PROGRESSIONS IN MATHEMATICAL REASONING: A CASE STUDY OF TWO TEACHERS' LEVELS OF UNITS COORDINATION

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Progressions in Mathematical Reasoning: A Case Study of Two Teachers' Levels of Units Coordination

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ABSTRACT

This dissertation case study examines how teachers' levels of units coordination afford their ability to reason multiplicatively and fractionally, as well as use their mathematics knowledge for teaching to analyze student multiplicative and fractional reasoning, when a two-year intervention is put into place to evoke new mathematical reasoning. The purpose of the study is to link levels of units coordination and teachers' growth in order to predict teachers' ability to successfully construct new mathematical reasoning and pedagogy through professional development and intervention. This qualitative case study addresses three research questions: 1) What pathways of reasoning, markers and transitions, may teachers go through? That is, what changes in their multiplicative and fractional reasoning schemes could be inferred? 2) To what extent, and in what ways, do teachers' levels of units coordination affect their learning pathway? and 3) How do teachers' levels of units coordination affect their ability to recognize levels of units in their students? The study involves two participating elementary teachers, Nancy and Marsha, who engaged in a project titled, *Student-Adaptive Pedagogy for Elementary Teachers (AdPed)*, in which specific interventions were put into place to foster a conceptual understanding of multiplicative and fractional reasoning. Four main contributions to the field are discussed: 1) Expansion of the mathematical knowledge for teaching construct to include the analysis of teachers' levels of units coordination and the effect on their specialized knowledge of content, 2) a new methodological approach to examining mathematical knowledge for teaching that includes the collection of qualitative data through one-on-one intervention and coaching sessions, 3) a theoretical extension to the mathematical knowledge for teaching construct that includes a constructivist lens, specifically a linkage with the hypothetical learning trajectory construct, and 4) Implications for researchers in the field.

The form and content of this abstract are approved. I recommend its publication.

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INTRODUCTION

In this dissertation study, I address the problem of how teachers' ability to coordinate multiple levels of units may affect their construction of multiplicative and fractional reasoning schemes, as well as their ability to analyze their students' mathematical reasoning in the classroom. By addressing this problem, I intend to contribute to the field's current knowledge of teachers' mathematical reasoning, as well as their mathematical knowledge for teaching (referred to as MKT) (Hill & Ball, 2004). As the field becomes more knowledgeable of teacher reasoning, we can discover strategies for helping teachers move from a procedural understanding of mathematics to a conceptual understanding, specifically through the construction of assimilatory units, which in turn can improve children's mathematics reasoning. As teachers' mathematical knowledge for teaching improves, their mathematical pedagogy improves as well, which I see as stemming from the teachers' ability to coordinate multiple levels of units simultaneously (Hill & Ball, 2004; Norton et al., 2015). As teachers build up, and assimilate, more units into their assimilatory schemes, their mathematical reasoning transitions and affords them the ability to recognize their students' reasoning and levels of units coordination. By pinpointing possible linkages between levels of units coordination and teachers' learning trajectories, researchers and mathematics educators may be able to predict teachers' ability to engage in specific mathematical tasks and interventions, as well as possible success in constructing new mathematical schemes. In turn, teachers' improvement in reasoning, specifically the assimilation of more levels of units, is likely to also foster their students' reasoning (Hill & Ball, 2004; Tzur et al. 2015).

Research in the field of mathematics education has shown that teachers often do not have the level of mathematical knowledge needed to support teaching their students mathematics using quality instruction. According to Hill & Ball (2004), many teachers are lacking knowledge of "mathematical reasoning, … the meaning of mathematical ideas and procedures, and … how ideas and procedures connect" (p. 331). Thus, to teach mathematics, teachers typically rely on curricula and procedural understanding rather than on conceptual understanding underlying those curricula. The need for such conceptual understanding has led many schools and districts in the USA to implement professional development opportunities for teachers so they may begin working on developing a better understanding of the mathematics they teach (Hill, Rowan, & Ball, 2005; Arbaugh et al., 2010; Tzur et al., 2015).

Teachers may benefit from professional developments (referred to as PD) that help them construct new mathematical reasoning, specifically PDs that target levels of units coordination, multiplicative reasoning, and fractional reasoning. Research shows that many students entering into middle school are lacking the ability to assimilate multiple levels of units (Steffe, 2007). If that is the case, then many adults (teachers included) may also struggle to assimilate multiple levels of units. Therefore, teachers who need to help students construct more levels of units may not have the ability to do so. This inability to assimilate multiple levels of units may also constrain a teacher's ability to construct new mathematical knowledge, specifically multiplicative and fractional reasoning, which both require the assimilation of three levels of units (discussed further in Chapter II). Again, if teachers do not have the reasoning themselves, it will constrain their ability to help their students construct the reasoning as well. By studying how units coordination in the teacher population affects their construction of new mathematical schemes, researchers may begin to determine new ways for promoting teachers' construction of new ways of reasoning, which in turn will positively affect their MKT.

One such professional development, recently implemented by a research team, was designed to promote *Adaptive Pedagogy for Elementary Teachers* (referred to as AdPed) (Tzur et al., 2015). The research team examined teachers' mathematical knowledge and put interventions into place to foster teachers' construction of multiplicative and fractional reasoning, that is, conceptual understanding in those areas, as well as their mathematical pedagogy, throughout the two-year professional development. The PD focused on helping the participating teachers construct new multiplicative and fractional schemes, and their transitions from perception-based pedagogy towards a more conception-based pedagogy, based on previous research done in the field. The project's PD program drew on six multiplicative reasoning schemes articulated by Tzur et al., (2013); the fractional schemes drew on eight schemes articulated by Tzur (2014) based on researchers' work on fractional reasoning (Hackenberg & Tillema, 2009; Norton & Hackenberg, 2010; Steffe, Liss, & Lee, 2014; Steffe & Olive, 2010). I will further explain these 14 schemes in Chapter II.

The AdPed project PD program included two week-long summer institutes, monthly "buddy-pair" teaching sessions, and quarterly grade-level workshops. Throughout these intervention sessions, the researchers taught mathematics content and led the teachers through tasks designed to elicit new conceptual understandings for the teachers, and to help them transition to a more conceptions-based pedagogy. As the two-year PD program progressed, it seemed some of the teachers were able to construct the new conceptual understandings earlier and deeper than other teachers, despite receiving the "same" interventions (Hodkowski, 2018). This led to the teachers having differing ability levels for noticing their students' reasoning in the classroom. While there may be other factors at play, in this dissertation, such differences are linked with the teachers' available (assimilatory) schemes, specifically their levels of units coordination (further discussed later in this chapter). The focus on the teachers' assimilatory schemes stems from past research done on the connection between teachers' mathematics and their pedagogical teaching (Hill & Ball, 2004; Hill, Rowan, & Ball, 2005; Tzur et al., 2015). The similarities and differences we observed in the learning of two specific teachers became of great interest to the project's team and constitutes the heart of this dissertation study. A critical reason for using these two teachers as exemplars is that both worked as partners while teaching same-grade classes. Both teachers (Nancy and Marsha, pseudonyms) were inferred to manifest key transitions in their reasoning (transitioning from not knowing to knowing) from the start to the end of the PD program. However, their transitions differed greatly, due to their different assimilatory schemes coming into the project.

The similarities and differences in the two teachers' reasoning became apparent as the researchers worked with them in their classrooms, as well as during content-specific workshops in which the teachers were asked to discuss their thinking while solving mathematical tasks. This phenomenon became an interesting topic of discussion for the researchers and led to this study, which examines (compares and contrasts) how the two participants progressed through the learning trajectory differently. The framework by which I shall compare and contrast the two teachers will focus on their ability to coordinate multiple units (further discussed in Chapter II). This study examines Nancy and Marsha's mathematical knowledge, and its evolution throughout the AdPed study—particularly their units coordination schemes for multiplicative and fractional reasoning, as well as the evolution of their Mathematical Knowledge for Teaching.

Mathematical Knowledge for Teaching

Hill & Ball (2004) differentiated between two dimensions of teacher content knowledge – specialized knowledge of content and common knowledge of content. Specialized knowledge of content refers to a teachers' ability to identify students' computational methods and analyze them for effectiveness and generalizability. This dimension of knowledge requires a conceptual understanding of the mathematics in order for the teacher to deconstruct what the students are doing and how it is related to their thinking. A teacher's assimilatory scheme allows them to assimilate the mathematics they are teaching, as well as assimilate the mathematics they are noticing in their students. Hill & Ball did not address the issue of units construction in their research, but I find it to be foundational to MKT: A teacher's specialized knowledge of content can only go as far as their personal levels of units coordination allows them to go. An example of this would be when a teacher is looking at how a child is reasoning in a multiplication problem – is the child operating only on units of one, or are they able to coordinate units of one and composite units simultaneously (the notion of composite units will also be explained in Chapter II)? In this study, I examine how Nancy and Marsha are able to notice and analyze the units their students are assimilating, and operating on, and how transitions in their own assimilatory units affects their ability to recognize those units in their students.

Common knowledge of content refers to a teacher's ability to determine a student's computational work for accuracy – did the student use the correct procedures and get the correct answer? An example of common knowledge is the teaching of algorithms, such as the partial-products algorithm taught in schools (e.g., 15 x 4 is the same as (10 x 4) + (5 x 4) = 60). This type of content knowledge needs a procedural understanding of the content in order to determine if students are solving the problems correctly.

Hill & Ball claimed that both content knowledge dimensions are needed in order to teach mathematics effectively; teachers need to be able to analyze students' mathematical work for effectiveness, generalizability of novel methods, and accuracy all at once. Hill & Ball called this twofold knowledge *mathematical knowledge for teaching* (MKT), which encompasses all aspects of a teacher's ability to teach mathematics effectively and conceptually to their students. Testing for teachers' MKT on a large-scale is difficult because MKT is usually tested through interviews, teaching experiments, and open-ended questioning (Hill & Ball, 2004). There have been measurements created that attempt to analyze teachers' MKT; however, while those measurements analyze the level of conceptual understanding the teachers have, they do not analyze how the teachers develop, or do not develop, the mathematical concepts being tested. This study attempts to go beyond an analysis of teachers' conceptual levels by articulating, from a constructivist perspective, three sub-dimensions inferred as constituents of MKT: 1) inferences into teachers' multiplicative and fractional reasoning pathways, 2) teachers' assimilation of levels of units coordination, and 3) ways in which transitions in a teacher's levels of units coordination affect their specialized knowledge of content when considering students' units coordination and multiplicative/fractional reasoning.

Mathematics education researchers can gain further insights from detailing conceptual underpinnings of teachers' MKT—particularly its underlying units coordination schemes. In turn, such insights may help improve the mathematics being taught in elementary schools. Accordingly, in this study, I focus on Nancy and Marsha's specialized knowledge, particularly their levels of units coordination, how their assimilatory units changed throughout the PD, and how these transitions affected their mathematical pedagogy (recognizing and analyzing their students' assimilatory units). By targeting each teacher's specialized knowledge growth through the lens of their units coordination, we may gain better understanding of what a PD for teachers could set as goals for their learning – and how such learning may come about. Once the research field has a better of understanding of this, researchers may begin to develop targeted PD for teachers that will help them transition from a procedural understanding of mathematics to a more conceptual understanding, specifically in multiplicative and fractional reasoning. I believe this transition begins with understanding how teachers construct and assimilate different levels of units in their multiplicative and fractional reasoning. The hope is that as teachers construct these levels of units in their own reasoning, they will begin to identify and analyze those units in their students, in order to develop their own interventions for eliciting unit construction in their students. Because the lens of units coordination draws not on the general work about teachers' MKT, but on a constructivist framework, I turn to a brief discussion of that framework (more is discussed in Chapter II).

Conceptual Framework: Constructivism

This study draws on a constructivist theory of learning (Piaget, 1976). Specifically, it uses the lens of units coordination (Norton et al. 2015; Steffe, 1992) for examining how the teachers' reasoning (multiplicative, fractional) and levels of units coordination may evolve. I first present the two key constructs of the constructivist theory, assimilation and accommodation. Then, I relate the core notion of scheme to those two constructs. Finally, I further situate the notion of units coordination within scheme theory.

According to von Glasersfeld (1995), constructivism begins with the "assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience" (p. 1). This construction of new knowledge occurs through the coordination of cognitive structures and operations, which occur in cycles as external elements are integrated into those systems of abstracted experience (Piaget, 1985). When new elements are experienced, a learner may have a perturbation (e.g., an obstacle to how they previously accomplished their goal, or an unexpected success), which can lead their cognitive system into disequilibrium (Piaget, 1985; Steffe, 2011). Once in disequilibrium, the system functions to reequilibrate, that is, to restore equilibrium. If the reequilibration is successful, new schemes are constructed within the system. The key to the ideas of perturbation and reequilibration is the awareness that they result from how a person's available schemes afford and constrain what may become a perturbation and/or reequilibration. That is, a conceptual change is postulated to commence with assimilation of external elements into available schemes (Piaget, 1985).

Assimilation

Assimilation refers to the incorporation of external (or internal) elements – such as aspects of a mathematical task - into previously constructed schemes and concepts (Piaget, 1985). According to Piaget (1976), "…no behavior, even if it is new to the individual, constitutes an absolute beginning. It is always grafted onto previous schemes and therefore amounts to assimilating new elements to already constructed structures" (p. 17). In essence, assimilation is the process of integrating new experiences into already existing cognitive structures or schemes (von Glasersfeld, 1995). Such a process leads to the mental system setting a goal toward which it would direct activities on some objects (e.g., units) with an expectation for a particular result. I illustrate this central constructivist notion with an example using multiplicative reasoning.

For example, two students are asked to build three towers of six cubes each, with two of the cubes in each tower being one color and the other four cubes in each tower being another color (Figure 1.1). After building the towers, they are asked how many cubes they have in all (answer is 18) and how they got their answer. Student A assimilated the task into a scheme in which she needed to physically split up the three towers into nine towers of two cubes each (Figure 1.2). She explained that she solved the task that way so that each tower had an even number of cubes, and she could count by twos to find the total. In Chapter II, I return to this example while discussing the level of units coordination involved in this way of operating.

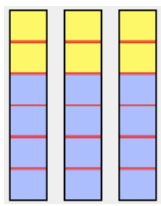

Figure 1.1 - Three towers of six cubes, each composed of 4 blue and 2 yellow cubes.

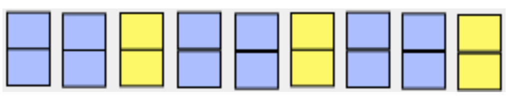

Figure 1.2 - Nine towers of two cubes.

Student B, on the other hand, assimilated the task into a scheme that allowed her to look at the original three towers and see them also as three towers of four and three towers of two. She didn't need to physically split up the towers into same-size units. Rather, she could mentally conceive of the towers of 6 as decomposed into towers of 2 and towers of 4. The student's assimilatory scheme did not require her new towers to be of equal size like Student A. Student B was then able to apply the distributive property to multiply 3x4 and 2x4 followed by the addition of the results (12+6 = 18). The two students assimilated the tasks into their available schemes, and then operated on the towers and cubes using different activity sequences. Student A's assimilatory scheme required composite units of the same size in order to then count by 2's. Student B's assimilatory scheme allowed for composite units of differing sizes in order to use the distributive property to multiply two compilations of towers and cubes and add their products together.

Accommodation

Assimilation may eventually lead to accommodation, that is, to the modification (reorganization) of an existing scheme into a new scheme (Piaget, 1985). Once new information is assimilated into existing schemes, an internalized activity and its corresponding result (further explained in the next section) occur that is either positive or negative (von Glasersfeld, 1995). If the result does not fit the one expected by the person, then a perturbation occurs and the original assimilation may be reviewed while the triggering characteristics will no longer result in the same activity (von Glasersfeld, 1995). If the result fits the expected one, the characteristics become "a new recognition pattern" (von Glasersfeld, 1995, p. 65) within a new scheme (von Glasersfeld, 1995). The accommodation of the old scheme into a new scheme helps the learner regain their cognitive equilibrium.

Accommodation can be further illustrated by revisiting the example above of the multiplication task solved by Student A and Student B. The explanation given by Student B for the task may have caused a perturbation for Student B, and possibly an accommodation of her previous way of operating. When asked to do another, similar problem, Student A engages in the activity differently the second time around. When asked to build five towers of thirteen cubes each, Student A builds her towers, so each tower has ten black cubes and three white cubes. She is already anticipating the easiest way to decompose the thirteen – into a ten and three. From there, she solves the problem by first counting the black cubes by 10 – 10, 20, 30, 40, 50 – and then multiplying three times five. This time, Student A's towers were not of equal size, and she did not seem to need to physically split the towers up in order to "see" them in a new way. This shift in her reasoning from the first (3 towers of 6 cubes) to the second (5 towers of 13 cubes) task is due to an accommodation in her way of operating, which seemed more compatible with Student B's solution to the first task. Student A assimilated new information from Student B's earlier explanation into her existing scheme, which resulted in a perturbation for her. This perturbation led to Student A to review her original assimilation and creating a "new recognition pattern" (von Glasersfeld, 1995, p. 65) based on the new way of reasoning brought forth by Student B.

Scheme Theory and the Reorganization Hypothesis

According to von Glasersfeld (1980; 1995), a scheme is an interiorized, mental structure that underlies observable actions and consists of three parts. The first part is when an individual experiences and recognizes a situation, which sets a goal that would direct her actions. The second part is a mental activity triggered to accomplish that goal. Being able to recognize the situation, which triggers a previously known activity, is a result of assimilation. The third part of a scheme consists of a result expected to ensue from the goal-directed activity.

The construction of multiplicative and fractional schemes, the backbone of one's reasoning in these areas, is hypothesized to come about as reorganization of currently constructed number sequences. The *reorganization hypothesis* (Steffe & Olive, 2010; Steffe et al., 2013) is an expansion of the notion of accommodation, while focusing on the linkage between additive reasoning with whole numbers and how it serves as a basis for constructing multiplicative and fractional schemes. The reorganization hypothesis can be used to reorganize previous understandings of natural numbers, rather than just extending those previous number schemes. Steffe & Olive (2010) claim that "children's fraction schemes can emerge as accommodations in their numerical counting schemes" (p.1), through the reorganization hypothesis. Through this reorganization, children use the operations they already understand from their previous numerical counting schemes, in new ways and in new situations brought forth in the new fractional reasoning schemes.

Reorganization hypothesis is relevant to this study as I will examine Nancy and Marsha's mathematical reasoning to determine how their previous understandings of number either interfered with or were reorganized into new understandings. For example, if we go back to the example from earlier, Student A experienced a reorganization of her previous reasoning scheme after hearing Student B's explanation using the distributive property. After experiencing the perturbation from Student B's explanation, Student A began to use her previous operations and counting schemes in a new way, resulting in an accommodation of her previous schema.

The hypothesis that constructing schemes for multiplicative and fractional reasoning can arise as reorganization of whole number schemes has been expanded by Tzur and his colleagues (Simon & Tzur, 2004; Simon et al., 2004; Tzur, 2011). Specifically, they proposed a model, termed reflection on activity-effect relationship (Ref*AER), that accounts for the cognitive change process underlying reorganization. In this dissertation study, I draw on the Ref*AER model, as it helps specify changes I attributed to Nancy and Marsha's schemes. I elaborate on this model of scheme reorganization in Chapter II.

Units Coordination

The mathematical lens I will be using to analyze Nancy and Marsha's construction of schemes for multiplicative and fractional reasoning is that of levels of units coordination (Norton et al., 2015; Steffe, 1992). While I provide a detailed explanation of this model in Chapter II, I briefly explain here how it relates to my study of Nancy and Marsha's mathematical knowledge and their MKT (specialized knowledge of content). Units coordination focuses on the ability of an individual to assimilate— notice and keep track of— different types of units simultaneously (Norton et al., 2015; Steffe, 1992). Such assimilation, including a goal-directed activity involving coordination of these units, requires one to understand that smaller units can be embedded within larger units. In our example of Student B's solution above, she was operating on 3 units of six 1s (Ones, units of 1) as a simultaneous composition of 3 units of four 1s and 3 units of two 1s. Furthermore, she could keep track of all the units while operating on them separately (one tower of four is four, two towers of four is eight, three towers of four is twelve; one tower of two is two, two towers of two is four, three towers of two is six) (Norton et al. 2015; Steffe, 1992). This would require operating on three levels of units simultaneously: 18 (one level) consisting of three units of six 1s (second level), each of which consisting of 3 units of four plus two 1s. Student A, on the other hand, operated sequentially (not simultaneously) on the three units of six 1s, essentially decomposing them into same-size units of two 1s each (first and second levels).

This study examines the hypothesis that teachers' assimilatory schemes, and the level of units coordination involved, may serve as a basis for their construction of more advanced schemes. Examining such a hypothesis can contribute to the literature in that, to date, most studies on levels of units coordination have been conducted with children. Only a few studies (Lovin et al., 2018; Stevens et al., 2018) have recently examined adults' (particularly teachers') reasoning through the units coordination lens. Furthermore, this study sheds light not only on different "markers" in a teacher's advance from one, to two, and to three levels of units coordination, but also on the plausible process of change, that is, on the conceptual transformation (reorganization) involved (Tzur, 2019).

Research Questions

This qualitative, case study addresses three research questions:

1. What pathways of reasoning, markers and transitions, may teachers go through? That is, what changes in their multiplicative and fractional reasoning schemes could be inferred?
2. To what extent, and in what ways, do teachers' levels of units coordination affect their learning pathway?
3. How do teachers' levels of units coordination affect their ability to recognize levels of units in their students?

CHAPTER II

LITERATURE REVIEW

This study examines conceptual transitions in teachers' mathematical knowledge and how those transitions affect their MKT, with a lens on their units coordination (described later in this chapter), during a two-year professional development program aimed at providing tasks to elicit conceptual change in the teachers' mathematical reasoning. This study is important for the field of mathematics education research, because it is a first attempt to link three lines of work into one study: Hill et al.'s (2005) mathematical knowledge for teaching (Hill & Ball, 2004; Hill et al., 2005; Tzur et al., 2016), units coordination (Norton et al., 2015; Steffe, 1992), and transition research (Tzur, 2019). Mathematical knowledge for teaching (MKT) has been studied in both practicing and pre-service teachers; however, the studies have only looked at what Tzur (2019) considered as conceptual markers, rather than transitions in teachers' reasoning. Furthermore, research on teachers' MKT did not include analysis of teachers' levels of units coordination—an analysis that, to date, has all been focused on students and pre-service teachers – and has not linked how a teacher's levels of units coordination may affect their MKT. Most of these units coordination studies have also been marker studies, rather than transition studies. The linkage my study seeks to establish focuses on transitions in teachers' MKT in terms of shifts in their reasoning, from specific levels of units coordination to the next.

