Description
1: Comprehension Check	Play video animation of the situation. State if you understand the situation. If yes, move to question group 2. If no, explain why.
2: Select Graph and Explain	Select a graph (ABCD) that represents a relationship between attributes in the situation. Explain your choice.

Task Based Interviews: Description and Rationale

We hypothesized that students' covariational reasoning could explain their graph selections. Through task based interviews (Goldin, 2000), we were able to gather evidence of students' reasons for their graph selection, and make inferences about their covariational reasoning. If students were selecting correct graphs for reasons not associated with covariational reasoning, for instance, by appealing only to physical features of graphs rather than quantities being represented, we could use that information to revise assessment items.

Data Collection

Participants were undergraduate students enrolled in college algebra at a university located in the metro area of a large U.S. city, serving large percentages of students of color and first generation to college. We selected three groups of 10 participants, interviewed in consecutive spring, summer, and fall semesters. The interviews took place midway during the semester. Student volunteers left for part of class to go to a nearby office for the interview. Our selection method resulted in a not entirely random sample; students who participated in interviews may have been those more invested in the course or more willing to talk about their thinking. Johnson conducted 30 individual interviews, which were video and audio recorded. One graduate research assistant (GRA), either Gardner or Smith, observed each interview. The GRA engaged in two activities: monitoring the video camera and writing field notes. Johnson used a semi-structured protocol for the interviews, to gather evidence of three areas: students' comprehension of assessment questions; students' engagement in covariational reasoning; and students' experiences with the technology. GRAs used the protocol to organize their field notes. Because we designed the covariation assessment to work on smartphones, tablets, and computers, we wanted students to work across a range of devices. Students could bring a device of their choosing. If students did not bring a device, or if students wanted a different device, we had a tablet and a laptop computer available for use. We allowed students to choose a device, because that is what students would do during the actual assessment.

Data Analysis

Qualitative analysis. Johnson led the qualitative analysis, adapting the four-item coding framework from Johnson et al. (2020). Table 2 shows the codes, descriptions, and examples. The COV and VAR codes are bolded, because they represented conceptions of graphs in terms of quantities and relationships. When students reasoning was coded as COV or VAR, they provided evidence of engaging in reasoning consistent with at least the levels of gross variation (VAR) or gross coordination of values (COV), per Thompson and Carlson (2017). When students' reasoning was coded as MO or IC, they provided evidence of a conception of the object's literal motion (Kerslake, 1977) or an object's physical appearance (Leinhardt et al., 1990).

Table 2: Four Item Coding Framework, Adapted from Johnson et al. (2020)

Code	Description	Example
COV	**Relationship between two attributes capable of varying and possible to measure**	**As the toy car's total distance traveled increases, the toy car's distance from the center decreases then increases.**
VAR	**A single attribute capable of varying and possible to measure**	**The Ferris wheel cart's distance from the center gets larger, then smaller.**
MO	Literal motion of an object	Nat walked back and forth, so the graph goes back and forth
IC	The shape of an object	The graph is shaped like a fishbowl.

Johnson watched each student's interview video, drawing on what students said and did as sources of data. For each item, in the order they appeared for each student, Johnson assigned students a single reasoning code. If students demonstrated multiple forms of reasoning, Johnson assigned the most sophisticated one as the overall code, to indicate if students were conceiving of quantities and/or their relationships. For example, if a student said that the graphs showed how the toy car moved around the track (MO), then went on to explain how each of the distances were changing together (COV), Johnson coded the reasoning COV. Prior to our quantitative analysis, Johnson shared the codes and notes with another researcher as a member check.

Quantitative analysis. Analysis of variance (ANOVA) was used to analyze device effects. A simple linear regression model, at 95% confidence, was used to analyze the relationships between students' total number correct and evidence of their covariational reasoning.

Results

Students used a range of devices on the assessment: 13 used a computer, 10 a tablet, and 7 a smartphone. The choice of device did not impact students' reasoning. Students who selected a computer or tablet expressed a preference for a larger screen size or touch screen capability. Students who selected a smartphone preferred to use their own device, which they had with them. None of the students expressed dissatisfaction with the interface on their device.

Across the four items, students selected a correct graph 48% of the time; 55% of student responses provided evidence of covariational reasoning (COV), 23% variational reasoning (VAR), 25% motion reasoning (MO), and 1% iconic reasoning (IC). Across the 30 students, 10% got all four items correct; 23.33% three items correct, 23.33% two items correct, 40% one item correct, and 0.34% zero items correct. The correlation between the total number correct and students' evidence of covariational reasoning was .709, which indicated a significantly high degree of correlation ($p < .01$). The regression analysis also showed that 50.2% of the total variation in the dependent variable, students' total number correct, can be explained by their covariational reasoning, which is also statistically significant ($p < .001$).

Discussion

This fully online assessment, containing built-in video animations, and designed for the undergraduate college algebra population, is the first of its kind. Our results demonstrate its validity, for assessing covariational reasoning, at least at the level of gross coordination of values, per Thompson and Carlson (2017). Furthermore, distinguishing between students engaging in variational or covariational reasoning, rather than motion or iconic reasoning, could be useful for diagnosing students' figurative or operative thinking (Moore et al., 2019) on graphing tasks.

We are encouraged that students' choice of device did not impact their reasoning. Given the prevalence of mobile phones, the feasibility of the assessment for use on this type of device can increase its potential for usability.

Continued validation work with a larger sample size is a vital next step. We are expanding the assessment to include six items and examining the qualitative coding scale via Rasch modeling to quantitatively corroborate its hierarchical nature.

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