In this chapter, I examine previous research that informs my study. I begin with an overview of MKT, its importance, and how to measure it. I then summarize research on units coordination and tie it to schemes for multiplicative and fractional reasoning. These specific schemes are important for this study as they were mathematical domains the AdPed project promoted in teachers. That is, those schemes provided a context for analyzing transitions in the participating teachers' levels of units coordination. I end with a discussion on *how* transitions in conceptual reasoning can be explained through a constructivist lens on learning—particularly the reflection on activity-effect relationships framework (Simon et al., 2004; Simon & Tzur, 2004).

Mathematical Knowledge for Teaching

A teacher's own mathematical reasoning is important for them to effectively teach mathematics to their students, including the key activity of analyzing their students' reasoning (Hill et al., 2005; Tzur et al., 2016). Without such analyses, teachers cannot inform their own instruction with what students already know, nor make changes in their mathematics teaching practices (Tzur et al., 2016). That is, a teacher's MKT affords or constrains what they can notice and take as a basis for what and how they teach. In turn, what a teacher can notice in their students' assimilatory reasoning gives evidence into the teachers' own mathematical reasoning. In other words, teachers can only notice in their students' reasoning that which they have constructed for themselves. Therefore, it is essential for mathematics education researchers to determine what teachers' MKT is and how to improve teachers' conceptual reasoning in mathematics. This study attempts to go beyond previous research in MKT by articulating, from a constructivist perspective, three sub-dimensions inferred as constituents of MKT: 1) inferences into teachers' multiplicative and fractional reasoning pathways, 2) teachers' assimilation of levels of units coordination, and 3) ways in which transitions in a teacher's levels of units coordination affect their specialized knowledge of content when considering students' units coordination and multiplicative/fractional reasoning.

One sub-dimension of MKT that many teachers need to improve is their multiplicative and fractional reasoning (Tzur et al., 2016). Many elementary teachers seem to simply think of multiplication as repeated addition or equal groups of items (Fischbein et al., 1985; Greer, 1992; Lamon, 2012; Thompson & Saldanha, 2003). What they seem to lack is a way of reasoning multiplicatively in which the focus is on the distribution of one unit (a single unit) over another unit (a composite unit) which results in the creation of a third unit (this way of reasoning is further explained below). Lacking multiplicative reasoning, including the ability to coordinate (distribute) different types of units, is also postulated to underlie limited understandings of fractions (Hackenberg, 2013, 2010; Hackenberg & Tillema, 2009; Steffe & Olive, 2010).

A majority of teachers may only understand fractions procedurally as part-of-whole, which is limiting to their reasoning beyond basic unit and non-unit fractions (Lovin et al., 2018; Tzur & Depue, 2014a, 2014b). When fractional reasoning goes beyond that, for instance, into improper fractions and fractions-of-fractions (schemes underlying these terms will be explained later in this chapter), they may be unable to reason at that level. Such a limitation in one's fractional reasoning may also constrain their ability to reason about decimals, ratios, percent, and algebraic reasoning (Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsak et al., 2012; Norton & Boyce, 2013; Norton et al., 2015; Olive & Caglayan, 2008; Steffe & Olive, 2010). For example, Tzur et al. (2016) worked with a teacher (pseudonym Annie) whose understanding of fractions as part-of-whole constrained her understanding of decimals and equivalent fractions as anything more than a procedural understanding tied to place-value. Annie did not see the decimals and fractions as quantities that result from a whole being determined by its fractional parts (a scheme underlying this understanding will be explained in detail later).

While there have not been studies focused on practicing teachers' fractional understandings, there have been several studies that analyzed pre-service elementary teachers' (PSTs) fractional reasoning. These studies found that PSTs have a number of difficulties with fractional computations (Armstrong & Bezuk, 1995; Ball, 1990; Newton, 2008; Tirosh, 2000). Newton (2008) found that PSTs struggled with operating on fractions and made computational mistakes when adding, subtracting, multiplying, and dividing fractions. When adding and subtracting, PSTs often added or subtracted across both the numerators *and* denominators (i.e., 1/4 + 1/5 = 2/9). When multiplying fractions with the same denominators (e.g., 2/15 * 7/15), they would often omit multiplication of the denominators. In contrast, they would multiply the denominators if they were not the same (i.e., 2/15 x 1/4= 2/60). Newton concluded that those teachers were overgeneralizing fractional computation rules when adding, subtracting, and multiplying fractions. When dividing fractions, many mistakes were made by the PSTs, including errors in cross-multiplying and inverting the dividend. Critically, even when PSTs can correctly solve such problems, they are usually unable to justify *why* the algorithms work (Borko et al., 1992; Tirosh et al., 1998; Toluk-Ucar, 2009).

PSTs especially struggle when the fractions are embedded within a story problem. They often revert back to memorized procedures, seemingly without understanding the procedures due to their lack of a conceptual understanding of fractions and how they work (Lovin et al., 2018). These studies about PSTs informed my research problem, as they give clues about how practicing teachers may reason about fractions themselves. However, there is a lack of research that delves into practicing teachers' ways of reasoning about fractions, particularly how those ways of reasoning may be afforded or constrained by their levels of units coordination.

Adults, specifically teachers, can greatly benefit from transforming their fractional reasoning from part-of-whole reasoning to an understanding of fractions as a multiplicative relationship between the whole and its fractional parts. However, this type of reasoning may be challenging for adults to construct, as they are likely to assimilate fractional tasks into their deeply ingrained part-of-whole scheme (Tzur & Depue, 2014a). Said differently, adults may be constrained by their part-of-whole reasoning when attempting to learn new schemes rooted in conceptualizing fractions as multiplicative relations.

As we work with teachers to increase their MKT so it includes higher levels of units coordination, they may get stuck (experience perturbations, see more below) that can lead to folding back onto their old ways of reasoning (Pirie & Kieren, 1994). Folding back occurs when an individual is presented with a task that is not currently within in their assimilatory scheme, and "one needs to *fold back* to an inner level in order to extend one's current, inadequate understanding" (Pirie & Kieren, 1994, p. 69). However, teacher educators may foster reorganization of old schemes into new fractional reasoning (Steffe & Olive, 2010; Steffe et al., 2013). As Tzur & Depue (2014a) found, a

> Constructivist-informed task design for teaching unit fractions promotes adults' (teachers') desired combination of conceptual understanding and computational mastery … Improving this fluency…indicates not only enhanced performance on rule-based, timed-test questions but also that 'it is never too late to re-learn.' (p. 6)

At issue is how might teachers re-learn so they can assimilate new mathematical concepts into their current schemes? This question will be further explored later in this chapter.

A stance on MKT that currently seems missing and called for by other researchers (Silverman & Thompson, 2008; Hackenberg, 2010), is a constructivist framework to help "understand how teachers transform such understandings into pedagogically powerful tools, whereby they can design instruction in this area for students" (Hackenberg, 2010, p. 429). Simon (2006) greatly contributed to this aim through his notion of *key developmental understanding* (KDU), which is a "change in the learner's ability to think about and/or perceive particular mathematical relationships" (p. 993). Silverman & Thompson (2008) expanded on this notion by asserting that teachers developing MKT must be able to transform their own KDUs to include an understanding of how their own KDUs may empower compatible KDUs in their students and how their (teachers') actions may support the development of those KDUs in their students.

When we examine MKT through this constructivist lens, we can characterize MKT as a second-order model which teachers may construct and use to understand a "subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi). In other words, when a teacher is able to use their KDUs to better understand their students' reasoning and their possible pathways through that reasoning, they are building stronger MKT and constructing a second-order model of their students. Through this study, I attempted to provide empirical evidence in support of these theoretical calls for the inclusion of a constructivist framework within MKT, which was used to explain student reasoning and is here forth expanded to examine teachers' KDUs. Specifically, the KDUs I examined in this study were Nancy and Marsha's progressions through multiplicative and fractional reasoning schemes, as well as their levels of units coordination.

Before digging into *how* teachers may construct new schemes, that is, learn conceptually, I turn to explaining foundational components of their MKT that are at the forefront of this study – units coordination, multiplicative reasoning, and fractional reasoning.

Units Coordination

Whereas MKT research has delved into practicing teachers' mathematical reasoning, it seemed to lack a focus on teachers' levels of units coordination. This study's focus on units coordination brings a new lens to the MKT research and the field of mathematics education research. The study examines two new sub-dimensions of MKT related to levels of units coordination – how teachers may assimilate levels of units and transition to higher-levels of units coordination over time, as well as how teachers' levels of units coordination affords or constrains their MKT, especially when considering their students' multiplicative and fractional reasoning. In this section, I review current research on units coordination and explain how it may relate to teachers' understandings of mathematical concepts. As noted above, prior research on units coordination has focused on students, with no research on practicing teachers' levels of units coordination. This study examines levels of units coordination in a new population – practicing teachers. Understanding teachers' levels of units coordination may give researchers a new understanding of teachers' reasoning, including how units coordination may serve in promoting their construction of new mathematical schemes, as well as their ability to identify and make inferences into their students' assimilatory units and reasoning.

"Units coordination refers to students' abilities to create units and maintain their relationships with other units that they contain or constitute" (Norton et al., 2015, p. 111). For instance, to multiply 4 x 3, a student would have to distribute 4 units of 1 across units of 3 in order to get a total quantity of 12, which is a composition of three units of four 1's. The coordination of two and three levels of units is essential for students to develop mathematical understanding in counting, multiplication, integers, fractions, and algebraic reasoning (Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsak et al., 2012; Norton & Boyce, 2013; Norton et al., 2015; Olive & Caglayan, 2008; Steffe & Olive, 2010). Steffe (1992), stated that units coordinating is "the mental operation of distributing a composite unit across the elements of another composite unit" (p. 279).

Three stages of units coordination have been identified when people operate with/on whole numbers (Norton & Boyce, 2015; Norton et al., 2015). My examples to illustrate each of those stages are related to a visual found in those scholars' work (Figure 2.1). In stage 1, students may be able to coordinate two levels in activity, and one level taken as given. The term "in activity" means the student needs to carry out the full operation in order to assimilate the unit into their current scheme. The term "taken as given" refers to a student's ability to assimilate the unit into their current scheme without the need to carry out the activity (Norton & Boyce, 2015; Norton et al., 2015). For example, a child who is at this stage and is asked how many units of 3 are in the number 12, would be able to assimilate only the single units (1s) making up the 3 and 12 as given. They would need to actively group units of three in order to figure out how many there are in the number 12. This would be done through some sort of activity sequence since only the single units were taken as given. In other words, they would have to take their units of one, group them into units of 3, and then figure out how many of those units (four) they needed to get to 12 (e.g. 1, 2, 3 = 1st unit of 3; 1, 2, 3 = 2nd unit of 3; 1, 2, 3 = 3rd unit of 3; 1, 2, 3 = 4th unit of 3; then they would count the 4 units of 3) .

Students working at stage 2 may be able to coordinate three levels of units, with one unit in activity and two levels taken as given. For example, a student at this stage would be able to take the single units of 1 as well as the units of 3 as given. They would then need to count their units of 3 in activity to find the total number of three's in 12 (e.g. 3, 6, 9, 12).

At stage 3, students can coordinate three levels and switch between structures while taking all three levels as given (Norton & Boyce, 2015; Norton et al., 2015). For example, a student at this stage would automatically know that there were 4 units of three in the number 12. They would not need to count anything in activity, because all three units (single units, 4 units of 3, and 12) would be taken as given.

According to Steffe (2007), an estimated 30-50% of students entering into middle school are operating at Stage 1, which significantly affects their ability to operate on fractions, integers, and algebraic reasoning (Hackenberg, 2013; Norton et al., 2015; Steffe & Olive, 2010). Transitions from one stage to the next require substantial conceptual leaps as students coordinate more units. Norton et al. (2015) developed a written instrument that can be used to assess units coordination with large populations of students. This instrument allowed them to determine how many units a child (or a teacher) could coordinate when working on a task, and therefore operate on, in order to evaluate their units coordination stage. I describe this assessment here, as it can help illustrate how students may reason with multiple units at different levels.

In the assessment, students were given a picture of three bars – a large blue one, a medium-sized yellow one, and a smaller red one (see Figure 2.1). The students were then given the following problem: "If the small red bar fit into the medium yellow bar three times, and the medium yellow bar fit into the long blue bar four times, how many times would the small red bar fit into the long blue bar?" (Norton et al., 2015, p. 112-113). At Stage 1 students may iterate the

Figure 2.1 - Norton et al., 2015

red bar in order to determine its relation to the length of the blue bar, but they would not see it as a multiplicative relationship; they would see the red bar as a unit of 1, but not see the blue bar as a composite unit that is n (e.g., 12) times as much as the red bar. This is because they can only take one level of units (the 1's) as given and coordinate it with another unit (the composite unit) in activity. At Stage 2 students may iterate the yellow bar four times, seeing each iteration as a composite unit made up of three 1's (e.g., 3, 3, 3, and 3 equals 12). They are able to take the first two units (the red bar of 1's and the yellow bar made up of three 1's) as given. However, the third unit (the blue bar made up of yellow bars) must be built up in the activity of iterating the yellow bar (which they conceive of as 4-units-of-1). They can assimilate the multiplicative relationship between the bars but can only coordinate the third unit in activity. At Stage 3 students would understand the multiplicative relationship as a coordination of a unit of units of units. Specifically, they would see the blue bar as four yellow bars, which are each made up of three red bars (i.e., 4 units of 3 units of 1). Importantly, this conception releases the need to perform any activity (e.g., iteration), as the three units are coordinated mentally (abstractly).

The mental operations underlying units coordination include *unitizing*, *iterating*, *partitioning*, and *disembedding* (Norton & Boyce, 2015; Norton et al., 2015). Unitizing refers to the chunking of single units into new units (Glasersfeld, 1981; Ulrich, 2015). For example, chunking together three single units makes a new unit of 3 that can now be operated on. Iterating refers to the repeated production of a unit to create a new unit (Hackenberg &Tillema, 2009). For example, a unit of 3 can be repeated, or iterated, four times to create a new unit of 12. Partitioning refers to the subdivision of a unit into smaller, equal-sized units (Hackenberg & Tillema, 2009; Kieren, 1980). For example, the number 12 can be partitioned, or split into four equal units of 3. Disembedding refers to the ability to reverse the embeddedness of units within other units, without destroying the larger unit (Steffe & Cobb, 1988; Ulrich, 2016). For example, the number 12 has four embedded units of 3 within it. These units of three can be compared to the unit of 12, and reflected upon as their own unit, while maintaining the embeddedness of the unit of 3 within the unit of 12.

A student at Stage 3 would be able to coordinate three levels of units within 12 by mentally partitioning it into units of 1, disembedding one of those smaller units and iterating it three times, unitizing that into a new unit of three 1's, and finally iterating that new composite unit four times to reproduce the original unit of 12 as a unit of four units of three 1's. A student at Stage 2 would be able to unitize sets of three 1's into units of one 3 in order to iterate the composite unit. They are also able to disembed the 1's from the 12 and the composite units of 3, and they can disembed the units of 3 from the four 3's within the 12. Stage 2 is different from Stage 3, however, because in Stage 2 the student has yet to maintain their new units as they iterate disembedded units. For instance, a student who has disembedded and iterated a unit of 3 two times to produce 6 would not be able to determine how many 3's are in two 6's because they do not take as given that 6 is also two units of 3; the student would have to start over and build up from 3 all over again (Norton & Boyce, 2015; Norton et al., 2015).

A student at Stage 1 may be able to produce a composite unit of 12, but it would not be a unit made up of four units of three 1's. Instead, it would be a sequence of 1's which were segmented into units of 3, but the student would not be able to work with both the 1's and the composite units of 3's simultaneously. The student would be able to iterate the 1's into groups of three or one group of 12 but would not be able to partition the 1's into units of 3. The student may be able to iterate a partitioned unit of 3 four times, but would not be able to unitize each unit of three 1's into one unit of 3 in order to iterate it and simultaneously see 12 as a unit of twelve 1's and four 3's. Therefore, a student at Stage 1 is only able to iterate and partition with single units (Norton & Boyce, 2015; Norton et al., 2015).

The examples I presented above illustrate units coordination in multiplicative operations on whole numbers. Yet, units coordination is required for conceptual understanding in many mathematical domains (Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsak et al., 2012; Norton & Boyce, 2013; Norton et al., 2015; Olive & Caglayan, 2008; Steffe & Olive, 2010). The mathematical domains in which I will use units coordination to analyze Nancy and Marsha's conceptual transitions are multiplicative and fractional reasoning (explained in the next section).

Units Coordination Within Multiplicative and Fractional Reasoning

Assimilatory schemes with two to three levels of units coordination are necessary for conceptual understandings in most mathematical areas, including multiplicative and fractional reasoning (Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsak et al., 2012; Norton & Boyce, 2013; Norton et al., 2015; Olive & Caglayan, 2008; Steffe & Olive, 2010). This dissertation study examines Nancy and Marsha's transitions in levels of units coordination as they progressed through multiplicative and fractional reasoning schemes, while engaging in specific tasks designed for those reasoning schemes. In this section, I first describe the current research in multiplicative reasoning. I then briefly explain the link between multiplicative and fractional reasoning, before also describing current research in fractional reasoning. I section culminates with an explanation of how teachers might construct these reasoning schemes and what types of constraints they may experience as they assimilate new reasoning into their current schemes.

Hackenberg's Multiplicative Reasoning Schemes

Hackenberg (2010), defined multiplicative reasoning "as the functioning of a person's multiplicative operations, multiplying schemes, and multiplicative concepts in ongoing interaction in her experiential world" (p. 391). According to researchers, students develop three multiplicative constructs, or schemes, which require the production and coordination of composite units, or units of units (Hackenberg, 2013; Hackenberg & Tillema, 2009; Steffe, 1992, 1994). Hackenberg (2010) outlined three multiplicative constructs and the levels of units coordination required to reason in each construct.

The first construct (MC1) does not require the students to take a composite unit as a given. This is the simplest construct and only requires the coordination of two levels of units. This coordination is done in activity and does not require the use of interiorized units

(Hackenberg, 2010; Steffe, 1994; Ulrich, 2015; Ulrich, 2016). An example of an MC1 problem would be: "A classroom has 6 rows of desks with 7 desks in each row; how many desks are there in all 6 rows?" To solve this problem, an MC1 student would coordinate the two units by inserting (activity) a composite unit of seven into each of the other composite units of six. To do this in activity, the student would have to complete the coordination to the end of the activity (sequentially), because the results would not already be available to the student; they could not anticipate the results prior to acting. A student who has interiorized the two levels of units (MC2) would be able to anticipate the results and would not have to actually enact the insertion in order to coordinate the two units. They would be able to take the insertion as given and could re-present it. The student who has not interiorized the structure of 42 as a unit of 6 units containing 7 units is not able to assimilate this multiplicative concept. For example, if they were told that a second classroom has 4 rows with 7 desks in each row and were asked to find the total number of desks in the two classrooms, they would be able to find the total number in each classroom and unite them to find the total, but would see them as two separate problems. This is due to the limitation of not having interiorized both levels of units. Thus, they cannot retain the quantities as structures involving three levels of units. A student who has interiorized three levels of units (MC3) would assimilate the follow-up problem into the old problem as a starting point for coordinating all levels of units. They would be able to solve the follow-up problem by combining the units of the two classrooms (6 units of 7 and 4 units of 7) into 10 units of 7 in order to determine that the total number of desks is 70. These students can coordinate the three levels of units and operate further on them in order to find the total number of desks.

Students who have constructed MC2 reasoning are able to coordinate two levels of units (Hackenberg, 2010; Ulrich, 2015; Ulrich, 2016). For example, they may be able to operate on a unit of three units, each containing five units, without having to make the partitions within the three units. These students can create a third level of unit in activity, this means that they must carry out the operation to obtain the third unit. Students working at the MC3 level are able to coordinate three levels of units (Hackenberg, 2010). For example, they may be able to operate on a unit of three units, each containing 5 units, and can mentally construct this before having to operate on it. They take the structure as a given and can manipulate or switch the structure to a different structure, such as five units, each containing three units (Hackenberg, 2013).

When one relates Norton et al.'s (2015) units coordination to Hackenberg's (2010) multiplicative reasoning constructs (MC1, MC2, MC3), it can be determined that Norton's Stage 1 of units coordination corresponds to Hackenberg's MC1 construct, because in Norton's first stage, students are able to coordinate two levels of units in activity, and one of those levels is taken as given. Hackenberg's MC1 construct requires that a student be able to coordinate two levels of units in activity, without the use of any interiorized units. The one difference between Norton's Stage 1 and Hackenberg's MC1 is the Stage 1 requires the students to have one level taken as given, while MC1 does not require any units to be interiorized. For the students to have one of the units "taken as given," they would have to be able to assimilate that one unit into their current scheme without having to fully engage in the activity. In MC1 however, the students do not need to interiorize any units, therefore, the operations do not yet need to be assimilated into the scheme for use in further operations; for Hackenberg, the units do not yet need to be "taken as given." Units coordination Stage 2 corresponds to MC2, because in Norton's second stage, students are able to coordinate three levels of units in activity, with two of those levels being taken as given. Hackenberg's MC2 construct requires students to coordinate two levels of units and create a third level in activity. The coordination of units in both Norton's Stage 2 and

Hackenberg's MC2 requires students to coordinate three levels of units, but only two of those units are taken as given, or interiorized, and assimilated into the current scheme; the third unit must be carried out in activity.

Stage 3 of units coordination corresponds to MC3, because in Norton's third stage, students are able to coordinate three levels of units and switch between those three levels while taking all three as given. Hackenberg's MC3 construct also requires that students be able to coordinate all three units and switch between them. MC3 also requires that all three levels be taken as given and interiorized into their assimilated scheme.

Tzur et al.'s Multiplicative Reasoning Schemes

Tzur et al. (2013) further developed a progression of multiplicative reasoning schemes, which include six schemes for multiplicative and divisional concepts. The first scheme, termed *Multiplicative Double Counting* (mDC), requires the student to coordinate a given number of composite units when each composite unit contains the same amount of 1's in order to find the total number of 1's within the whole set. This requires that students do a simultaneous double count to keep track of the number of composite units and the number of total 1's counted. An example of this would be a student who was asked to find the total amount of cubes when they have 3 towers with 4 cubes in each tower. To solve this problem, the student would have to coordinate the towers and embed the 4 cubes into each tower, while keeping track of the total number of cubes in all towers. Their double count would be, "1 tower of 4 cubes is 4, 2 towers of 4 cubes would be 8, and 3 towers of 4 cubes would be 12." This scheme may correlate to Hackenberg's MC1 scheme and Norton's Units Coordination Stage 1, if students coordinate two levels of units in activity (composite and individual units), but the units do not yet need to be taken as given. That is, students may use double counting and must iterate composite units in order to complete the problem (Ulrich, 2015; Ulrich, 2016). If a student can assimilate both quantities and anticipate the double-count without executing it, mDC is closer to MC2.

Tzur et al.'s second scheme is *Same Unit Coordination* (SUC), in which students use additive operations on two compilations of composite units to compare the two composite unit compilations. An example of an SUC problem would be: "I have 7 towers of 4 cubes each, and you have 12 towers of 4 cubes each. How many more towers do you have than me?" To solve this problem, the student would have to disregard the 1's making up the composite units in order to solve for the difference in composite units. This scheme correlates to Hackenberg's MC1 scheme, and Norton's Units Coordination Stage 1, because students are still coordinating two levels of units in activity, while having to understand the iterated composite unit being used in the problem in order to compare the two sub-compilations (Ulrich, 2015; Ulrich, 2016).

In the third scheme, *Unit Differentiation and Selection* (UDS), students use additive operations to compare two sub-compilations, which they accomplish through a coordinated-count to find the difference in 1's. An example of a UDS problem would be: "I have 199 six-packs of soda cans. You have 203 six-packs of soda cans. How many more soda cans do you have?" One way to solve this problem, would be to first use additive reasoning to find the difference in six-packs (4), and then use MDC to find the difference in sodas (4 x 6 = 24). Tzur (2019) referred to this way as "difference first," as the student's focus is first on the difference in composite units. A student who did not fully have the UDS scheme would first find the total number of soda cans in each set of packs and then find the difference in soda cans (to which Tzur, ibid, referred to as "total first"). This scheme correlates to Hackenberg's MC2 scheme, and Norton's Stage 2 Units Coordination Stage, because the students must be able to coordinate all three levels of units (composite units, individual units, and total compilation), with two of the units (composite and individual units being taken as given and the third level being created in activity (Ulrich, 2015; Ulrich, 2016).

The fourth scheme, *Mixed Unit Coordination* (MUC), underlies students' use of multiplicative and additive operations in order to make a global compilation of composite units. An example of an MUC problem is: "I am making bags of candy that each contain 6 pieces of candy. I have already made 4 bags of candy, and I still have 18 more pieces of candy. When I finish, how many total bags of candy will I have?" In order to solve this problem, the student would need to segment (or partition) the 18 pieces of candy into 3 more bags, and then add those 3 bags to the previous 4 bags in order to find a total of 7 bags of candy. This scheme correlates to Hackenberg's MC3 and Norton's Units Coordination Stage 3, because all three units must be coordinated and students must be able to effortlessly switch between those units (Ulrich, 2015; Ulrich, 2016). This is a very complex scheme, as it requires students to purposely switch their focus of attention between all three interiorized units in order to solve any MUC problems. All three units must already be constructed outside of the activity and taken as given before they can switch between the units and coordinate them.

Tzur et al.'s fifth scheme, *Quotitive Division* (QD), requires students to anticipate the partitioning of a given total of 1's into given-sized composite units. For example, a student solving the problem, "You have 21 cubes, and I would like you to put them into towers of 3. How many towers will you make?" This scheme may be considered the reverse operation of mDC, so a child could use a coordinated double count to solve for the total number of cubes, as long as they know that they must stop when they have reached 21 cubes, or 7 towers. Due to the required anticipation of two units taken as given (total compilation of 21 cubes; units of 3), and a third unit (7 towers) constructed in activity, this scheme correlates to Hackenberg's MC2 scheme and Norton's Units Coordination Stage 2 (Ulrich, 2015; Ulrich, 2016).

The final scheme, *Partitive Division* (PD), requires students to partition a given amount of 1's into a given number of composite units. An example of a PD problem would be: "You have 21 cubes, and you need to make 7 towers. How many cubes will you put into each tower?" This type of problem demands that the student figure out how big to make their towers without using a distribution-by-1s strategy to solve it. This scheme correlates to Hackenberg's MC3 scheme and Norton's Units Coordination Stage 3, because the student would have to coordinate three levels of interiorized units outside of the activity (total, number of composite units, and size of each composite unit to be placed in those) in order to figure out how many individual units should go into each composite unit before carrying out the activity (Ulrich, 2015; Ulrich, 2016). Next, I link the levels of units coordination to fractional reasoning.

Multiplicative and Fractional Reasoning Links

Hackenberg (2013) and Steffe & Olive (2010) have linked fractional reasoning schemes to multiplicative reasoning. They had originally identified three fractional schemes (more schemes will be discussed later) that students develop in relation to multiplicative reasoning schemes. The first scheme is the *Parts-Within-Wholes Fraction Scheme* (PWWFS). This scheme does not require a student to disembed any portion of the fraction, therefore MC1 students are able to perform at this reasoning level. The second scheme is the *Partitive Unit Fraction Scheme* (PUFS) and is within the ability level of MC2 students. The third scheme is the *Iterative Fraction Scheme* (IFS; see also Tzur, 1999); MC3 students are able to do problems involving operations in this scheme.

Students with only MC1 cannot conceive of fractions as anything other than parts within a whole, because they must actively partition a given length, for example, into six parts and somehow identify four parts to see the fraction four-sixths. The second scheme (MC2) is required to begin conceiving of fractions as measurable units and can begin to understand problems involving a partitive fraction scheme. The third scheme (MC3) is more advanced as it requires for students to construct fractional knowledge involving the addition, subtraction, multiplication, or division of fractions, and the understanding of equivalent and improper fractions (Hackenberg, 2013, 2010; Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Tzur, 2019).

Steffe's Fractional Reasoning Schemes

Steffe & Olive (2010) and Steffe et al., (2013) have identified eight fraction schemes through which they inferred students' progress through as their reasoning and number sequences develop. According to Steffe et al. (2013), children use their number sequences to construct fractions by partitioning wholes into parts. Children are able to do this because the child's number sequence allows them to "willfully create their own countable items using elements of their number sequence and count these elements using the same number sequence that was used to create the countable items" (p. 53), while also being able to coordinate three levels of units (at least two levels as given and one level in activity). When these number sequences and their related levels of units coordination are related to the construction of fractions, students are partitioning a whole into countable parts, which they can then use to iterate and count when reconstructing the whole. It allows students to construct an understanding of the multiplicative relationship between any whole and its fractional parts.

Drawing on Steffe's progression of eight fractional schemes, Tzur (2013) further articulated how a child reorganizes a previous fractional scheme on the progression to create the next scheme at an anticipatory level. The anticipatory level, which I shall discuss in more details later in this chapter, refers to Tzur & Simon's (2004) distinction of two qualitative distinct conceptions within a single scheme. Children are able to anticipate the effects of their activity-sequence without having to actually complete the activity and without the need of prompts when at the anticipatory stage. Here, I provide a brief explanation of Tzur's (2014) expansion of Steffe's fractional reasoning schemes.

According to Tzur (2014; 2019b), the first iterative fraction scheme identified by Steffe et al. (2013) and Steffe & Olive (2010), is equipartitioning, which is rooted in the partitioning of composite units. In this scheme, students are able to partition a whole into equal parts and disembed one of the parts without mentally "destroying" the whole. If the child wants to check to make sure the disembedded piece is equal to the other pieces, they iterate the piece and create a new segmented whole to compare to the original whole. For example, a student may partition a whole into three equal pieces and disembed one of the pieces to create a 1/3 piece. To check that this piece is equal in size to the other pieces and fits three times within that whole, the student would iterate the 1/3 piece to construct a new whole which can be compared to the original whole (Fig. 2.2).

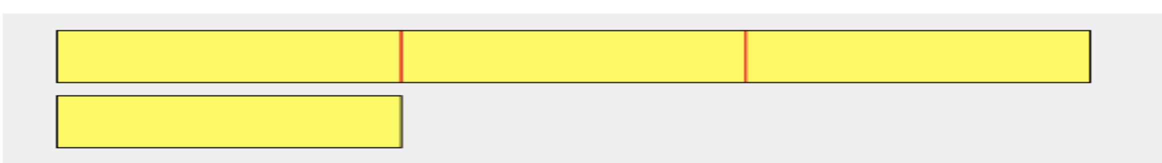

Figure 2.2 - Bar partitioned into three parts and one disembedded part.

The equipartitioning scheme requires the student to use both operations of partitioning a given whole into equal pieces and then iterating one of the pieces to reconstruct a new whole of the same size. Their understanding of a composite unit as something that can be partitioned, and its pieces disembedded and reiterated, allows the student to reorganize their understanding of composite units into a fractional understanding of wholes and their relationship to fractional parts (Steffe & Olive, 2010). By reorganizing their current number sequence, the concept of 3, in the problem above, can be used "as a template for partitioning" (Steffe & Olive, 2010, p. 316), and the sequentiality of operations allows one to compare the partitioned pieces for equality. The partitioned piece can be disembedded without destroying the original whole unit. According to Tzur (2014), the reorganized anticipation means that a child must use their number sequence in order to simultaneously partition the composite unit while iterating a unit fraction (1/n) in order to reproduce the whole.

The *Partitive Fraction* scheme is the second fractional reasoning scheme and comes after equipartitioning, because it requires an understanding of a unit fraction and its multiplicative relationship to the whole (Steffe & Olive, 2010; Steffe et al., 2013; Tzur, 2014). In this scheme students are able to use equipartitioning to find an iterable fraction of the whole, but then are able to further use that iterable piece as a measurable segment to create a new fraction of the whole. For example, a student may partition a whole into 6 pieces and disembed one of the pieces to find 1/6 of the whole. They are then able to iterate the 1/6 piece 5 times to create 5/6 of the original whole, understanding that the meaning of 5/6 lies in the multiplicative relationship between the fractional piece of 1/6 and the whole (Fig. 2.3).

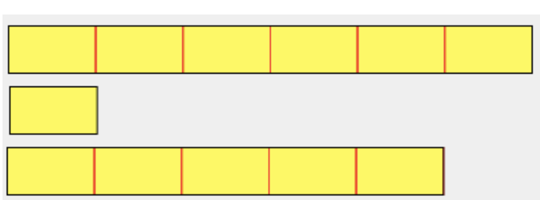

Figure 2.3 - Bar partitioned into 6 equal parts. One part is disembedded and iterated five times.

The student must understand that 5/6 is a composite fraction (Tzur, 2019b) made up of five units of 1/6, and they must be "explicitly aware of the multiplicative relation between the connected number as a composite unit item and any one of its parts" (Steffe & Olive, 2010, p. 111). This scheme requires one to understand that the composite whole can be partitioned into units, which can then be iterated to create a new whole or part of a new whole that carries the same multiplicative relationship as the original whole and disembedded unit (Steffe & Olive, 2010). For Tzur (2014), the reorganized anticipation for this scheme means that a child can already anticipate the equipartitioning results and then use those anticipated results as an input for the iteration of proper fractions (m/n). So, a child would disembed the unit fraction (1/n) and iterate it m times in order to create a proper fraction of the whole.

The third scheme, *Splitting*, may appear similar to equipartitioning and the partitive scheme. However, unlike the sequential nature of operating in those two prior schemes, in splitting those activities are performed simultaneously (Steffe & Olive, 2010; Steffe et al., 2013). For example, a student asked to draw a piece which is 1/5 as big as the original whole would be able to imagine a hypothetical piece that, when iterated five times, would constitute a 5-part whole

> That is, the child not only posits a hypothetical stick, but also posits the hypothetical stick as one of five equal parts of the given stick that could be iterated five times and sees the results of iterating as constituting the given stick. (Steffe et al., 2013, p. 55)

Using this scheme, a child could mentally construct a multiplicative relationship between a whole and its fractional unit. Like equipartitioning and the partitive scheme, the sequentiality of the units is key for identifying units of equal size that can be iterated to create a new whole or part of a new whole while retaining the same multiplicative relationship involved with the original whole and fractional unit. A child must already be able to anticipate the activity for partitioning the whole into n parts, disembedding a part (1/n), and iterating that part n number of times (n * 1/n) in order to reproduce the original whole. This requires that the equipartitioning and partitive fraction schemes are already anticipatory for the child (Tzur, 2019b).

The three schemes above all require only two levels of units to be coordinated at a time (Lovin et al., 2018). The partitive fraction scheme involves three levels of units, but only two of the units need to be coordinated at a time due to the individual being able to work "within a referent whole" (215). Therefore, they may be working only with two levels interiorized and one level in activity. For example, in PFS a learner might be asked to create 5/6 of a whole, which would require them to coordinate the whole (6/6) with the iterable unit fraction of 1/6 by disembedding the 1/6 unit. Through activity, the learner would then iterate the 1/6 unit five times to create a new unit of 5/6. Starting with PFS, all the following schemes require three levels of units coordination interiorized (Lovin et al., 2018).

The fourth, *Iterative Fraction Scheme* (IFS, see Steffe & Olive, 2010; Tzur 1999) requires a student to use the splitting scheme while operating on three levels of units in which a given fraction, such as 8/7, is made up of given units of units of units. In this scheme, a student can produce and understand fractions greater than the whole ("improper") while understanding that the whole contains hypothetical units representing the unit fraction. The unit fraction "becomes a fractional number *freed from its containing whole* and available for use in the construction" (Steffe & Olive, 2010, p. 116), of a fraction greater than the whole (e.g., 8/7). In other words, the student must understand that this fraction is made up of anticipating that 1/7 could be iterated eight times. To produce 8/7, they must produce the unit fraction of 1/7 from the whole (7/7), iterate it 8 times to create 8/7, and coordinate this new unit of 8/7 back with the original whole. (Fig. 2.4).

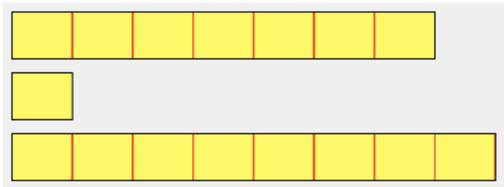

Figure 2.4 – Whole partitioned into 7 equal parts. One part is disembedded and iterated 8 times to create 8/7.

Three levels of units are required to understand that 8/7 is a composite unit made up of eight 1/7 pieces that came from a whole, which was partitioned into seven pieces and had one piece disembedded (Steffe & Olive, 2010; Steffe et al., 2013). It is important for an individual working in this scheme to retain the unit fraction and its multiplicative relationship to the whole. They must understand that the unit fraction is 1/7, and not revert back to a part-of-whole understanding and begin thinking that the unit fraction is 1/8 now that there are eight pieces. This scheme requires the previous schemes (splitting, hence equipartitioning, partitive) to have already been constructed, because it requires the simultaneous partitioning and iterating of unit fractions to create a new unit that is greater than the whole, while retaining the relationship between the unit fraction and the whole. Accordingly, the sequentiality of whole numbers needs to be reorganized for use with fractional units, which can be iterated to create a new unit that is larger than the original whole. In order to extend the unit fraction (1/n) into any multiple iterations (m * 1/n) to create improper fractions such as 9/8, the child must reorganize their anticipatory partitive and splitting schemes (Tzur, 2013).

The four schemes discussed so far were constructed on the basis of operating on an iterable unit. Whereas, the following schemes were constructed on the basis of partitioning, specifically the results of partitioning previous partitions (i.e. partitioning a unit fraction into new, smaller units).

The fifth, *Recursive Partitioning Scheme* (RPS), is made up of the partitioning of a partitioned piece with an understanding of the relationship between the two partitions and the whole (Steffe et al., 2013; Tzur, 1996). For example, a student would partition a whole into 4 pieces, then disembed one of the pieces to extricate a 1/4 piece. That piece (1/4) would then be partitioned into 5 pieces, and if they disembed 1 of those pieces they would now have the fraction 1/20 (Fig. 2.5).

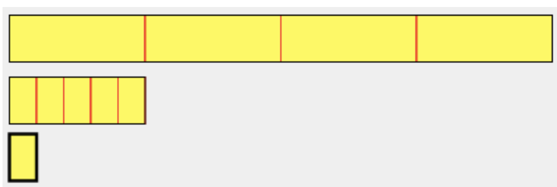

Figure 2.5 - Bar partitioned into 4 equal parts. One part is disembedded and partitioned into 5 equal parts. One of those parts is disembedded.

> This activity requires the child to distribute the partitioning operations that produced the partial partitioning across the parts of the original partition. But this is not all, because the child must also use the partial result of the second of the two partitions (the one that is not fully implemented) in the service of another goal, which in this case was a fractional goal (Steffe et al., 2013, p. 57)

In order to carry out this mental activity, the student must anticipate that partitioning one of the original segments into 5 new segments could potentially be done to all of the original segments which would give them 20 total segments. To this end, they must understand the multiplicative relationship occurring between the original fractional unit(s) and the partitions of those units. This scheme builds off of the previous four schemes, because it requires an understanding of unit fractions and non-unit fractions and their multiplicative relationship to the whole, including the ability to simultaneously partition and iterate within a *hypothetical* whole. Without that understanding, one cannot find the fractional part of a fractioned whole. This scheme involves the coordination of two fractional number sequences; however, an individual may be able to use the splitting scheme, along with recursive partitioning, to mentally distribute the second partition into all of the remaining original partitions without having to enact the full activity (Steffe & Olive, 2010). Recursive partitioning requires a child to reorganize the previous two anticipatory schemes in order to partition a unit fraction and link the part of the unit fraction back to the original whole (Tzur, 2013).

Steffe et al. (2013) considered the sixth, *Unit Fraction Composition Scheme* (UFCS), to be the first fraction multiplication scheme. This scheme requires one to compose the fractions of a fractional unit into a unit of one that has its own multiplicative relationship to the whole. For example, a whole is partitioned into 4 pieces and one of those pieces is disembedded, then partitioned into 5 pieces; this partitioning of each of the fourths into fifths would create a new fractional piece of 1/20. This part of the scheme uses the operations of recursive partitioning, but the scheme then goes further to require an understanding that the 1/4 piece is commensurate with 5/20. A student working within this scheme would have to understand that each 1/4 piece was composed of 5/20, and that there are units of five inserted into (distributed over) each of the four units of 1/4 (Fig. 2.6).

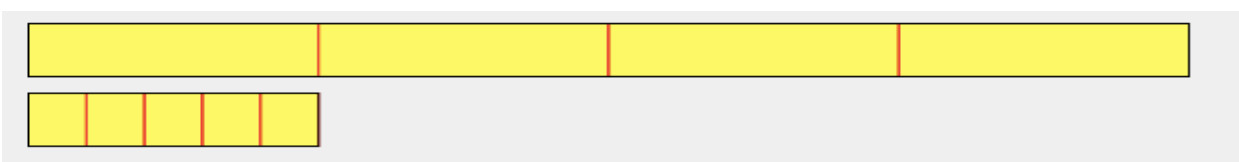

Figure 2.6 - Bar partitioned into 4 equal parts. One of those parts is disembedded and partitioned into 5 equal parts.

There has to be an intentional, anticipatory shift between the two units in order to see their relationship to each other and the whole (Steffe & Olive, 2010). The iteration of the composite unit of five over the composite unit of four, requires sequentiality; however, it is possible to distribute the iterable composite unit of five into the composite unit of four mentally without having to carry out the activity to completion (Tzur, 1996). The unit-fraction composition scheme requires the reorganization of the anticipatory recursive partitioning scheme in order to reverse the process of iterating unit fractions (Tzur, 2019).

The seventh, *Distributive Partitioning Scheme* (DPS), requires distributive reasoning in which students partition *"n* items among *m* shares by partitioning each of the *n* items into *m* parts and distributing one part from each of the *n* items to the *m* shares" (Steffe et al., 2013, p. 57). An example of this scheme would be a child who is sharing four pizzas of different sizes among five people. The student would first partition each pizza into five pieces and distribute one piece from each pizza to each of the five people (Fig. 2.7). They would then have to understand that each person's full share could be replicated, or multiplied, five times to recreate the original four pizzas (Steffe & Olive, 2010). Distributive partitioning requires that a child use their anticipatory splitting scheme to distribute the results of partitioning onto separate fractional parts of the whole (Tzur, 2013).

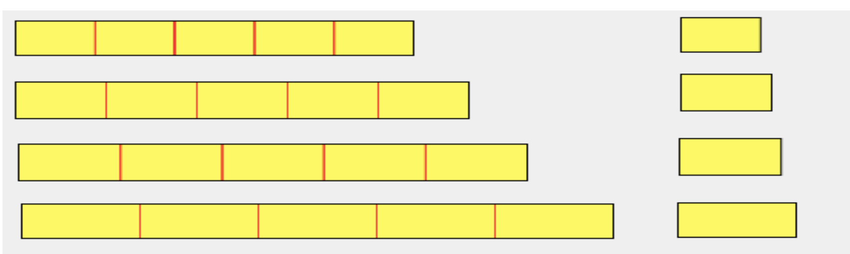

Figure 2.7 - Four bars partitioned into differing n parts. One part of each bar is disembedded.

The eighth scheme in the progression is the *Fraction Composition Scheme* (FCS, see Steffe & Olive, 2010; Tzur 2019). This scheme requires distributive partitioning in which a child partitions an unmarked composite (non-unit) fraction into a certain amount of parts and then disembed some of those parts to further split each part and disembed a certain number of those partitions in order to find the fractional part that the partitions make of the original segment (Steffe & Olive, 2010). For example, a child who is asked what two-sevenths of three-ninths is would have to split a whole into nine parts, disembed three of those parts, split each of those three parts into seven partitions and disembed two of those partitions from each of the three parts to find the answer of 6/63 (Fig. 2.8).

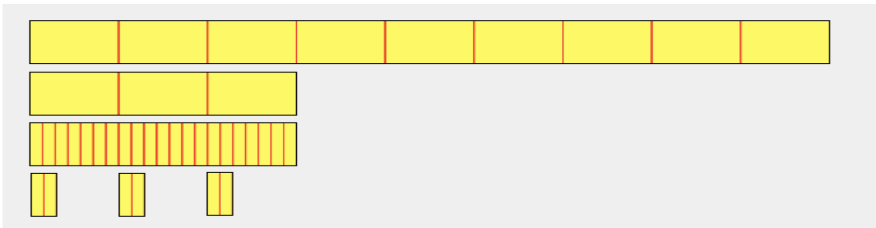

Figure 2.8 - Bar partitioned into 9 equal parts. Three parts are disembedded, and each part is partitioned into 7 parts. Two partitions from each of the three parts are disembedded.

A child working within this scheme must coordinate “two sequences of composite units” (Steffe et al., 2013, p. 59), – the composite fraction made up of three 1/7 units, which are each partitioned into seven parts all within the composite unit of nine segments. This scheme requires the student to mentally coordinate “the distributive partitioning scheme and the recursive partitioning scheme into the splitting scheme” (Steffe & Olive, 2010, p. 333). The fraction composition scheme also requires a child to coordinate their anticipatory distributive partitioning scheme with a reversed iterative fraction scheme in order to operate on disembedded proper fractions (Tzur, 2013).

Teachers’ Construction of Fractional Reasoning

One question that has been addressed recently by a handful of researchers (Lovin et al., 2018; Tzur & Depue, 2014a; Tzur & Depue, 2014b; Tzur et al., 2016) is: Can the fractional reasoning schemes identified by Steffe & Olive (2010), Steffe et al. (2013), and Tzur (2019), be adopted, and perhaps adapted, for promoting and studying adults’ fractional reasoning? These scholars suggested that it may be possible to attribute the same fractional reasoning schemes to adults attempting to assimilate new fractional reasoning into their previous part-of-whole reasoning. This dissertation study is based on that suggestion, further examined below, while looking to extend it into practicing teachers’ reasoning.

Lovin et al. (2018) conducted a study in which they validated the fraction scheme hierarchy with PSTs (N=109) who were getting their licensure in pre-K-8 education. They assessed those PSTs' fractional reasoning using task items previously developed for use with elementary and middle school students (Norton & Wilkins, 2012; Norton et al., 2016; Wilkins & Norton, 2011). This assessment required the PSTs to solve tasks ranging from the part-of-whole scheme through the iterative fraction schemes. Lovin et al. also asked the participants to provide brief explanations of their solutions in order to gather more data on how the PSTs were reasoning. Their results found that, "PSTs' fraction development seems to be largely consistent with the established hierarchy in that each lower-level fractional understanding appeared to be a prerequisite to higher levels of understanding" (Lovin et al., 2018, p. 221). They concluded that PSTs follow the same fractional reasoning trajectory path as children do.

Going back to Tzur et al.'s (2016) study with Annie, they attempted to help her begin to construct the recursive partitioning scheme in order to improve her understanding of decimals and equivalent fractions. The main researcher started this work from her assimilatory understanding of the equivalence between 9/10 and 90/100. He provoked a perturbation in her reasoning as a means to commence her construction of the recursive partitioning scheme (as explained above). This was his goal so she could better conceptualize the connection between those two fractions and the unit fraction of a unit fraction involved in partitioning 1/10 into 1/100. Through their work, Annie began to construct an early level of understanding of the recursive partitioning scheme. This opened the door to future development on her recursive partitioning scheme, leading to an understanding at the anticipatory level, and later the assimilation of the scheme into the latter schemes. I consider that study, and the one further discussed below, as a preliminary line of work to which my dissertation may contribute further understandings.

Tzur & Depue (2014a; 2014b) used assessments to determine individuals' conceptions of fractions before and after (pre-post) a conceptual-based intervention designed to promote initial equipartitioning reasoning. They found that all (100%) participants' reasoning at the start of the study was as part-of-whole. The intervention seemed to promote an understanding of fractions as a multiplicative relationship (at least as explained by the equipartitioning scheme). This reorganization of their previous scheme into equipartitioning had a significant impact on their numerical comparisons, involving both whole numbers and fractions. The key findings suggested that the intervention led to a 20% decrease in the time it took the participants to compare unit fractions. These studies give credible evidence to the claim that the fractional reasoning schemes identified in children may be adopted for use when working with adults to improve their fractional reasoning MKT.

Anticipated Issues/Challenges in Teachers' Construction of Fractional Schemes

As teachers work through the equipartitioning scheme, two common issues are likely to arise similarly to how they arose in children – understanding the direction and the magnitude of change (Hunt et al., 2016; Tzur & Hunt, 2015). The "direction of change" refers to one's decision whether to next try creating a larger/smaller fractional piece based on one they already have (either created or given). For instance, consider an individual attempting to find a fractional piece that will fit into the original whole four times. After iterating the piece four times it comes out shorter than the original whole. To improve the estimation next time, they would need to adjust their piece by making it larger. If after iterating their original piece four times it was larger than the original whole, they would need to adjust that piece and make it smaller.

The magnitude of change refers to the amount of adjustment that should be done to the previous piece, whether it needs to be larger or smaller. For instance, if the original piece was too small, then they would need to take the piece leftover on the original whole, partition it into four pieces and distribute one of those pieces to the original piece and iterate the new piece four times. If the original piece was too large, then they would need to take the amount of the overage, partition it into four pieces, and shorten the original estimate by that amount; they would then iterate the new piece four times. It isn't until a learner reflects on activities designed to promote their reasoning about both the direction and magnitude of change that they will begin to construct an understanding of "the unique, multiplicative 'fit' between each unit fraction (1/n) and the whole (n times as much of 1/n), and of the inverse relationships among unit fractions (to fit more pieces – each must be smaller" (Tzur & Depue, 2014b, p. 299). Once this reasoning is constructed, a teacher can begin to reorganize their previous part-of-whole scheme into new fractional reasoning (e.g., equipartitioning).

Before the equipartitioning scheme has been constructed at an anticipatory level, a teacher is likely to fall back (Tzur & Depue, 2014b) onto their part-of-whole reasoning to solve fractional reasoning problems similar to the following (Figure 2.9):

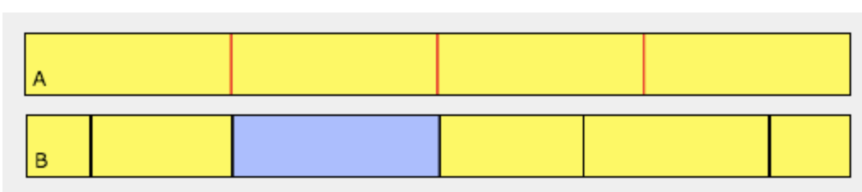

Figure 2.9 - "Sticks A and B are equal in length. A contains 4 equal parts. The shaded part on B is equal to the part above it on A. What fraction, if any, is the shaded part of B? Of A? Why" (Tzur & Depue, 2014b, p. 299).

As Tzur and Depue (2014a, 2014b) demonstrated, a teacher with only part-of-whole reasoning (not yet equipartitioning) would most likely claim that the shaded part of B is 1/6 of B. Their part-of-whole reasoning makes this sensible; the shaded piece is 1 part out of 6 total parts, or they may claim that they cannot determine the fractional part because the partitions in bar B are not equal in size. They may also claim that it isn't any fractional part of A, because it is not part of the same whole. For a teacher like this, the fractional part *has* to be a part of the whole in order to be a fraction of the whole (Tzur & Depue, 2014b). They may also reason about fractions by simply looking at how many parts are shaded out of the total amount of parts within the whole. The teacher will reason about this problem by analyzing the multiplicative relationship existing between the whole and the fractional piece once they have established the equipartitioning scheme at an anticipatory stage. They would be able to reason that the shaded part of B is 1/4 of B because the whole is four times as much as the shaded piece. They would also be able to reason that the shaded part is also 1/4 of A because A is also four times as much as the shaded part. This reasoning focuses on the multiplicative relationship between the given whole and part at issue, not on whether or not the part is "of-the-whole" (Tzur & Depue, 2014a; Tzur & Depue, 2014b).

Similarly, to studies about children, a teacher who only understands fractions as part-of-a-whole is likely to have difficulty with the iterative fraction scheme (i.e., improper fractions). For example, consider a task presented to the AdPed project teachers, in which they were shown a bar partitioned into 6 pieces and asked to write down as many fractions they could identify in that diagram (see Fig. 2.10). Many of them gave the fraction 6/6, or 5/6, or 1/6, and the like. But none initially could see, say, 6/5.

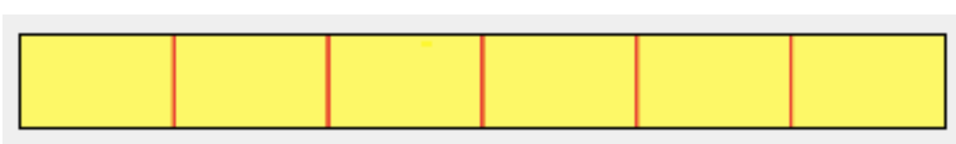

Figure 2.10 - Bar partitioned into 6 pieces. What fractions can you identify in this this diagram?

Only a teacher who understands fractions as a multiplicative relationship and has the iterative fraction scheme would be able to also see that bar as representing other fractions, such as 6/5, 6/4, 6/3, or 6/2 – let alone as 3/2 or 4/3. A teacher with this reasoning could understand that the unit fraction may be 1/5, 1/4, 1/3, or 1/2 and that the bar is representing six iterations of that unit fraction.

Another example of how a teacher with part-of-whole reasoning may struggle with this scheme would be if they were shown a fractional piece representing 1/4, then the piece was iterated five times and asked what the fraction of one of the pieces is, they may say that the piece is now 1/5 because they would see it as one piece out of five total pieces, rather than as still seeing it as the original unit fraction of 1/4. Their view of the whole adjusted based on the number of pieces that were iterated, rather than retaining the original multiplicative relationship between the fractional unit piece and the whole.

While these schemes are conceptual markers, the focus of my study is *how* Nancy and Marsha transitioned from one scheme, or marker, to the next and *how* their levels of units coordination either afforded or constrained those transitions, as well as how far they were able to progress during the project. The next section provides a framework to understand and examine such transitions.

Learning as a Cognitive Change

With the conceptual progressions articulated above, a key question arises: How may students and teachers (e.g., Nancy and Marsha) advance in their levels of units coordination, and multiplicative and fractional reasoning schemes? There has been much research around the cognitive processes involved in the construction of new mathematical learning (Piaget, 1980; Simon et al., 2004; Simon & Tzur, 2004; Tzur, 2011; von Glasersfeld, 1995). At the heart of this research is the attempt to articulate the core mechanism of reflective abstraction (Piaget, 1985), which would also inform my study of conceptual transitions in Nancy and Marsha's conceptions. In this section, I explain key constructs in the conceptual framework I use, constructivism: especially assimilation, accommodation, and reflective abstraction (Piaget, 1976; 1980; 1985; von Glasersfeld, 1995). I use these constructs to explain how cognitive change may occur in individuals, and how I will explain the specific cognitive changes in Nancy and Marsha's mathematical reasoning. I begin with an explanation of Piaget's (1976; 1980; 1985) constructs of assimilation, accommodation, and reflective abstraction, as those are the foundation for the more recent theories I present later in this section.

Assimilation, Accommodation, and Reflective Abstraction

As stated in Chapter I, the conceptual framework for this study is rooted in Piaget's (1976) constructivist theory of knowing and learning – specifically his constructs of assimilation, accommodation, and reflective abstraction. Assimilation refers to the mental system's integration of external, or internal, elements into its available cognitive structures (Piaget, 1985). These integrated elements may come, for example, from a mathematical task or an activity sequence in which an individual is engaged. One's available cognitive structures afford and constrain what and how they are able to assimilate. When the external/internal elements are experienced by an individual, they may experience a perturbation, which may push their cognitive structure into disequilibrium (Piaget, 1985). This disequilibrium causes the system to carry out a process for restoring equilibrium, that is, reequilibration. If reequilibration is successful and involves a change to the perturbed scheme, a new scheme is constructed as an accommodation of the previously available schemes (Piaget, 1985). Accommodation is the modification of existing schemes, resulting from the reequilibration process (Piaget, 1985).

How does this cognitive transformation – the accommodation of existing schemes – occur? According to Piaget (1980), the change is due to a mechanism he termed reflective abstraction. Piaget (1980) believed that reflective abstraction was key to constructing new knowledge. He stated that "All new knowledge presupposes an abstraction, since, despite the reorganization it involves, new knowledge draws its elements from some preexisting reality, and thus never constitutes an absolute beginning" (p. 89). Reflective abstraction occurs when the learner executes goal-directed, mental operations, which results in the individual reflecting on the cognitive processes they just experienced (von Glasersfeld, 1995). When the reflection shows a negative result (i.e. the result didn't meet the goal), perturbation occurs (von Glasersfeld, 1995). When the reflection shows a positive result (i.e. the result met the goal), a new recognition pattern may be created by transforming the existing scheme – accommodation (von Glasersfeld, 1995).

Reflection on Activity-Effect Relationships

According to Simon et al. (2004), Piaget's reflective abstraction construct needed further elaboration in order to help mathematics education researchers, or teachers, further articulate and promote the process of conceptual change. To elaborate on Piaget's reflective abstraction, they postulated that learners use a cognitive mechanism they termed *Reflection on Activity-Effect Relationships* (Ref*AER). This conceptual change process begins when the learner assimilates a task, which brings forth a goal they set to accomplish. This goal arises from their interpretation of the task and is thus different than the teacher's goal for the student's learning. The learner then implements a mental activity sequence to reach that goal. As the learner implements the activity sequence, they are focusing on the results it brings about to determine if the sequence is getting them closer to or farther away from their goal. Based on their results, the learner may make adjustments to their activity sequence to get them closer and closer to the intended goal. These results, including adjustments the learner makes, are referred to as *effects*. As these adjustments occur, the learner is making comparisons between the different adjustments and their subsequent results and noticing patterns (Simon et al., 2004; Simon & Tzur, 2004; Tzur, 2011). That is, the learner abstracts "the relationship between activity and effect" (Simon et al., 2004, p. 319).

An example of Ref*AER can be found in the multiplication tasks involving Student A and Student B (Chapter I). In the first task (3 towers, each made of 4 blue and 2 yellow cubes), Student A's goal was to figure out a way to find the total number of cubes. She initiated an activity sequence, which included the act of physically splitting the three towers of six into nine towers of two. The effect of her activity was a count (by 2s) of all cubes in the 3 towers. Then, she witnessed Student B's activity sequence of separating each tower into two unequal towers of 4 cubes and 2 cubes. Student A's solution to the follow-up question (towers of 13 cubes) was an assimilation of Student B's solution into a scheme that allows decomposition of composite units into unequal-size units. The number choice for that task (13) further supported Student A's assimilation, as it involved thinking about 13 as composed of 10 and 3. For a goal similar to the one she set in the previous task (figure out the total of cubes), she then brought forth an adjusted mental activity of decomposing the given composite units (into towers of 10 and 3) in a way that could help her compute the total as partial products (5x10 + 5x3). Whereas at this point it would not be possible to determine the extent to which Student A's change would persist, this shift illustrates the adjustment of an activity to accomplish a goal that is explained by Ref*AER. My analysis of transitions in Nancy and Marsha's schemes would focus much attention to such adjustments.

After using this activity sequence, Student A began constructing a relationship between the activity and the effect. This relationship is one in which the towers (of 13 cubes) can be decomposed into uneven towers (five towers of 10 and five towers of 3) that are easily multiplied to find the answer (5x10 + 5x3). This construction of a new relationship helped Student A anticipate the effects of her tower decomposition (Tzur & Simon, 2004). This anticipation ties to von Glaserfeld's (1995) scheme theory, especially the third part of a scheme - "The expectation that the activity produces a certain, previously experienced result" (p. 65).

I chose to use Ref*AER in this study for a twofold reason: 1) It can help explain cognitive changes that occurred in Nancy and Marsha and 2) It can help identify what may have helped or hindered this process for them. Specifically, Ref*AER consists of two critical distinctions beyond Piaget's (1980) reflective abstraction – two stages in the construction of a new scheme and two types of reflection that constitute reflective abstraction.

Tzur & Simon (2004) identified two stages that an individual may progress through as they are constructing new schemes through the accommodation of their previous schemes. They termed these two stages *participatory* and *anticipatory* stages. Accordingly, they also pointed out two types of reflection underlying the constructive process, which Tzur (2011) articulated further. He termed these as *Type I* and *Type II* reflections. Both of these distinctions will be explained further in the following two sections.

Stage Distinction: Participatory and Anticipatory

Tzur & Simon (2004) expanded on Ref*AER by identifying two stages through which learners construct a new mathematical scheme. The first one they termed the *participatory* stage. At this stage, the learner is able to anticipate the activity sequence results but are only able to do so "in the context of the activity" (p. 296), or by being prompted through their activity – either self-prompting or prompting from others (Tzur & Lambert, 2011). They termed the second stage the *anticipatory* stage. At this stage the learner no longer needs prompting in order to anticipate the effects of their activity sequence. They are able to activate their previously constructed activity-effect relationship spontaneously without having to act out the activity sequence. That is, the relationship between a goal-directed activity and its effect has become a regularity the person can bring forth upon assimilation of, say, a task.

An example of these two stages can be explored once again through the multiplication task that Student A and Student B engaged in. On the first task, Student B was able to spontaneously anticipate the effect of her activity sequence – mentally decomposing the towers into three towers of two and three towers of four – and did not need any prompting to do so. Therefore, she was working at the anticipatory level of a scheme involving at least two levels of units coordination (and potentially a third level in activity) for this task. Student A was not able to anticipate such a decomposition on her own, as indicated by, instead, her organization of the towers into units of 2. Student A had to physically split all of her towers into towers of two before she was able to solve the problem. However, Student B's explanation in the first task prompted Student A's activity to solve the follow-up task (towers of 13 cubes each), and that time she was able to anticipate the effect of her activity sequence (intentionally multiplying 10x 5 separately from 3x5, then add them). Therefore, Student A was using that scheme at the participatory level.

These two developmental stages in the construction of new mathematical schemes are relevant to this dissertation study, as I conjecture both Nancy and Marsha may proceed through these stages as they construct their new understandings. These stages are a conceptual tool for me as a researcher, in that they direct my focus of attention onto changes in the teacher's anticipation of the effects of her goal-directed activities.

Type I and Type II Reflections

As noted above, to explain possible processes leading to the construction of each stage, Tzur (2011) elaborated further on the cognitive mechanism of Ref*AER by postulating two types of reflection that constitute the Ref*AER process. Type I reflection involves the learner making continual comparisons between their goal and the effects of their activity-sequences (Simon et al., 2004). This type of reflection occurs when the learner is at the transition from available to the participatory stage. Type II reflection involves the learner making comparisons across multiple "(mental) records of experiences" (Tzur, 2011, p. 612). He thus postulated this type to be necessary for a transition from the participatory to the anticipatory stage of a new scheme. The comparisons across multiple experiences may lead to the learner's construction of a regularity in their activity that enables anticipating what would happen in similar situations requiring the same type of cognitive functioning. He thus suggested the second type of reflection, and the resulting anticipatory stage, are needed for transfer of what someone learned to another context. The two reflection types are relevant to this study as they provide me, as a researcher, with a tool to analyze how a participatory or an anticipatory stage in Nancy and Marsha's schemes might have come about.

In order to analyze Nancy and Marsha's cognitive change, I need to know where the changes were occurring in their cognitive structures, particularly their anticipation of effects ensued from their activities. In other words, I will focus on schemes they were assimilating elements into, and how those schemes were being accommodated into new reasoning through Ref*AER. I now turn to two additional accounts, Simon's (1995) hypothetical learning trajectories (HLT) and Tzur's (2019) transitions research, as ways to further identify how Nancy and Marsha's cognitive changes came about.

Hypothetical Learning Trajectories

Simon (1995) developed his construct of *hypothetical learning trajectories* (HLT), to articulate how teachers may use their knowledge of students to create a potential path for learning. HLT is made up of three components: 1) a learning goal for the students, 2) learning tasks and activities that may lead students to the learning goal, and most centrally 3) a hypothesis of a potential learning path the students may evolve through as they engage in the learning tasks. The third step in this process involves specifying conceptual markers (Tzur, 2019) that learners may progress through as well as the process of transition from previous to more advanced ones. For example, students may proceed from conceptual marker of counting-all in order to advance to counting-on. Studying conceptual markers is important in order to understand the potential pathways of learners and what goals are possible for the learner. If a student is currently able to reason with one level of units coordination, then the next goal would be for the student to work through a scheme in which they can use a second unit in activity. Yet, conceptual markers aren't enough for understanding learners' growth; researchers must also examine the transition that occurs between these markers (Siegler, 1995; Simon, 2018; Tzur, 2019).

Most research in mathematics education has focused on the conceptual markers and not on the transitions occurring as learners move from one marker to the next (Simon, 2018). This transitional explanation is important for understanding the cognitive processes involved in learning (Simon, et al., 2018; Tzur, 2019). That is, a transition study "foregrounds the specification of conceptual transformations involved in progressing from less to more advanced markers" (Tzur, 2019, p. 60).

This study is comprised of both foci, on markers and on transitions, in Nancy and Marsha's reasoning (Simon, 2018; Tzur, 2019). Specifically, I will examine the extent to which Nancy and Marsha move along markers similar to those found in research with K-12 students when it comes to multiplicative reasoning, fractional reasoning, and levels of units coordination while also examining the transitions that occurred as they progressed from one conceptual marker to the next (Tzur, 2019). In this way, my study can extend Tzur et al. (2016) and Tzur & Depue's (2014a; 2014b) work on learning experiences designed for adult learners based on HLT's for K-12 students. The extension is important in that those prior studies only examined two specific conceptual schemes of learning: 1) utilizing fractional parts to understand decimals (via recursive partitioning) and 2) understanding unit fractions as a multiplicative relationship (i.e., equipartitioning). My study examines a wider range of conceptual markers experienced by Nancy and Marsha, while also inferring into the possible transitions that occurred when they moved from one marker to the next.

Summary Remarks

The research done by Tzur et al. (2016), and Tzur & Depue (2014a; 2014b) is just "the tip of the iceberg" when it comes to examining adults' fractional reasoning; much more work seems needed to determine how adults construct fractional reasoning beyond part-of-whole. There also seems to be a gap in the research about how adults, particularly practicing teachers, may construct new multiplicative reasoning and whether schemes (markers) and processes (transitions) found in research with children can guide research with teachers. There is evidence to suggest that the construction of the multiplicative and fractional reasoning schemes in children may be similar to the schemes that adults may construct. In order to better help teachers construct new schemes based on their assimilatory schemes, researchers need to delve into how adults reason as their schemes are being constructed.

The connections between the units coordination stages, multiplicative reasoning schemes, and fractional reasoning schemes that I laid out in this chapter seem of critical importance for teachers to construct in order to effectively teach their students. In order for students to progress through the multiplicative schemes, they must be able to eventually coordinate three levels of units and take each unit level as a given. In order to develop new schemes, students first need an anticipatory understanding of multiplicative reasoning schemes. Without these previous concepts, students may struggle to understand fractions as a multiplicative relationship and will only ever be able to understand fractions as a part of a whole. To support such growth in students, understanding how teachers advance in such reasoning seems imperative.

CHAPTER III

METHODS

In this longitudinal, qualitative study, occurring over a two-year timespan, I use a cross-case analysis method defined by Powell et al. (2003), which outlines a specific model for analyzing videotape data. In this chapter, I will be outlining those research methods for collecting and analyzing the data. I begin with a review of my research objectives and questions for the study. Then, I define my research design and sample criteria. I describe my two participants and my data collection procedures involved in my study. I end with a full description of my analysis methods and the limitations and ethics of this study.

Research Objectives and Questions

The objective of this qualitative case study is to analyze changes in teacher reasoning, in terms of units coordination, as they participated in the AdPed PD program, which was designed to help them construct schemes for multiplicative and fractional reasoning. It also explores the effects of transitions in teachers' reasoning on their MKT and ability to recognize their students' reasoning in-regards-to their levels of units coordination. The research questions for this study are:

1. What pathways of reasoning, markers and transitions, may teachers go through? That is, what changes in their multiplicative and fractional reasoning schemes could be inferred?
2. To what extent, and in what ways, do teachers' levels of units coordination affect their learning pathway?
3. How do teachers' levels of units coordination affect their ability to recognize levels of units in their students?

Research Design, Sample, Instrumentation, and Procedure

This research is a case study (Creswell, 2013; Yin 2017) involving two participants – Nancy and Marsha (pseudonyms). The research design I chose provides a tool to analyze the phenomenon of mathematical reasoning of those two teachers as they participated in a PD created to help them construct new schemes based on their previous schemes. For example, teachers may have joined the PD program with a "repeated addition" scheme of multiplication, indicating 1 level of units coordination. For such teachers, the goal for their learning might be to construct the multiplicative double-counting (mDC) scheme (2 levels). To address changes in teachers' unit coordination schemes, Nancy and Marsha are used as exemplars that can illuminate this phenomenon. I chose Nancy and Marsha because of growth in their levels of units coordination I could notice and infer throughout the 2-year period. Specifically, they illustrated differing transitions and progressions throughout the project (Creswell, 2013; Yin, 2017).

Sample

I chose Nancy and Marsha from a sample of practicing teachers who experienced a two-year PD aimed at improving teachers' fractional reasoning (40 teachers total). My choice reflects a critical case sampling (Miles & Huberman, 1994), using two participants who had specific characteristics that gave insight into the phenomenon being studied (Onwuegbuzie & Collins, 2007). This sample provided the opportunity for intensive analysis of growth in levels of units coordination along the spectrum. Specifically, Nancy and Marsha indicated characteristics of making significant transitions in their reasoning and MKT. Most importantly, Nancy and Marsha have been working together on the same team of third grade teachers in one school. Within the context of the project, this included working as a buddy-pair within that team (see more about the buddy-pair PD method below). In such a context, I chose those teachers for my case study because their different levels of transition allow delving into factors that may have contributed to it. While this started as a convenience sample (Onwuegbuzie & Collins, 2007), it became evident the two participants were representative of larger groups of teachers – some of whom participated in the same PD.

The decision to do a multiple-case study using two contrasting cases came from the deliberate effort to strengthen a qualitative study. According to Yin (2018), using two or more contrasting cases within a study strengthens "your findings compared with those from a single-case study alone" [especially when] "findings support the hypothesized contrast" (p. 61). Having strong findings from the contrasting cases may then lead toward "a strong start toward theoretical replication" (p. 61). Postulating that, while working and learning together, Nancy and Marsha experienced very different levels of transition during the AdPed project, they seemed helpful contrasting cases for this study.

Participants

Nancy and Marsha are white, female teachers who taught in a diverse, urban school setting in the USA. At the start of the project, Nancy had been teaching for six years and Marsha had been teaching for nine years. These demographics help situate the exemplar each of them is, in that they represent the majority of teachers in the school and district they taught in. Neither of them was new to teaching and had several years' experience in the classroom. In the first year of the project, they both taught third grade. In the following (second) year they both moved ("looped") with their students to fourth grade.

Based on a survey the participants filled out at the project's Summer Institute PD components, both Nancy and Marsha indicated they had previously taken three to five college-level mathematics courses and one or two methods of teaching mathematics courses. In the survey, participants were also asked to rate their own views about mathematics and their abilities to teach it. Table 1 shows Nancy's responses for this part of the survey and Table 2 shows Marsha's responses. I offer these tables as an insight into Nancy and Marsha's attitudes towards their own mathematical understandings and confidence in teaching mathematical content to their students at the onset of the project. Their responses indicate that both entered the project with somewhat low self-perception in-regards-to their mathematical abilities. The choice to include them in this study, is further strengthened by these data – they both began with similar feelings about mathematics and their confidence in teaching it.

Table 3.1 - Nancy's Responses to Survey Questions Regarding Her Views About Mathematics

		Strongly Disagree				Strongly Agree
Q1. Please indicate how well each of the following statements describes your attitude toward *teaching* mathematics.	a. I enjoy teaching mathematics.	1	2	3	4	5
	b. Mathematics isn't my strongest subject to teach.	1	2	3	4	5
	c. I consider myself a "master" mathematics teacher.	1	2	3	4	5
Q2. Please indicate the extent of agreement with the following statements about your *knowledge* of mathematics.	a. Overall, I know the mathematics needed to teach this subject.	1	2	3	4	5
	b. I have strong knowledge of fractions and decimals.	1	2	3	4	5
	c. I have strong knowledge of *all areas* of mathematics.	1	2	3	4	5
	d. My knowledge of fractions and decimals is adequate to the task of teaching these subjects.	1	2	3	4	5

Table 3.2 - Marsha's Responses to Survey Questions Regarding Her Views About Mathematics

		Strongly Disagree				Strongly Agree
Q1. Please indicate how well each of the following statements describes your attitude toward *teaching* mathematics.	a. I enjoy teaching mathematics.	1	2	3	4	5
	b. Mathematics isn't my strongest subject to teach.	1	2	3	4	5
	c. I consider myself a "master" mathematics teacher.	1	2	3	4	5
Q2. Please indicate the extent of agreement with the following statements about your *knowledge* of mathematics.	a. Overall, I know the mathematics needed to teach this subject.	1	2	3	4	5
	b. I have strong knowledge of fractions and decimals.	1	2	3	4	5
	c. I have strong knowledge of *all areas* of mathematics.	1	2	3	4	5
	d. My knowledge of fractions and decimals is adequate to the task of teaching these subjects.	1	2	3	4	5

Data Collection

The project researchers collected data from the entire group of teachers participating in the AdPed project over the two-year timespan. For this study, I obtained data from the various components of the project's PD program, as well as its research efforts (which I explain in the subsequent sub-sections). I analyzed all data the project team collected, including:

- Summer Institute 1 video and transcripts – involving only Nancy (Marsha did not attend it)
- Summer Institute 2 video and transcripts – involving both Nancy and Marsha

- Account of Practice (AOP) videos and transcripts – four involving Nancy and two involving Marsha
- Buddy-Pair videos and transcripts – ten involving Nancy and eight involving Marsha
- Grade-level workshops – nine involving both Nancy and Marsha
- Extra coaching sessions – five involving Nancy

Summer Institutes

As part of the AdPed project PD program, the researchers conducted two Summer Institutes (SI). Both SI-1 and SI-2 were one week-long and held during the summer when the teachers were free from their classrooms. SI-1 took place at the very beginning of the project, as the first intervention put into place for the teachers. The focus of SI-1 was on fostering teachers' construction of conceptual understanding of number as a composite unit and thus a necessary requisite for their multiplicative reasoning. SI-2 was held during the summer between the first and second years of the AdPed intervention. In SI-2, the team focused on fostering new mathematical schemes for fractional reasoning within the teacher participants. In both SIs, the researchers focused also on participants' understanding of conceptual progressions in their students' reasoning and instructional methods to promote such progressions. The Graduate Research Assistants (GRA) on the AdPed team video-recorded all sessions of both SI's. Being a GRA on the project, I served as a videographer during both SI's.

Serving in this role helped my analysis, as it required, at the time, to make choices on what the camera captures. Later on, it supported me in getting familiar with video records and identifying data segments relevant for this study. Specifically, for this study, I looked for instances when Nancy and/or Marsha shared anything that seemed to be evidence of their mathematical reasoning and/or MKT.

Account of Practice

Throughout the two-year AdPed study, the researchers employed a data collection method developed by Simon & Tzur (1999), called *Account of Practice* (AOP). An AOP consists of either a full set or partial set, in which teachers are interviewed and observed teaching a lesson. In a full unit set, the researchers collect data over five events – a pre-interview, a lesson observation, another interview preceding the first observation, another lesson observation, and a final post-interview. In a partial unit set, the researchers collect data over three events – a pre-interview, a lesson observation, and a post-interview (at times, one of the two interviews may have been dropped).

The AOP strategy was originally designed to collect data on teachers' rationales for making specific pedagogical choices when teaching mathematics. Therefore, interviews in the AdPed project typically focused on teachers' pedagogical choices and what they noticed about student reasoning before and during the lesson being observed. However, these interviews often times included conversations between the researchers and the teachers about their understandings of the math involved in the lesson.

These data sets provide a plethora of evidence about Nancy and Marsha's reasoning throughout the two years of the study, as well as their MKT. Nancy participated in both partial and full sets, while Marsha participated in partial sets only. This was due to their own (IRB-approved) consent for being a case study on the AdPed project; case study participants committed to do full sets once a year during the project. Nancy consented to be a case study, while Marsha did not; therefore, Marsha only had to do partial sets.

The AdPed PI, co-PI's, and the GRA's conducted the AOP's. Either the PI or one of the GRA's on the project conducted Nancy's AOP sets, with me present for majority of them. I conducted all of Marsha's AOP sets. Either another GRA or I video-recorded each AOP set. I transcribed parts of Nancy and Marsha's AOP sets that were relevant for diagnosing their mathematical reasoning.

Buddy-Pair Sessions

In addition to the SIs, the AdPed project team engaged teachers in multiple, job-embedded, buddy-pair sessions. The project PI and a project GRA paired each participant with a "buddy" for the school year (or semester) – another teacher in their grade-level. Joined by a project team member, they visited each other's classrooms to either co-teach a lesson, or the visiting teacher would observe the other teacher's lesson. During each session, the teachers and researchers interacted with students, asking them questions about their reasoning, and taking notes on their findings. Following each buddy-pair taught/observed lesson, the two teachers and the researchers would conduct a reflective debrief (Murtadha-Watts & D'Ambrosio, 1997), in which they would discuss the mathematical reasoning they observed in the students and possible next steps for instruction. The PI or one of the GRA's always led the debriefs. A GRA on the project video-recorded each buddy-pair session (lesson and debrief). I was oftentimes the GRA recording the sessions involving Nancy and Marsha.

Originally, Nancy and Marsha were paired-up as buddies for these sessions, because they were teaching students at similar mathematical ability levels. Eventually they were paired-up with other grade-level team-members, because the team switched up their teaching assignments. Nancy was teaching the highest ability-level and Marsha switched to the lowest ability-level. The research team decided that it would be best for them to be paired with a teacher who was teaching similar students. In the fall of 2016, we conducted two buddy-pair sessions with Nancy and Marsha. We again paired up Nancy and Marsha for five buddy-pair sessions during the spring of 2017. However, Marsha missed one of the sessions, so she participated in a total of four sessions and Nancy participated in five sessions. In the fall of 2017, the research team paired Nancy and Marsha with other teammates, and each participated in one session. In the spring of 2018, Nancy and Marsha were again paired-up with other teammates, with Nancy participating in two sessions and Marsha participating in one session.

These buddy-pair sessions, ten in total for Nancy and eight in total for Marsha, would occasionally focus on the teachers' reasoning. Thus, the debrief discussions gave glimpses into their mathematical reasoning and/or MKT. Accordingly, I will be analyzing video sections that provide evidence into Nancy and Marsha's mathematical reasoning and MKT as it progressed throughout the AdPed project. To this end, I transcribed any sections of the videos pertinent for this study.

Grade-Level Workshops

The school at which Nancy and Marsha taught received nine workshops from the AdPed research team. The PI designed these workshops to address issues that arose during buddy-pair sessions, and to help the teachers develop their mathematical and pedagogical reasoning. We held the first workshops in the school's library, with all participants present at the same time. They sat in their grade-level teams to discuss and complete activities that pertained to their teaching.

Those workshops were usually half-day workshops led by either the PI or a GRA on the team. However, as the teachers began to develop their own ways of reasoning mathematically and using grade-level appropriate activities with their students, the research team realized they needed more differentiated workshops that met each grade-level team's special needs. Therefore, different workshops were developed for each grade-level team and what their specific needs were. These workshops became grade-level two-hour workshops designed specifically for each team of teachers.

I recorded all of these workshops, which provide a plethora of evidence about the teachers' mathematical reasoning. In each workshop the team members engaged the teachers in specific mathematical tasks to help them progress in their mathematical reasoning as they constructed new multiplicative and fractional reasoning schemes. I transcribed any sections of video that gave me evidence into how Nancy and Marsha were thinking about the mathematics, paying close attention to any instances of changes in thinking. I note that, from a data collection and analysis point of view, the video records of the PD workshops are richer than the SI data while not as in-depth as the AOP. Using those three sources further support triangulating (Creswell, 2013) my data and improve the credibility of my analysis.

Extra Coaching Sessions

In addition to the SIs, buddy-pair sessions, and grade-level workshops, the team conducted five extra coaching sessions with Nancy. Another GRA from the AdPed project led those extra sessions, which included co-teaching fractional reasoning lessons in Nancy's classroom. The GRA did this in response to Nancy's request. Once Nancy's students reached the fractional reasoning schemes, she did not feel confident enough in her own mathematical abilities to teach the lessons on her own. So, she requested these extra sessions to help her effectively teach the lessons to her students, as well as learn fractions herself. Just like in buddy-pairs sessions, the GRA and Nancy conducted a debrief about each of those lessons. Because these lessons and debriefs were video recorded, they provided me with extra evidence of Nancy's mathematical learning progressions, especially in fractional reasoning. Marsha did not participate in these extra sessions. We offered these to all participants in the project, but only teachers who requested the extra coaching received sessions like these. Marsha did not request any extra coaching. I transcribed all valuable data that gave evidence about Nancy's ways of thinking mathematically.

Mathematical Tasks

Over the course of the project, the researchers engaged teachers in solving and teaching different mathematical tasks found to help foster the construction of the mathematical and fractional reasoning schemes. The two main tasks were *Please Go Bring Me* (PGBM) (Tzur et al., 2013) and the *French Fry* activity (Tzur & Hunt). While both tasks were originally designed for promoting new mathematical reasoning in students, the AdPed team used these tasks with the teacher participants with the hope that it would foster the teachers' construction of the same multiplicative and fractional reasoning schemes that the tasks foster in students. The researchers would engage the teachers in the tasks just as they encouraged the teachers to use the tasks with their elementary students. As the teachers would progress through the activities and begin to construct the reasoning schemes, the researchers would then help the teachers bring the same activities into their classrooms. Often times, the researcher would engage the students in the tasks while the teacher would observe the lesson. Over time, the teacher would take the lead in teaching the lessons and use the tasks in the classroom even when the researchers were not present. I now present descriptions of the two activities.

Please Go Bring Me

Please Go and Bring for me (PGBM) was designed by Tzur to promote the construction multiplicative reasoning schemes in students that are found in Tzur et al. (2013). The game "fosters multiplicative reasoning by engaging children in tasks conducive to carrying out and reflecting on coordinated counting activities" (p. 88). The game involves pairs of students, with one student acting as a “Sender” and the other student acting as the “Bringer.” The game begins with the sender asking the bringer to “please go and bring for me a tower with *x* cubes in it.” The bringer walks to a bin containing connecting cubes and brings back a tower with the specific number of cubes in it. The sender then asks the bringer to go and bring another tower made of the same number of cubes as the first tower. This is repeated as many times as needed for the sender to obtain a desired number of towers. An example of this activity, involving three towers of four cubes each, would have the bringer in going to the box of cubes three times, each time bringing back a tower of four cubes. Once the sender has the towers they wanted, they ask the bringer four questions: 1) How many towers did you bring? 2) How many cubes are in each tower? 3) How many cubes are there in all? and 4) How did you find your answer? The purpose of the first two questions is to elicit reflection of the bringers’ towers (composite units) and cubes (single units). The third and fourth questions are used to help students coordinate the simultaneous counting of the composite units and the single units.

The sequence of the PGBM task changes over time as the students begin to assimilate the composite units and single units (see Tzur et al., 2020). When they begin playing the game, they use concrete items (e.g., connecting cubes). This allows the students to operate on the tangible items while coordinating the units. Once the students have exhibited success with the tangible cubes, the activity shifts to covering the cubes before asking the four questions. The purpose is to help the students move to using figural representations of the units. They may substitute the cubes by using their fingers or drawing the towers and cubes. Teachers guide students to specific drawings of the units, where they first draw the towers by showing cubes in them. Eventually, it moves to simply drawing sticks to represent the towers and cubes. Once students are able to anticipate the coordination of the units with figural representations, they are introduced to abstract representations (equations). There are different variations of this task used as individuals progress through the multiplicative reasoning schemes. Once they have constructed the multiplicative reasoning schemes, they may be introduced to the *French Fry* activities to begin constructing fractional reasoning schemes.

French Fry Activity

The *French Fry* task (Tzur, 1996; Tzur & Hunt, 2015) was designed to foster and "solidify children's multiplicative notions of unit fractions through a core activity of unit iteration (i.e., using a single item, such as a paper strip of specific length, and repeating it a number of times to create and/or 'measure' another unit)" (Tzur & Hunt, 2015, p. 150). Like PGBM, the students work through a variety of tasks all deriving from a basic task involving strips of paper that represent a French fry. Students are given two strips of paper of different lengths. The longer strip is the French fry, and the other strip is used as a tool for helping them iterate units. The students are first asked to share a French fry strip equally among two people. They are allowed to do whatever they would like in order to share the fry, including the folding of the strip of paper in half. The students are then asked what their strategy was and how they know the two halves are equal in size. The goal of this first task is to bridge from the students' segmenting operations, rooted in their concept of number as a composite unit, into the following tasks that foster construction of unit fractions (1/n) as multiplicative relations.

Next, the students are asked to share a new fry equally among three people. However, at this point the teacher presents the students with rules they must follow. Specifically, they cannot fold the strip of paper, nor use a ruler. They may use the smaller strip as a tool to help them determine the size of one person's share and iterate that share across the whole French fry strip. As students are working, the teacher asks them whether their guesses are too short or too long (direction of change) and what they might do to get a better guess of one person's share (magnitude of change). Once students are able to use the tool strip to accurately guess one person's share and iterate three times to fit the whole, they move on to sharing a new French fry strip among four people, five people, and so on. The purpose of this task is to foster the iterating operation in the student's assimilatory scheme.

Analysis

For this study, I employed a cross-case analysis (Miles & Huberman, 1994; Yin, 2018), a methodological approach that helps "retain the integrity of the entire case and then to compare or synthesize any within-case patterns across the cases" (Yin, 2018, p. 196). First, I analyzed Nancy and Marsha's progressions throughout the project separately. Then I compared my findings from each analysis to draw on similarities and differences within their mathematical and MKT transitions.

My analysis of the video recording from all data collection sessions draws on Powell et al.'s (2003) analytical model for investigating learners' mathematical development over time. Consistent with Glaser and Strauss' (1967) grounded theory methodology, these researchers outlined a specific, 7-step process for analyzing recorded videotapes. Next, I present those steps along with detailed descriptions of my actions throughout each step.

Step 1 – Data Viewing

The first step of the process is to attentively view the video data "without intentionally imposing a specific analytical lens on [the] viewing" (p. 416). The purpose of this step is to give the researcher a chance to become familiar with the full data sets. For this step, I watched all videos involving Nancy and Marsha. Those videos included recordings from both Summer

Institutes of the AdPed project, all buddy-pairs the two participated in, the grade-level workshops, Nancy and Marsha's AOP sessions, and the extra coaching sessions with Nancy.

In watching each video, I began to get an idea of which videos/segments would be used for further analysis, based on data they may provide. For instance, the Summer Institute videos provided no individual evidence of Nancy and Marsha's mathematical knowledge. This was due to the fact that the Summer Institutes involved a large number of teachers and there were no opportunities for recording individual teachers and their reasoning. Therefore, I eliminated those videos from further analysis. These initial viewings also provided some insight into which videos might provide specific findings. The buddy-pair sessions, AOP's, and extra coaching sessions provided possible evidence into the teachers' reasoning, as well as their MKT. The grade-level workshops only provided evidence into Nancy and Marsha's mathematical knowledge, but not their MKT.

As I went through the collection of videos, it soon became obvious that I needed some sort of organization system in order to remember which videos occurred at which point in the project, including which involved just Nancy, just Marsha, or both of them. So, I created a spreadsheet (Table 3.3) to keep track of this information, in order to easily identify the videos in my collection. This spreadsheet was first organized by session type – AOP videos, buddy-pair videos, workshop videos, and extra coaching videos. Then, I listed the videos in each category in order by the date they occurred. Each listing also included who (Nancy, Marsha, or both) was present in the video.

Since my study is longitudinal, it was important to put the videos in order by date, so I could later re-watch them in the same order they occurred and analyze the progressions in Nancy and Marsha's mathematical reasoning and MKT. Because knowing the order of the videos was important, I designed a plan for the order of viewing in Step 2.

Table 3.3 – Spreadsheet of Videos

Session Type	Nancy	Marsha	Both
AOP	4/26/16	5/1/17	
	5/3/17	4/24/18	
	2/9/18		
	5/7/18		
Buddy-Pair			10/25/16
			11/4/16
			1/19/17
			1/26/17
			2/10/17
			2/23/17
	11/6/17		
		12/7/17	
	1/29/18		
		2/15/18	
	3/19/18		
Workshops			9/26/16
			11/7/16
			4/3/17
			4/21/17
			10/26/17
			11/30/17
	1/26/18		1/26/18
	4/26/18		
Extra Coaching	2/17/17		
	5/4/17		
	5/9/17		
	12/4/17		
	12/7/17		
	12/8/17		

Step 2 – Data Descriptions

For Step 2, the researcher describes the video data, using only factual observations. The descriptions should be brief, time-stamped intervals of up to 5 minutes each. This step allows the researcher to go deeper into the data, while also identifying specific episodes within the data. Researchers often refer to this step as data logging (Easterby-Smith et al., 2008). For this step, I did a second viewing of each of the videos in the same order they occurred. While watching, I developed brief descriptions ("logs") of specific intervals that I wanted to revisit for Step 3. This step helped me narrow down which videos I would analyze further (Table 3.4), based on any key evidence I thought they provided – specific moments in which Nancy or Marsha provide evidence into their reasoning or MKT. I was able to eliminate videos that did not provide any significant data. For instance, I eliminated the workshop from 9/26/16, because there were no portions of the video in which Nancy or Marsha were neither doing any math tasks nor talking about their students' math. I added the descriptions to my spreadsheet (Table 3.3), for easy access in Step 3.

Table 3.4 - Remaining Videos for Step 3

Session Type	Nancy	Marsha	Both
AOP	4/26/16	5/1/17	
	5/3/17	4/24/18	
	2/9/18		
	5/7/18		
Buddy-Pair			10/25/16
			11/4/16
			1/19/17
			1/26/17
			2/10/17
			2/23/17
	11/6/17		
	1/29/18		
		2/15/18	
	3/19/18		
Workshops			4/3/17
			4/21/17
			10/26/17
			11/30/17
	1/26/18		1/26/18
	4/26/18		

Step 3 – Identifying Critical Events

The third, key step is to identify critical events (Maher, 2002; Maher & Martino, 1996a, 1996b, 2000; Powell et al., 2003). A critical event is marked by a significant leap in a learner's understanding (as the observer noticed it) – leading to a change from previous understanding to a new understanding (Kiczek, 2000; Maher, 2002; Maher & Martino, 1996a; Maher et al., 1996; Powell et al., 2003; Steencken, 2001). In this step, I re-watched the videos identified in Table

3.4, looking specifically for critical events in which Nancy and/or Marsha provided evidence into their current ways of reasoning, their MKT, or transitions in those. An example of one of these key moments occurred in the Buddy-Pair session from 10/25/16. During the debrief of this Buddy-Pair, Nancy was explaining what she noticed in her students' reasoning during the lesson she taught, where students played the Please Go Bring Me game (Excerpt 3.1). This explanation was identified as a critical event, because it provided evidence into Nancy's MKT as she used her own mathematical knowledge to identify and analyze her students' reasoning. This excerpt is further analyzed in Chapter IV.

Excerpt 3.1 - *Lesson Debrief: Nancy's Noticing of Student Reasoning*

0:10 minutes into the debrief

R: Tell me about how the lesson went.

Nancy: I noticed similarities in a lot of kids putting together a couple, or it's like, for instance, if it was 3 towers with 6 cubes in each, then it was like, well, I took 6 and 6 and I put them together and it was 12 and then I added on another 6 and that was 18.

R: When they did that – did they go 13, 14, 15, and counted on?

Nancy: Yes.

R: They doubled then counted on.

Nancy: Right. So, they're doubling, and maybe they know their doubles, maybe that's something they've just memorized. Then counting on. I had one girl even say, I put these two sixes together and then I counted on from there the last group, the last tower. And I noticed that happened again with another student where she said that I know that 7 plus 7 is 14 and then I took the 14 plus another 14 is 28. So, I guess this is where I start to break down in my understanding, because I see that they are putting the groups together in some sort of either doubling in putting them together or some form of repeated addition where they are saying 7 plus 7 plus 7 more, whether they are counting on they know that more fluidly from things that they've memorized, or actually know. Then they are putting those groups together. That's kind of what I say was a trend.

R: So, one of the things that you are paying attention to is really important, are they operating, the doubling suggests that they are operating on some level as a composite unit.

Nancy: Yes.

R: As a thing as opposed to ones versus sometimes, they operate on the ones. That's a very important distinction to make. All the time look for what are they operating on? Or are they operating on both, which will be the direction of where the gain is leading.

N: That's my question. When she said, I said could you have done this another way and I think she did, my interpretation was, she wasn't quite sure how to do it another way. She knew she could add it another way, but she didn't know she could stick with units in, *units within units* [italics added], another way. That when she said I could break off, and she's breaking off groups and so in my head I'm saying, I'm thinking, okay if someone initially started to do that is it because they are more comfortable working with 9 and 9. Are they making groups that are more comfortable for them to work with, or are they just trying to add it another way?

This excerpt is just one example of the many critical events identified in the videos. Some videos provided more critical events than others. For instance, Marsha's Buddy-Pair session on

2/15/2018 provided only one small critical event, while her AOP session on 4/24/2018 provided several critical events. Once the critical events were identified, I moved onto transcribing those videos for Step 4.

Step 4 – Transcribing Critical Events

In Step 4, the researcher transcribes the critical events data to allow the choice of transcription excerpts that can be "theoretically guided" (Erickson, 1992, p. 219) and hence, "necessarily selective" (Heritage, 1984, p. 12), by their research questions. The use of transcripts provides the researcher with an opportunity to analyze the "flow of ideas…[and] provide evidence for important theoretic or analytic matters relative to [their] guiding research questions" (Powell et al., 2003, p. 423).

For this step, I transcribed each of the critical events I identified in Step 3. These events were chosen based on the evidence they provided into how Nancy and Marsha were reasoning mathematically on their own, or how they were identifying and analyzing their students' reasoning (a part of their MKT). This was guided by my three research questions and my theoretical framework, which focused on how teachers' multiplicative and fractional reasoning, specifically their levels of units coordination, affected their MKT. When transcribing, I made sure to include any hesitations when the teachers were thinking in order to identify moments in which reflection or perturbations might have been occurring. I also included descriptions of pertinent gestures (e.g., hand movements) that provided more evidence into their reasoning. For example, there were many moments in which the teachers used their fingers to mimic their own reasoning or the reasoning of their students. Excerpt 3.2 is an example of the inclusion of hand movements in my transcripts that may have provided additional evidence into the teachers' ways of reasoning and analyzing student reasoning.

Excerpt 3.2 – *Lesson Debrief: Nancy's Noticings of Students' Units*

32:53 minutes into debrief

R1: I have a question. Let's say you have two kids, and you have 6 towers of 3. One does 1, 2, 3 (counts on one finger) 4, 5, 6 (counts on another finger) all the way to 6 towers but ends up with the answer of 17. The other one says I know that 3 times 6 is 18, and I have 6 towers, that's 18. Which one would be more, in terms of reasoning in multiplicative double counting?

Nancy: Well, I would say (to the child), how did you know?

R1: To the last kid?

Nancy: Yeah, and I would have to hear what they said.

R2: I just know, 6 times 3 is 18.

Marsha: Six towers, well what about the 6 towers?

Nancy: I think it's 19. Prove it to me.

R1: So, what about the 17?

Nancy: I mean, he's still (swipes index finger on left hand over each finger of the right hand to show the student keeping track of the towers), he could have missed one, he could have been going quick, he could have just left one off on accident. He still knew to stop. He knew how many towers to stop at. Could it have been just like a clerical error? Or did he not know how many to put in the last one?

Excerpt 3.2 illustrates Nancy's use of her fingers to show how her student was keeping track of composite units. I included the description of her swiping an index finger on her left hand over each finger of her right hand as she was mimicking what she believed the student was doing to track his units. This hand movement seemed like critical detail, which needed to be included in the transcription, because it gave some evidence not only into Nancy's reasoning but also what she was paying attention to in her students' reasoning. This excerpt if further analyzed in Chapter IV.

Step 5 – Coding of Critical Events

Once the data are transcribed, coding of critical events may begin (step 5). When I got to the coding stage in my analysis, I used emergent coding and analyzed those codes using the constant comparison analysis method as defined by Glaser & Strauss (1967). Although these codes were directed by my research questions and theoretical lens for the study, they emerged out of my focus on the critical events discovered within the data. To assist in the coding step, I used the NVivo computer software device (Powell et al., 2003).

As I began to dig into my transcriptions, I would code anything that seemed to provide evidence for my three research questions. I first began with two main categories of codes – 1) Codes for events that provided evidence into Nancy and Marsha's mathematical reasoning, and 2) Codes for events that provided evidence into Nancy and Marsha's MKT. Over time, the codes became more specific. Under the first category, I looked for their multiplicative and fractional reasoning (e.g., multiplication as repeated addition vs. multiplication as the distribution of units; fractions as part-of-a-whole vs. fractions as a multiplicative relationship) as well as their levels of units coordination (e.g., the assimilation of two levels of units as given). Under the second category, I looked for the specific reasoning they noticed in their students (e.g., noticing that students are tracking multiple units). Table 3.5 outlines the specific codes for Category 1, as well as exemplar transcriptions for each code. Table 3.6 outlines the specific codes for Category 2, as well as example transcriptions for each code.

Table 3.5 – Category 1 Codes and Examples

Code	Example
Multiplication as equal groups or repeated addition	Nancy: So, what I was doing now was to kind of push them from pictures to maybe even numbers or repeating that addition and seeing that connection.
Multiplication as the distribution of units	Nancy: Now I think about the units made up of smaller units, and that we're tracking them simultaneously. When we get to 72 that it's 8 times larger than where we began.
Fractions as part-of-a-whole	Marsha: Why is it 1/4? Because there's four pieces and one of the four pieces is yellow.
Fractions as a multiplicative relationship	Nancy: I drew the fraction bar with four pieces, and I said that the whole is four times as large or that that is how many times that can be like iterated into the whole (Uses her fingers to show a piece being iterated in the air), or fits into the whole, or repeated into the whole.
Procedural mathematical understanding	Marsha: I did 10 times 6 is 60 and 10 times 6 is 60. I did 120 and then I did 3 times 12 is 36. I added the 120 to the 36.
Teachers' understanding of fractional direction and magnitude of change	Marsha: Because I have left-over to give away. Because these pieces are all the same, so if I give that (points to a partition of the left-over piece) to that piece (points to first iteration), and that (points to a partition of the left-over piece) to that piece (points to second iteration), and that (points to a partition of the left-over piece) to that piece (points to third iteration) then it would be even.
Teachers' assimilation of one level of units as given and one level in activity	Marsha: I needed to see the towers and I started snapping them and then realized that 2 times 9 is 18 as well.
Teachers' assimilation of two levels of units as given and one level in activity	Marsha: Towers (swipes up her index finger) and then the number of cubes (mimics counting six times on her index finger).
Teacher's assimilation of three levels of units as given	Nancy: Well, I first knew I needed to figure out how many were in each box. So, if there were 10 bags of 10 each, I knew that 1 box had 100. So, 4 boxes with a hundred in each would be 400 boxes. And then I knew there were 10 in each bag, and I had 6 bags, so I had 60 within the bags. And then 19 single apples. So, I combined them.

Table 3.6 – Category 2 Codes and Examples

Code	Example
Students' counting methods	Marsha: Some of them are still doubling. Some of them are skip-counting. Some of them still have to count out. If we do it together, they can do like that picture on their fingers. They go 4, 8, 12, 16 (shows counting each of these on the fingers of her left hand).
Students' understanding of fractional direction and magnitude of change	Nancy: But particularly French fry, you know, it was a little difficult for them to understand how much larger they had to make each of their pieces, or how much smaller they had to make each of their pieces. That they had to divide that last piece up into how many partitions there were.
Students tracking multiple units	Marsha: You're skip-counting all the way at the bottom, but you're keeping track of two things. You're keeping track of towers and you're keeping track of cubes to figure out the total and apply it to your math work.
Students' assimilation of units	Nancy: I saw kids when I gave them a chance to go back to their seats and work with the manipulatives and represent their thinking. I kind of, just working at the one table that I did, I saw three things happen. I saw one student working with ones, breaking up the pile of cubes into ones, the 42 cubes (referring to the problem: You have 42 cubes, and you want to create 6 towers. How many cubes will you have in each tower?). Then I saw another student start with 5 in each, and then just add on what they had by ones, the remainder that they had. Then I saw David, who went back to drawing towers and cubes in each.
Students' assimilation of composite units	Nancy: I would look for students to be using knowledge of keeping track of both units to take those 7 candies and spread them across, to make composite units with them until they reach 56.

Some transcriptions were coded under multiple codes. For instance, Excerpt 3.3, from a buddy-pair session in February of Year 1, illustrates coding into four codes – students tracking multiple units, students' assimilation of composite units, students' assimilation of units, and multiplication as the distribution of units.

Excerpt 3.3 – *Lesson Debrief: Nancy's Focus on Units Coordination*

2:50 minutes into debrief

R: How do you think the lesson went?

Nancy: I was impressed by the way that they (referring to the students) were able to *use their fingers and keep track of the units* [italics added] and know when to stop. Like, Felipe (pseudonym) knew that three towers of three were nine, and then he just added one more tower of three. So, seeing them either *keeping track of the towers* [italics added] (swipes a finger on her right hand over each finger of her left hand to "show" the composite units) and counting one tower at a time. Or even doubling and having them start from the three groups of three and then adding on one more was a lot more complex than I thought. I didn't think they would do that. So, I was happy to see that. I thought they might do one or two (referring to the towers) and then count on.

This critical event was coded with those four codes, because Nancy is noticing her students' assimilation of the composite units as they are tracking single units and composite units simultaneously. This excerpt was also coded with *multiplication as the distribution of units*, because Nancy's MKT in this event gave evidence into her own multiplicative reasoning. This excerpt will be further analyzed in Chapter IV but is provided here to illustrate how multiple codes could be applied to certain critical events, if they were present in the data.

This overlapping of multiple codes is due to the interrelationship between some of the codes. For instance, in order for Nancy to identify the tracking of multiple units in her students, she would inherently be identifying their assimilation of composite units and single units, which they are tracking. This also implies that Nancy is assimilating the units as well, suggesting that she understands multiplication as the distribution of units. Once the coding was completed, I moved into Step 6.

Step 6 – Constructing a Storyline

After the coding step, the researcher moves onto constructing a storyline (step 6), in which they make sense of all previously coded data. The researcher must make sense of, and organize, the critical events in order to perceive *traces* – the coded and interpreted data that provide a glimpse into a learner's mathematical development (Maher & Davis, 1996; Maher & Speiser, 1997; Powell et al., 2003).

For this step, I attempted to use my data to identify the changes that occurred in Nancy and Marsha's own mathematical reasoning, as well as their MKT. I found the Coding Stripes function within NVivo to be very helpful in this step, because I could visually see how the codes changed over time. I noticed that, as the project progressed, both Nancy and Marsha's coding stripes changed significantly. For example, Nancy's stripes changed from more Category 1 codes at the beginning to more Category 2 codes later on. Marsha's Category 1 codes changed from more procedural codes (multiplication as equal groups or repeated addition, fractions as part-of-a-whole, and procedural mathematical understanding) to more conceptual codes (multiplication as the distribution of units and fractions as a multiplicative relationship) over time. Her Category 2 codes also transitioned from *students' counting methods* codes at the beginning to *students tracking multiple units* and *students' assimilation of units* over the course of the project. These coding transitions and their significance are further explained in the Chapter IV Analysis and the Chapter V Discussion. My noticing of the coding transitions guided my work in Step 7.

Step 7 – Composing the Narrative

The final (seventh) step is to compose a narrative made up of small segments of interpreted data and then composing all of those small segments into an interpretation of the whole data set. In my study, this step was done for each individual teacher, which informed my comparison between them. The upcoming Chapter IV Analysis provides the results of Step 7. I first composed a narrative of Nancy's transitions throughout the two-year project, and then composed the narrative of Marsha's transitions. (For Chapter IV analysis of those narratives, however, I first presented Marsha's case and then Nancy's.) Focusing on one teacher at a time allowed me to dig into their transitions for the within-case analyses. I was able to outline Nancy's data throughout the project and interpret what those findings suggested about her mathematical and MKT growth over the two years. For example, Nancy's narrative began with an analysis of her pre-intervention data, which provided a baseline of her reasoning and MKT before receiving any of the project interventions. The narrative then moves into the Year 1 data and analysis, followed by a summary of the Year 1 findings. Then, the narrative goes into the Year 2 data and analysis, followed up with a summary of the Year 2 findings. Marsha's narrative follows a similar structure, except for the pre-intervention data that are not available for her. From there, I was able to compare the two cases for the cross-case analysis, which is detailed in the Chapter V Discussion when I explain the significance of the findings.

Furthermore, I took my findings and compiled them into two different types of visual summaries. First, I compiled Marsha and Nancy's analyses separately into conceptually clustered matrices (Miles & Huberman, 1994), in order to provide a detailed connection between their levels of units coordination, their multiplicative and fractional reasoning, and their MKT over time. The matrices (Tables 4.1 and 4.2) allow one to get an integrated view of what was occurring in those categories at any point-in-time during the two-year project. For example, one can see that the findings from Nancy's first buddy-pair show she was assimilating at least 2.5 levels of units, had constructed the mDC scheme at the participatory stage, and her MKT seemed focused on identifying the units her students were assimilating and how they were tracking those units. In this sense, these matrices reflect my attempt to connect the three research questions – addressing learning pathways in teachers' multiplicative and reasoning schemes, teachers' levels of units coordination, and impacts on teachers' MKT and their ability to identify and make inferences into students' assimilatory units.

The second visual representation I used was a growth gradient (Miles & Huberman, 1994) for Marsha (Figure 4.4) and Nancy (Figure 4.8) separately, as well as one for the cross-case analysis (Figure 4.9). These growth gradients made it possible to visually represent how Marsha and Nancy's levels of units coordination and multiplicative and fractional reasoning schemes transitioned together. For example, one can see that as Marsha's levels of units coordination progressed from 1.5 levels of units to 2 levels of units, her multiplicative reasoning transitioned from pre-mDC to mDC at the participatory stage. These tables and graphs provided a visual timeline for framing the whole picture over time.

Summary

This study involves two case study participants, Nancy and Marsha, who were chosen due to the comparative growth in their levels of units coordination throughout the project. A triangulated data corpus, from five different sources (SIs, AOP sets, buddy-pairs, workshops, extra sessions), was collected as video-recordings over a two-year timespan. All of the video data were analyzed using Powell et al.'s (2003) seven-step method for analyzing critical events in video data. When looking for critical events, I focused on segments in which I inferred Nancy and Marsha experienced some sort of transition in their mathematical reasoning and/or MKT. This can contribute to a greater understanding of how teachers' mathematical reasoning may evolve between already identified markers of understanding.

Ethics

To protect the identities of all participants, their names have been changed to pseudonyms. Both participating teachers were aware that all sessions were being recorded for data collection and had previously signed an informed consent form allowing data to be used by the researchers. Our Colorado Multiple Institutional Review Board (COMIRB) provided formal approval for the AdPed project, and all studies connected to it (including this dissertation study). In return for being a participant in the study, the teachers received instruction and coaching to help them construct strong mathematical reasoning, which improved their mathematics teaching and pedagogy. They also received monthly coaching sessions to help them identify their students' mathematical reasoning and to learn activities they could use with their own students. In addition, each participant received a stipend for their participation in the project. Participants who chose to be case studies (e.g., Nancy) received a larger stipend than teachers who were not case studies (e.g., Marsha).

CHAPTER IV

RESULTS

In this chapter, I present and analyze data collected over the two-year timespan of the AdPed project. The data include AOP sessions, buddy-pair sessions, and workshops in which Nancy and Marsha participated. As I explained in Chapter III, the project team also collected other data at different points in the project, but those data did not produce usable evidence of Nancy and Marsha's own mathematical reasoning, as well as evidence into their ability to analyze their own students' reasoning.

I begin with an analysis of Marsha and then proceed to Nancy. My analysis indicates that Nancy and Marsha entered the project at different levels of reasoning. Accordingly, the evidence collected from the two of them shows different progressions of reasoning transformation throughout the project.

The Case of Marsha

In this section, I present Marsha's progression. Marsha came into the project with a different level of mathematical understanding, and therefore had a different pathway in her growth. Marsha also participated in fewer interventions than Nancy; she missed the first Summer Institute, as well as several grade-level workshops and buddy pair sessions. It is also important to note that the researchers did not have a chance to collect pre-intervention data from Marsha due to her entering into the project after pre-intervention AOP's were conducted. Accordingly, the first pieces of data from Marsha were collected through a buddy-pair session in Year 1 of the project. An AOP data set was not conducted with Marsha until the end of Year 1 (spring 2017). I first present data and evidence of Marsha's progressions during Year 1 of the AdPed project, then I present her progression during Year 2.

Marsha's Progression in Year 1

In Year 1, Marsha participated in six buddy-pair sessions (with Nancy as her partner), one workshop, and one AOP data set. Her first experience with the AdPed project occurred in October of 2016 – a buddy-pair session involving her, Nancy, and the researchers. This buddy-pair session is the earliest data set we have, which is thus used to analyze her baseline mathematical understanding shortly after the beginning of the project. My analysis of Year 1 data indicate that Marsha came into the project with a procedural understanding of multiplication as repeated addition or groups of things. This way of seeing multiplication continued through most of Year 1, with some growth towards understanding multiplication as the distribution of single units over composite units to create a third unit. There is also evidence to suggest that Marsha assimilated one unit (1s) and another in action (composite unit) at the beginning of the project, and advanced to assimilating both types of units as given by the end of that year. I now present the Year 1 data, beginning with Marsha's first buddy-pair session.

Buddy-Pairs #1 and #2. The first buddy-pair session occurred in October of 2016. The lesson was conducted in Nancy's room, with Marsha and the researchers present. The lesson was on multiplicative reasoning, with the students playing Please Go Bring Me (Tzur et al., 2013). The second buddy-pair session occurred a week and a half later. Again, the lesson took place in Nancy's room, with Marsha and the researchers present. The second lesson was another multiplicative reasoning lesson in which the students again played Please Go Bring Me.

At both post buddy-pair debriefs, Marsha did not say much. This session was right after she had missed the first Summer Institute, and she repeatedly stated that she felt behind in her understanding due to missing that week-long professional development. When she did make comments, however, her focus seemed to be on whether or not students were getting the correct answers. She describes multiplication as repeated addition or the building of arrays. About forty minutes into the first debrief, the conversation turns toward a discussion about a student who had switched up his units during the task. Instead of working with five towers of seven, as asked, he worked with seven towers of five. The researcher asks Nancy and Marsha why that distinction was important in the student's reasoning. Then, about eleven minutes into the second debrief, Marsha asks about the memorization of multiplication facts. She seemed very concerned about pushing students to memorize their facts. Marsha's responses, in Excerpt 4.1, indicate her current understanding of multiplication as repeated addition and her focus on correct answers and memorized facts (R stands for the Researcher).

Excerpt 4.1 - *Lesson Debrief: Initial Glimpse into Marsha's Reasoning*

0:40 minutes into the first debrief

R: Later you will see, we will bring it in with the unit differentiation and selection. Please go bring me 7 towers of 5, please go bring me 5 towers of 7. How are they similar, how are they different? Yeah, they both have 35, but they are very different arrays of things.

Marsha: That's what I was thinking. Why it was important to me was because we start talking about arrays and if you ask them to build it, they look different. And *if you write the multiplication sentence with that, so if they get asked that on a test it's going to be wrong because that's not the array they asked for. Again, it's just following directions* [italics added].

0:47 minutes into the first debrief

R: When she goes to bring the towers, I am working with you, let's use the fingers and see if you can figure it out before the towers come in, so you have the answer.

You're the sender, right? And you need to know that if she tells you it's 37, no it's 35. Why is this important?

Marsha: It helps them to have a different strategy because again, on tests or homework, they don't always have the towers available and so if they have another outlet if they don't know to draw it, that they can use their fingers as tools and that kind of tells me how fluent they are at adding. Like how far they can go by 7's. Like some of them can do a 3 and then they start doing [moves her fingers like she is counting each finger]...so some of the other groups *I could tell how fluently they could add, like repeated addition by 3's* [italics added], but when you get to the more difficult numbers they can usually get the first 3.

0:11 minutes into the second debrief

Marsha: At what point do they just start memorizing their facts? These kids they can explain why 4 times 7 is 28. Like it's not just like, it's not like we started just memorizing facts. *They can explain and do the repeated addition and draw an array and do the equal groups. At what point do they just start memorizing the facts?* [italics added]

The first glimpse into Marsha's reasoning leads me to infer that she may initially understand multiplication as repeated addition and the building of arrays. This is evident when she explains how her students can "explain and do the repeated addition and draw an array and do the equal groups." Marsha also seems focused on her students getting the correct answer on multiplication tasks, which she seems to think could improve when they have memorized their multiplication facts. She brings up how the students need to memorize their facts, because they will not always have the towers in front of them when completing tests and homework assignments. For Marsha, the towers are there for the students to count when they don't have their facts memorized.

At this point in her reasoning, Marsha's own assimilatory scheme oriented her to notice whether students are getting the correct answers or not, and whether or not they are using their fingers (as opposed to memorized facts) to help solve the task presented to them. For Marsha, the inversion of the towers and cubes was significant, because it would lead them to build incorrect arrays and possibly get the incorrect answer on tests. She goes further to explain that she looks for students' fluency when using repeated addition to solve multiplication tasks. It is clear from this excerpt that Marsha is focused on the correctness of her students' answers and their fluency with repeated addition and math facts. Marsha's own assimilatory scheme (multiplication as a repeated-addition task) seems to mean that she is looking for the same reasoning in her students. She wants to see her students using repeated addition, the drawing of arrays, and memorized facts to solve multiplication tasks. With such foci, she seemed to operate mostly on the 1s and less on accounting for the composite units a child is adding. Marsha's next buddy-pair session (buddy-pair #4) further delves into her multiplicative reasoning.

Buddy-Pair #4. The fourth buddy—pair session occurred in January of the project's first year. For this session, one of the researchers taught a beginning mDC lesson (students playing PGBM) in Marsha's classroom, while Marsha, Nancy, and other researchers observed and took notes about what the students were doing. Excerpt 4.2 (from the post buddy-pair debrief) begins with Marsha discussing one of the student's strategies with one of the researchers. She seems to have a clear idea of this student's change in additive reasoning, from a counting-all strategy to a counting-on strategy, as the student was solving a PGBM problem using six towers of five cubes each. From there, the discussion turns to a comparison between two types of reasoning where the students both began counting from one. However, one of the students used cubes to count from one to eighteen, and the other student used their fingers to keep track of the ones while counting from one to eighteen. The researcher asked Nancy and Marsha to determine which strategy is more advanced in their mind. While Marsha's contribution is not expansive, she gives an indication of thinking about the use of figural towers and cubes (on fingers) as more advanced than using the concrete cubes.

Excerpt 4.2 - *Lesson Debrief: Marsha's Noticing of Student Additive Reasoning*

10:45 minutes into the debrief

R: She was counting to five, holding the first finger for the first tower. Then to five (holds up a second finger), and then realizes this is ten. So now she is moving to putting up two fingers for the next 10.

Marsha: So, *she was able to move onto a different strategy, not just counting all* [1s; italics added]. She realized when she got here, she could do two more.

R: Right, so what does that tell you? Is this good, bad, advanced, not advanced compared to others, compared to herself, what you knew before?

Marsha: I felt like that's good.

R: In what way?

Marsha: She knew that she could move on. It's almost like timesaving. Like she didn't have, she could get to the answer quicker. She realized, maybe, that was what she was safe with at first, then she was like, oh wait, I do know that. Now I can make 10 over here.

32:53 minutes into the debrief

Marsha: Did he count-all?

R2: So, let me bring a distinction, because you're really struggling with something that is really important to notice. We call it levels of units coordination. How many units is the child operating on at the same time? So, if the child did, the towers are here 1, 2, 3 (counts one tower on the table) 4, 5, 6 (counts second tower on the table) 7, 8, 9 (counts third tower on the table) 10, 11, 12 (counts fourth tower on the table) 13, 14, 15 (counts fifth tower on the table) 16, 17 (counts sixth tower on the table). Okay, the child is done counting all the ones on the cubes. Whereas there's a child who counts 1, 2, 3 (counts on one finger) 4, 5, 6 (counts on another finger) 7, 8, 9 (counts on a third finger) 10, 11, 12 (counts on a fourth finger) 13, 14, 15 (counts on fifth finger) 16, 17 (counts on sixth finger). Is there a difference between these two ways? Or are they just the same? In both instances the child is counting all the ones and making a mistake. It could be a child that counts all the way to 18 and doesn't make a mistake.

Nancy: Hold on, the first one you were doing 1, 2, 3 (counts on her index finger).

R2: They counted the cubes.

R1: They had the towers in front of them.

R2: In the other one, they counted, and they pointed to the fingers and they stopped when they were at six.

Nancy: Yes, that's more advanced.

R2: In what way?

Marsha: *Because he's using his fingers to represent the cubes, instead of using the cubes themselves to count* [italics added].

Excerpt 4.2 shows that at this point in the project, Marsha's own mathematical reasoning allowed her to identify students' ways of reasoning additively. Marsha seems to have moved from observing whether or not students were getting the correct answer, to making observations about the students' counting strategies. She also seems to begin engaging in some discussions about what types of reasoning may be more advanced than others – paying particular attention to the types of units on which children operate. This indicates growth in Marsha's reasoning from when she first entered the project. She is now able to analyze more in her students' mathematical abilities other than just getting the correct answer. Marsha is changing her focus to *how* students are reasoning.

In the excerpt, Marsha was able to identify that one of her students may have moved from a counting-all strategy to a counting-on strategy during the lesson. She states that the student was able to, "move onto a different strategy, not just counting-all." For Marsha, this seemed like a big deal, because the student was able to not just use a more efficient counting strategy but also was to create units of ten to work with. Marsha explains that the student may have begun with the counting-all strategy at first, because it was a safe strategy for her to use. Then, once she realized she could make units of ten from two towers, she could use that to find the total number of cubes. It seems as though this was significant for Marsha, because she believed the student had a change in reasoning as the lesson progressed, which led to a more efficient strategy. There is also some evidence here that Marsha was beginning to notice how her students were working with their units. It was important to Marsha that this student was able to create composite units to operate on. It is the latter realization on her part concerning the child's way of using units that led me to infer her own mathematics at this point includes reasoning with two levels of units – 1s and composite.

Specifically, later in the excerpt, Marsha differentiates between a student who is counting-all using only the single cubes from a student who is counting-all while keeping track of the (composite) units they are tracking on their fingers. Importantly, she identifies the latter type of counting as more advanced. For Marsha, the difference lies in the fact that one student was using concrete items to count, while the other student was using figural items to count. Marsha explains that the student using his fingers is more advanced, “Because he’s using his fingers to represent the cubes, instead of using the cubes themselves to count.” For Marsha, the figural-item-counting is more advanced than concrete-item-counting, and therefore the second student was more advanced. That is, Marsha seems to shift her analyses, from students’ obtaining correct answers to counting strategies they are using, as well as what they are using to aid in the counting (concrete vs. figural). The following buddy-pair (buddy-pair #5) gives us data that shows a shift in Marsha’s reasoning.

Buddy-Pair #5. The fifth buddy-pair occurred in February of the project’s first year. For that session, one of the researchers taught a lesson in Nancy’s room, with Marsha present (observing and interacting with students during group work). The lesson involved students in solving an mDC story problem (8 bags of M&M’s with 7 M&M’s each) and then a related QD story problem (56 M&M’s, with 7 M&M’s placed in each bag). After solving the problems, the students were asked to compare them for similarities and differences. The goal for the lesson was to help the students make a connection between mDC and QD.

The post buddy-pair debrief began with grouping the students based on how they solved the mDC problem. This was done by examining their math journal work and placing their math journals into specific piles based on the reasoning the students showed in the journal. The main categories were students who used the AdPed way of drawing towers and cubes to keep track of the units and students who simply made eight circles where they placed seven items in each in order to count all of the items.

In Excerpt 4.3, Marsha and the researcher are discussing the connection between mDC and QD. Her response indicates an understanding that mDC requires students to keep track of multiple types of units. She also struggles a little bit with the connection between mDC and QD. However, as the conversation progresses, she makes some progress towards this connection.

Excerpt 4.3 - *Lesson Debrief: Marsha's Noticing of Unit Tracking*

09:53 minutes into the debrief

R: Like we talked about last time, QD is taught first as a way for an mDC, like a prompt or support.

Marsha: mDC?

R: Multiplicative double counting. So, like this is a multiplicative double count (points to one of the student journals). She's tracking one bag…

Marsha: *Keeping track of double [both] numbers* [italics added].

R: By 8's, which I think is also really interesting here.

Marsha: She switched it to 7 bags?

R: Mm-hmm.

Marsha: But *how are you supposed to do the division with the lines* [italics added]?

R: So, what QD is supposed to promote, which is why we teach it first … [a bit later, after some exchanges with Nancy that further prompted Marsha's focus on counting composite units] Exactly. So, in a QD problem, they're given a count-by number, which works as a prompt for what they are doing.

Marsha: So, that number is 7 then? *Count by 7's, and they stop when they get to 56* [italics added]?

R: Exactly. Because they know how many go into a group.

Excerpt 4.3 leads me to infer that Marsha is developing an understanding of a linkage between mDC and QD at a participatory level. Critical for this linkage is her assimilation of two levels of units (e.g., the 1s in each unit of 7 and number of such groups, 8, which make up the given total of 56 ones). There appears to be a transformation in her understanding as she soon realizes that, when solving a divisional situation, students could be keeping track of composite units as they count. This is the first instance (in the project data) in which Marsha comes to this realization, which I consider a crucial shift in her own reasoning and thus in her ability to identify student reasoning. Once prompted by the researcher and Nancy, Marsha is showing evidence of her understanding of the mDC-QD connection. Marsha seems to realize that mDC is called mDC due to its requirement of a double counting sequence (e.g., "keep track of double numbers"). Later, after some exchanges with the researcher and Nancy, Marsha identifies how students can keep track of their units (the 7's they are counting) while solving the QD task. In this identification, Marsha seems to operate at two levels of unit coordination.

Accordingly, Marsha's questions here provide evidence that she is constructing the two (linked) schemes, seemingly knowing that multiplicative reasoning goes beyond repeated addition, arrays, or memorizing facts. Knowing this, Marsha is attempting to gain a better understanding by asking questions about the development and requirements of the schemes. When Marsha began this project, her own understanding and reasoning allowed her to determine students' counting strategies and the correctness of their solutions. At this point in the project, Marsha has constructed an understanding of multiplicative reasoning that allows her to look deeper into how students are reasoning in terms of units they focus on and operations they use with those units. Marsha's own reasoning seems to afford this leap in her mathematical pedagogy, because her own construction of the multiplicative reasoning schemes means she can better identify it in her students. In the next buddy-pair (buddy-pair #6) Marsha's multiplicative reasoning continues to transition.

Buddy-Pair #6. The sixth buddy-pair session, which occurred in February of Year 1, provides evidence of a great shift in Marsha's multiplicative reasoning, as well as her ability to analyze her students' reasoning. In this session, one of the researchers taught a lesson in Marsha's classroom, with Nancy present (observing and interacting with students). The lesson was a PGBM lesson in which the students were shown how to figuratively draw the towers and cubes. First the students were asked to find the total number of cubes when they had six towers of four cubes each. They were then shown how to connect that task to multiplication equations (abstract version of the task). The students were then asked to solve a problem figuratively for six towers of seven cubes each.

In Excerpt 4.4, Marsha, Nancy, and the researcher are discussing the different reasoning they observed in the students during the lesson. They are specifically discussing some students who flipped their units when solving the task for six towers of seven cubes each; instead, the students solved the task as seven towers of six cubes each. When first asked what she thought might be happening, Marsha's inclination is to rely on a procedural explanation. However, Nancy's conceptual explanation seems to serve as a prompt for Marsha, who begins to look more at the units the students were operating on. In Excerpt 4.5, this shift is further illustrated. In this excerpt, Marsha seems to begin looking deeper into *how* the students are reasoning on the units required for the task.

Excerpt 4.4 - *Lesson Debrief: Marsha Relies on Procedural Explanation*

06:38 minutes into the debrief

R: What do you think they're thinking as they're going through this?

Marsha: We tried to go over it when we were playing, and as soon as I explained, see, he started doing the number line. But as soon as I explained it to them, they were like, oh yeah okay. But then I don't know.

Nancy: This is the one where…

R: They're counting by 6's instead of 7's.

Marsha: Maybe *because we taught them the commutative property* [italics added]?

R: Did you see how they were counting?

Marsha: No

R: What do you think? (Speaking to Nancy)

Nancy: Well, I would think this is where we started to see kids fall off, not knowing which one represented the tower and which represented the numbers in each. I think they don't fully understand the units yet. I saw some kids around me counting with their fingers or one little girl was drawing circles and counting on.

Excerpt 4.5 - *Lesson Debrief: A Change in Marsha's Analysis of Student Reasoning*

13:45 minutes into the debrief

R: Do you want to talk about what you saw? Because we can talk about counting now.

Marsha: Luke (pseudonym) will use his fingers. *He can set it up, and he can tell me how many towers and cubes, but once it comes to the totaling, that's where he gets confused, and he still need to count-on* [italics added].

R: Are they able, when they start at one, having to go like 1, 2, 3, 4? Or do they know to start at 4?

Marsha: He knew it was 4. Well, like the first one, he knew [it] was four. *Then he can label (points to the tower label), label (points to the cubes in each tower). Then here (points to the total label) he had to go back to 4 and count up to 8* [italics added].

Nancy: (Speaking to Marsha) So if you do anything where they don't represent it figuratively, like in just PGBM, or if you ever give them an equation, where they just have to do it, when they use their fingers, do you feel like, how do you see them keeping track of it? Like with just using their fingers. How are they counting?

Marsha: Some of them are still doubling. Some of them are skip-counting. Some of them still have to count out. If we do it together, they can do like that picture on their fingers. They go 4, 8, 12, 16 [shows counting each of the towers on the fingers of her left hand].

R: So, they would have one hand for towers, and one hand for cubes?

Marsha: No. *Just one tower (points to index finger), 4 cubes. Second tower (points to middle finger), 8 cubes* [italics added].

R: What if they didn't know the 8?

Marsha: They're pretty good at their 4's though. But most of them can do it pretty quickly. The other one, *I think I see them doing this (counting four fingers) on their other hand* [italics added].

24:25 minutes into the debrief

R: (Discussing the next steps for the whole class) I think they need labeling, equations, and transfer to real-world word problems. Why would that make sense for them?

Marsha: *Because they're understanding that the towers are made up of something. The boxes contain something* [italics added].

R: So, what do you think for this group? (Referring to what the next steps should be for the group of students who flip-flopped their units)

Marsha: These two groups are kind of the same.

R: Why?

Marsha: Because they're both flip-flopping. I think they're just picking numbers and going through the process. *They're getting lost keeping track of the total. So, they're misinterpreting what's towers and cubes* [italics added]. Then they're miscounting.

Excerpts 4.4 and 4.5 lead me to infer that Marsha has constructed the mDC scheme at least at a high participatory stage and is assimilating at least two levels of units as given. This inference is drawing on multiple instances in which Marsha discusses how her students are keeping track of the units with their fingers, and she models how they did it (e.g., counting four 1s simultaneously with the fingers that stand for the towers). Marsha seems to understand composite units as units that also contain single units. She exhibits this when she explains how her students are, "understanding that the towers are made up of something. The boxes contain something."

This is also the first time (in the observed project data) Marsha talks about multiplicative reasoning as the tracking of units; she explicitly states how students should be keeping track of their units. This is evident in several places throughout the excerpts; for example, she explains how one of the students was having trouble with the task, because "He can set it up, and he can tell me how many towers and cubes, but once it comes to the totaling, that's where he gets confused." Marsha is analyzing this student's reasoning as assimilating the single units and composite units but seems to struggle with the third unit (the total of 1s in the compilation of composite units). In order for her to make this distinction in her students' reasoning, she would need to have constructed this reasoning for herself (the tracking of all three units simultaneously). In this sense, Marsha seems to have moved from examining students' additive counting methods (or fact memorization) to determining units on which students are operating. This seems like an important leap in her own reasoning, as well as her ability to recognize her students' reasoning.

It is interesting that, initially, Marsha used a procedural explanation of why the students may have flip-flopped their units (Excerpt 4.4), explaining that it was due to them learning about the commutative property. However, once prompted by Nancy's explanations, Marsha jumps right into analyzing the units students were working with and how they were operating on them (Excerpt 4.5). Critically, she identifies when students are doubling, counting-on, or keeping track of their single units and composite units on their fingers – a shift in her recognition of how her students are operating. She is also modeling how the students may be keeping track of those units on their fingers when she uses her own fingers to show that they may be counting "one tower (points to her index finger), 4 cubes. Second tower (points to her middle finger), 8 cubes." This shows that Marsha has both constructed this understanding for herself and that it gave her the ability to recognize it in her students. She is now able to look beyond the students' additive counting strategies while looking at their units and whether or not they can keep track of those units.

In order for Marsha to be able to notice two levels of units in the students' reasoning, she would have to also be assimilating at least two levels of units herself. This is apparent when she models keeping track of the composite units and single units on her hand. At this point, I infer that Marsha is assimilating at least two levels of units, but I do not have evidence as to how she may be assimilating a third unit – in activity or as given. At the end of year, Marsha participated in her first workshop (workshop #2) that gave us evidence into her fractional reasoning.

Workshop #2. The second workshop of the project occurred at the end of Year 1. This workshop was focused on helping the teacher participants construct the beginning fractional reasoning schemes. So, the teachers played the French Fry Game (Tzur & Hunt, 2015) and its expansion game – the Envelope Game. In this expansion game, the participants are shown a bar that represents the whole. They are then shown unmarked fractional unit pieces in comparison to the whole bar. They then guess what the fractional piece is by attempting to repeat the piece until they have rebuilt the whole. Importantly, this game involves operating multiplicatively on two levels of units – the whole and fractional units of which it could be composed (via iteration).

Marsha's procedural understanding of fractions and her part-of-whole reasoning seemed to affect her ability to engage with the different fractional reasoning tasks presented during the workshop. Up to that time, it looked as though Marsha was assimilating at least two levels of units in her multiplicative reasoning, but when she is presented with new fractional reasoning, she seems to have lost the second level of units as given. This will be illustrated in the following excerpts, as well as her struggles to think of fractions as a multiplicative relationship between the whole and the fractional pieces.

It is important to note that I do not present this evidence to suggest there was anything lacking in Marsha's reasoning, but to show what happened when she was initially presented with this new type of fractional reasoning. Specifically, the focus is on how two levels of units coordination in whole numbers is no guarantee for the same level in fractions. The evidence in this workshop also provides a baseline for comparing her reasoning later in the project. There will be quite a bit of growth in her fractional reasoning between this workshop and the end of the project.

In this first excerpt (Excerpt 4.6), Marsha becomes aware of her own fractional reasoning constraints. Prior to playing the Envelope Game, the teachers had been shown the picture in Figure 4.1 (Tzur, 2019d) and told/asked the following:

- Sticks A and B are exactly the same size.
- The yellow part is exactly the size of the part above it in A.
- What fraction is the yellow part of A?
- What fraction is the yellow part of B?

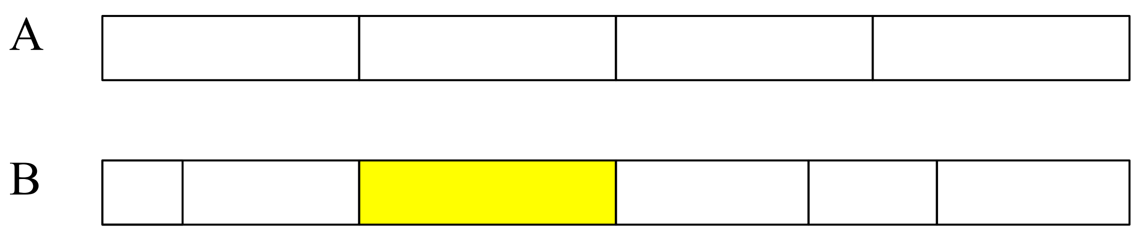

Figure 4.1 - Fractional Reasoning Task

While discussing this task, the teachers were informed that some adults do not "see" a fraction if the fractional piece is not actually inside the whole. For example, some adults are not able to determine what fraction is the yellow part of stick A, because it is not part of stick A. This is due to a part-of-whole reasoning where the part must be in the whole for it to be a fraction of the whole. From there, the teachers were shown a bar that represented the whole, and a fractional piece that represented 1/2 of the whole (Figure 4.2). The 1/2 piece was unmarked, and the teachers were asked to predict what fraction that piece was.

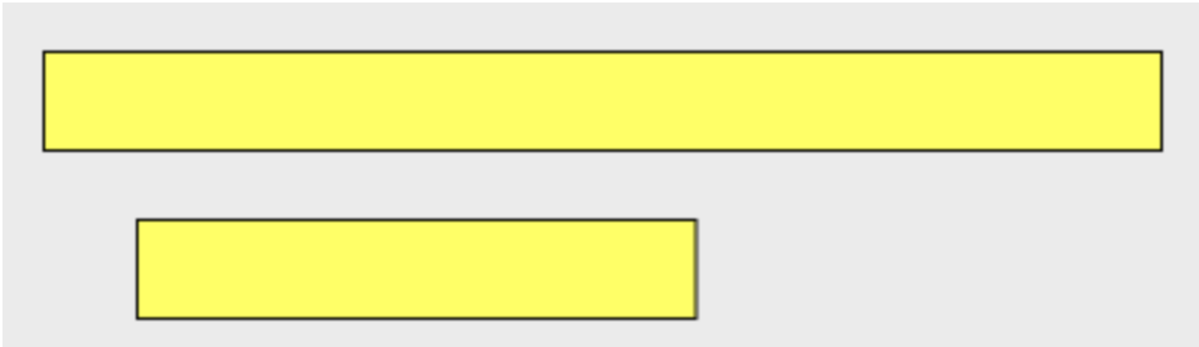

Figure 4.2 - Whole Bar and 1/2 Fractional Piece

Excerpt 4.6 – *Workshop #2: Marsha Realizes Her Fractional Reasoning Constraints*

24:50 minutes into the workshop

R1: The first piece is this. Predict what you think this piece is.

Marsha: (Looks confused) What do you mean?

R1: Is it like a fourth, an eighth?

Marsha: (Still looks confused)

R2: What's the fraction of the white bar to the yellow bar?

Marsha: *I'm one of those adults you were talking about (referring to the adults who can't "see" a fraction if it is not actually within the whole). I still don't know what that means* [italics added].

The evidence in Excerpt 4.6 leads me to infer that Marsha's reasoning is yet to include fractions as multiplicative relations, especially since prompting did not work for her. Marsha seems to realize that she also struggles to "see" a fraction if it is not part of the actual whole. This is evident when she refers to herself as one of those adults that cannot see the fraction when it is disembedded from the whole. Therefore, she does not seem engaged much in the rest of the Envelope Game, because the fractional pieces are never part of the whole, as seen in Figure 4.2. This seems like a constraint in her reasoning being brought on by her current part-of-whole reasoning. Furthermore, neither the task presented in Figure 4.1 nor the researcher's question (is it one-fourth? One-eighth?) work as a prompt for Marsha to engage with the Envelope Game.

In Excerpt 4.7, the teachers were working on partitive fractions, and the task has the teachers build non-unit fractions using connecting cubes. They are presented with a cube and are told it represents 1/3 of a whole (the whole thought of is a pizza and the cubes stand for its slices). They then add another cube and asked what fraction of the whole they have now. After working through a few scenarios like this, the teachers are asked why the denominators are not added together. For example, when you have two of the 1/3 pieces, why is it 2/3 and not 2/6? Excerpt 4.7 indicates that, when asked this question, Marsha seems to shift her focus from the whole to the individual cubes.

Excerpt 4.7 – *Workshop #2: Marsha Loses the Whole*

54:20 minutes into the workshop

R: Why when we are adding fractions with like-denominators, do we not add the denominators?

Marsha: *(Takes a unifix cube and holds it up) We were saying that you are still adding part of this whole. You're not adding this pizza (points at one cube) and this pizza (points at another cube). You're still adding this whole thing (makes a circling motion above one of the cubes)* [italics added]. There are still 8 slices, whether you're adding or subtracting, you're still adding in that (points to the cube again).

I infer that in this task, Marsha was assimilating one level of unit as given, and one level in activity. Marsha assimilated two levels of units as given when working through multiplicative reasoning tasks with whole numbers. However, when solving a fractional reasoning task, she seems to shift back to a focus on one unit at a time. Importantly, the teachers had already built the whole out of the cubes, but Marsha had taken her whole apart into single cubes. Once this happened, she seemed to focus on the individual cubes as wholes in their own right. Instead of referring to an imaginary whole and the cubes as fractional pieces of it, she used the single cubes as a proxy for the whole. This was an unexpected finding since Marsha had been assimilating at least two levels of units in her multiplicative reasoning. At the end of Year 1, Marsha's first AOP session occurred, and it gave us more evidence into her progress towards constructing multiplicative reasoning with whole numbers.

AOP #1. Year 1 of the project concluded with Marsha's first AOP data set; she did not have an AOP session earlier due to her late entrance into the project. For this session, Marsha taught a PGBM lesson with a focus on helping the students move from the concrete manipulatives (connecting cubes) to drawing figural representations of the towers and cubes. This AOP session is the first instance in the project in which Marsha taught her own lesson and was not receiving coaching. Due to this, some folding back to a reliance on procedural understanding and a push for her students to memorize their multiplication facts is observed. There is a moment in the interview in which she focuses on one student's inability to keep track of the correct total as they were counting their units. It is not clear whether this is due to her own understanding of the importance of keeping track of units, or if it is because her goal for the lesson was for students to find the correct totals. However, before this, Marsha had been exhibiting multiplicative reasoning and the assimilation of at least two levels of units with tracking of those units.

In Excerpt 4.8, Marsha is teaching her lesson to the students, engaging them in using drawings to solve the task, "" [FILL IN the info here]. She is trying to help the students differentiate between the towers and the cubes in the problems. In both questions she asks the students, she prompts them by explaining how the towers hold the cubes, which indicates her reasoning with both types of units.

Excerpt 4.8 – *AOP #1: Marsha Teaches a PGBM Lesson*

04:20 minutes into the lesson

Marsha: In this problem, what represents the towers? If we're pretending to play Please Go Bring Me, what's the towers and which one is the cubes? Which one holds the units? Which one holds the cubes?

08:22 minutes into the lesson

Marsha: Now I'm going to ask you again, in our problem, what is our towers? What is holding our units? Like our towers hold all of our cubes together. Our bags held our M&M's. What are our towers in this one? What object is doing the holding?

I infer that Marsha has an understanding of the units involved in multiplicative reasoning and can assimilate both units herself as given. However, she may struggle when it comes to promoting that understanding in her students. Throughout the lesson, Marsha asks her students questions like, "Which one holds the units? Which one hold the cubes?" when attempting to get her students to assimilate the composite units. This suggests that a teacher may need more than being at the anticipatory level in order to fully promote the concept, while beginning to teach the concept with some struggles through the explanations.

It is not clear why Marsha keeps referring to the towers as the container for the "units." Initially this led me to believe that perhaps she was assimilating only the single units, and was not assimilating the towers as units themselves, but that would contradict previous evidence collected in Year 1, where Marsha was assimilating at least those two units as given. My best inference to explain this, is that Marsha does assimilate the two units herself, but has not yet figured out how to explain this to her students. She seems to struggle with transferring her own understanding of the units to her teaching.

In Excerpt 4.9, Marsha and the researcher are analyzing some of the students' work during the post-lesson interview. Marsha first focuses on whether or not the students had the correct answers. She then looks at how one of the students was keeping track of his units, specifically the total number of cubes. She attempts to explain why it is important for the students to draw each tower fully for unit tracking.

Excerpt 4.9 – *AOP #1: Marsha Analyzing Student Work*

0:1:00 minute into the debrief

R: What does correct mean for you?

Marsha: Meaning they did the correct number, they did 6 boxes, not 7 boxes. They got the correct total. Then these ones (points to another stack of student work) are outliers. These ones did it backwards. They had the correct total, but they had the wrong number of towers. He got the wrong answer (looking at one student's work).

R: What do you think was going on with his thinking and his understanding?

Marsha: Even when he did it here, he got the wrong total. *He's not keeping track of his totals properly* [italics added]. Or he doesn't know how to.

34:42 minute into the debrief

R: Why is it important for you that they do it that way? The [way] that you showed them, and not doing 1, 2, 3, 4, 5, 6 (drawing all the towers first), then write 7, 7, 7, 7, 7, 7 (writing the number of cubes in each tower), then write all the totals.

Marsha: Because it's not following the organization of controlling your thinking. When we play Please Go Bring Me, they don't just go grab 42 cubes and start going like that.

> They go very systematically, 1 tower, 6 cubes, here's my total. Go back, *here's tower 2, 6 cubes, now I have 12* [italics added]. To correlate with what they've been playing with all year. Again, slow them down, because that's where I feel like they get messy.

Excerpt 4.9 further supports my inference from Excerpt 4.8 that Marsha understands multiplicative reasoning and can assimilate two units as given in order to track those units when operating multiplicatively. This is especially evident when she explains that when students are playing PGBM they should be keeping track of their units, such as, "here's tower 2, 6 cubes, now I have 12." In this, Marsha is explicitly describing the units students should be keeping track of (composite units made up of single units, which together create a third unit). Marsha's multiplicative reasoning allows her to reason about the task herself, as well as to be clear of what she is looking for in her students' reasoning.

Summary of Year 1

In Year 1, Marsha is beginning to work through the mDC scheme and seems to end the year with two levels of units coordination. When Marsha entered into the project, she used additive reasoning (repeated addition) for multiplication tasks and looked for the same reasoning in her students (Excerpt 4.1). Throughout the year, she is working to construct an understanding of multiplicative reasoning as the distribution of one unit over another unit, while simultaneously tracking two types of units. Her own growth in understanding seems to also afford her shift from looking for correct student answers (Excerpt 4.1) to a focus on how her students are reasoning (Excerpt 4.9). Her analyses typically focus on the counting sequences her students are using as they are working on tasks. She is able to identify if students are counting-all, counting-on, or doubling (Excerpt 4.2). While the workshops in Year 1 included work on fractional reasoning, Marsha still seemed to use part-of-whole reasoning (Workshop #2).

When Marsha's levels of units coordination are examined, she seems to move from assimilating one unit as given and another in activity in the beginning of the year, to assimilating at least two units as given by the end of the year (Excerpt 4.5). Her assimilation of the units is linked with her construction of the mDC scheme, and as she assimilates more units as given, her ability to reason within the mDC scheme grows (Excerpt 4.5). As this change in her levels of units coordination occurs, there is a shift in her understanding of how students should be reasoning. Next, I turn to the growth in Marsha's reasoning in Year 2.

Growth in Marsha's Reasoning: Year 2

In Year 2, there is a major shift in Marsha's own reasoning, as well as in her ability to analyze her students' reasoning. She seems to construct the mDC scheme at a solid (anticipatory) stage and is clearly assimilating at least two levels of units as given. This growth in her own reasoning leads to her ability to identify the units her students are assimilating and how they are operating on those units. I begin the Year 2 analysis with workshop #3, which focused on higher-level ways of multiplicative reasoning.

Workshop #3. Year 2 began with a grade-level workshop focused on using the Unit Differentiation and Selection scheme to solve tasks involving the distributive property of multiplication over addition. This workshop was significant, because it shows an actual shift in Marsha's reasoning within the UDS scheme. At first, Marsha struggles with the tasks and assimilating the composite units, but with prompting from the other teachers and the researcher she begins to construct the scheme.

In Excerpt 4.10, the teachers have been asked to build three towers of six cubes each, but to build them so that four of the cubes were one color, and the other two cubes were another color. They were then asked how many cubes they had altogether. Initially, Marsha said 18, which she got by counting 6, 12, 18. She is then asked to find the answer another way. Nancy asks if she can break the towers apart. This seems to prompt Marsha into physically breaking her towers into 9 towers of 2 cubes each.

Excerpt 4.10 – *Workshop # 3: Marsha Solving Task 1*

32::00 minutes into the workshop

R1: So, what would you do?

Marsha: Make 9 towers of 2 cubes.

R1: How?

Marsha: *I broke apart my towers of 6 into two's, so they were even and counted by two's* [italics added].

R1: And how did you end up with 9?

Marsha: Because that's how many I can make without being uneven or having any extra.

R2: What prompted you to do that? What was your goal?

Marsha: *I need to see the towers, and I started snapping them* [italics added] and then realized that 2 times 9 is 18 as well.

R2: Were the two white ones on top of the four any hint for you? Any reason it was helpful?

Marsha: I wanted the colors to match.

Excerpt 4.10 leads me to infer that, initially, Marsha is assimilating the composite units (towers) as composite units of size six, but not as units composed of four 1s of one color plus two 1s of another color. She also seems to assimilate at least one unit in activity since she needed to physically split her three towers of six into nine towers of two; therefore, finding the total number of individual units in activity. It is interesting that Marsha needed her towers to be of equal size, which seemed to have constrained her assimilation of the six single units and kept her from "seeing" this as units composed of two sub-units (4s and 2s). Importantly, at that point she seems to assimilate this task as an mDC task (ways to show how many cubes are there in total, e.g., nine equal composite units with units of two distributed over them), rather than a UDS task that would allow her to utilize the distributive property (i.e., three composite units made of four 1s plus two 1s each).

After Marsha explains her solution, Nancy gives an explanation that seems to cause a perturbation, and then a prompt, for Marsha. Nancy explains that she split her towers into three towers of four and three towers of two, then added the products of each together (3 x 4 = 12, 3 x 2 = 6, 12 + 6 = 18). This prompt seems to help Marsha in the following task, where they were asked to build five towers of thirteen cubes each, using two different colors for the towers. As seen in Excerpt 4.11, Marsha initially asks if two of the cubes in each tower were supposed to be the same color, but when working with her partner (here – a teacher other than Nancy), they build five towers of ten black cubes and five towers of three white cubes.

Excerpt 4.11 – *Workshop #3: Marsha Solving Task 2*

43:07 minutes into the workshop

R: Using two different colors again, build 5 towers of 13.

Marsha: So, *two cubes have to be the same* color [italics added]?

R: It doesn't have to be two. It can be whatever you want.

Marsha: (Builds five towers of ten black cubes and five towers of three white cubes).

R: How many cubes do you have altogether?

Marsha: 65

R: How do you get your answer? Marsha, do you want to start?

Marsha: *I counted by 10's. 10, 20, 30, 40, 50. Then I did 3 times 5, which is 15* [italics added]. Then I added 15 to 50.

Excerpt 4.11 leads me to infer that Marsha's thinking was prompted by what Nancy did in the first task (3 x 4 = 12, 3 x 2 = 6, 12 + 6 = 18). This time, Marsha decomposes her composite units of thirteen into separate composite units of ten and three, in order to utilize the distributive property to find her answer. Her question (about using two cubes of one color) indicated that she assimilated Nancy's prior solution. Yet, she already seemed to have a plan in which a different number would be used, which seems to have led her to actually decompose the composite unit as ten and three. My inference is that it came through her discussion with her partner as they were building their towers. I could not hear what ensued in that discussion, so I am not sure who suggested that decomposition first. To an extent, Marsha may still assimilate the decomposition of her units in activity, since she physically made her two sets of composite units separately in order to operate on them sequentially (first operating on the composite units of 10, then operating on the composite units of 3, and adding up the products). That said, it is clear from her solution that her assimilation of the task included three levels of units coordination, with 65 made of five units of 13 – each of which she conceived of as made of 10 and 3.

For the third task, the teachers were asked to build ten towers of thirteen cubes each, with ten cubes being one color and three cubes being another color. Marsha and her partner added five more towers of ten black cubes and five more towers of three white cubes to the towers they had already built for task two, which further supports the above claim about coordinating three levels of units. Excerpt 4.12 provides her explanation of how she solved the task.

Excerpt 4.12 – *Workshop #3: Marsha Solving Task Three*

1:11:32 minutes into the workshop

R: How did you find your total?

Marsha: I did 50 and 50, which is 100. Then 15 and 15 is 30. So, I got 130.

R: Did you have a different way?

Marsha: No (Thinks silently for a moment). I guess you could do 13 times 10.

R: And the 50 you knew from what you had done before?

Marsha: (Shakes head yes).

R: Could you demonstrate that using multiplicative double counting?

Marsha: (Does not answer).

R: Do you see how what you did maybe was not expressed in a way that uses multiplicative double counting?

Marsha: So, 10 times 10 and 3 times 10, (Points to the cubes as she is talking about them).

R: So, how would that be multiplicative double counting?

Marsha: I don't know. I don't know what that means, because you're not doing it (puts up her fingers like she is counting them).

R: Could you use your fingers to demonstrate it?

Marsha: Then I'm running out of fingers, unless I count by 13's.

R: With the 10 times 10, and the 10 times 3.

Marsha: *Oh, like separate? Yeah. 10, 20, 30, 40, 50, 60, 70, 80, 90, 100. And then 3, 6, 9. 12. 15, 18, 21, 24, 27, 30 (Counts the composite units on her fingers)* [italics added].

Excerpt 4.12 leads me to infer that once the numbers got above what she could count on her hand (13), Marsha experienced a perturbation and was not sure how to link her solution of decomposing with double counting. Her spontaneous reasoning was to double her products from task two, which is evident when she first explains that she, “did 50 and 50, which is 100. Then 15 and 15 is 30.” She knew that her compilation of composite units had doubled, so she doubled her products from the last task. When asked how to show that multiplicatively, Marsha struggles and says that she would run out of fingers. She did not assimilate the activity sequence of using the distributive property for the larger numbers in this task (10 x 10 = 100, 10 x 3 = 30, 100 + 30 = 130). Once prompted by the researcher, she was able to model how to use a double counting sequence to solve the two sets of composite units sequentially, explaining that you could do that separately as, “10, 20, 30, 40, 50, 60, 70, 80, 90, 100. And then 3, 6, 9, 12, 15, 18, 21, 24, 27, 30 (counting the composite units on her fingers).” She was able to simultaneously count the units within each individual compilation, while tracking her total number of cubes. In doing so with the researcher’s prompt, Marsha indicated at least a participatory stage of the operation on a given compilation of composite units (doubling it) while assimilating three levels of units as given (the total of 1s in just five 13s, the total of 1s when it is doubled, and the original, decomposed units of 10 and 3 in each).

For the final task, the teachers were asked to make twelve towers of thirteen cubes each, with the same configuration of cubes as in task three. Excerpt 4.13 provides Marsha’s explanation of how she solved this task. The discussion from task three seems to have worked as a prompt, because in this last excerpt, Marsha has decomposed the composite units and used the distributive property to find the total.

Excerpt 4.13 – *Workshop #3: Marsha Solving Task Four*

1:33:14 minutes into the workshop

R: How did you find your answer?

Marsha: *I did 10 times 6 is 60, and 10 times 6 is 60 again. Then I did 3 times 12, which is 36. I added the 120 to the 36* [italics added].

By the end of the workshop, Marsha has assimilated the goal of the tasks as the decomposition of the composite units in order to utilize the distributive property for finding the total number of cubes. In so doing, she seems to assimilate three levels of units as given. This is evident when she explains how she solved the final task by first multiplying 10 times 6, then again multiplying 10 times 6, then multiplying 3 times 12, and finally adding the products. By looking at the evidence across all four tasks, I infer that Marsha was originally pre-UDS and moved into the participatory level of the scheme by the end of the workshop – likely by expounding on the level of units she has been coordinating. In task one, Marsha was struggling to assimilate the composite units as something that could be decomposed for use with the distributive property. After some prompting, she decomposes her composite units, but has to do so in activity and does not seem to assimilate the prompt of the different colored cubes as a pointer to such reasoning. In task two, she is able to decompose the composite units based on their colors and solves the task using the distributive property. In task three, she experiences a significant perturbation, when the numbers she has to operate on are above ten (i.e., more than the number of fingers on both hands). She is able to decompose the composite units but isn not sure how to operate on them multiplicatively. Instead, she uses additive reasoning to find her total. Finally, by the last task, Marsha decomposes her composite units and uses the distributive property to operate multiplicatively on her units. This progression shows a critical shift in

Marsha's reasoning (assisted by prompting) and likely through Type II Ref*AER. By reflecting on her results through the four tasks, Marsha kept adjusting her activity sequences until she had constructed the scheme at least to the high participatory level. Not long after Workshop #3, the teachers participated in workshop #4 that provides more evidence into Marsha's fractional reasoning.

Workshop #4. This workshop occurred about a month after Workshop 3 and focused on fractional reasoning. This workshop was an incredibly powerful session for Marsha's transition in terms of her fractional reasoning. Specifically, there is a shift from part-of-whole reasoning to understanding fractions as a multiplicative relationship between the fractional unit and the whole. Throughout the workshop, Marsha experiences quite a bit of frustration and many perturbations, until she constructs an understanding of fractions as a multiplicative relationship, and thus the equi-partitioning scheme at least at the participatory stage.

As the workshop began, the researcher asked the teachers what challenges they had faced teaching fractions since the second Summer Institute (SI2) five months prior. Marsha looked through her notes from SI2 and explains to the researchers that she does not understand fractions conceptually, just like her students. She is struggling to remember and understand what she had learned from the SI. In Excerpt 4.14, Marsha's struggles with this and her realization that she does not remember the content. Throughout this discussion, Marsha is looking for the key phrase that she seems to believe needs to be said in order to explain fractions. This phrase is one that was said multiple times to the teachers throughout the SI – "The whole is x times as much as the fractional piece." Her dependence on the phrase (and notes) indicates she does not yet have the fractional reasoning required to understand unit fractions as multiplicative relations.

Excerpt 4.14 – *Workshop # 4: Marsha Trying to Remember Her Learning from SI2*

21::00 minutes into the workshop

R: What are some of the challenges you've had teaching fraction to students?

Marsha: (Looks through her notes from SI2)

R: Are you looking at the Summer Institute notes?

Marsha: (Shakes head yes. Reads from her notes) The number of times…the smaller the number, the larger the piece (makes a face as though she doesn't understand what she just read).

Nancy I think, just really understand what's actually happening. You know, this summer my understanding developed, and I feel now I'm like, what are fractions again? When we go there, that will be helpful.

Marsha: Just like multiplying and dividing. Sure, they can rattle off facts, but do they know what they are actually talking about? Sure, they can split the pie into 8 pieces and tell you what's 2/8, but do they actually know what it means? Apparently, I don't know what it means after this summer.

R: (About to give the teachers a question on fractions) Don't look at your notes (referring to Marsha's notes from SI2).

Marsha: I'm looking for that key phrase that he (referring to the project PI) constantly said.

R: Don't worry about the key phrase. Just what you think right now. What is a fraction? Can you give an example with ¼? Write it down. We are going to share.

Marsha: Are you going to tell him (again referring to the project PI) that I said it was part of a whole?

R: No.

Nancy: I'm just really trying to think of how to say this.

Marsha: If it can't be part of a whole, what else can it be? Part of a part? *It's something like how many times it fits into the whole.* [italics added]

Excerpt 4.14 illustrates Marsha's fractional reasoning at this point, which is part-of-whole reasoning and is reliant on terminology learned in SI2. Marsha's current part-of-whole reasoning has caused her to assimilate the fractional learning from SI2 as being the key phrase learned from the researchers. She says that "It's something like how many times it fits into the whole," but she does not seem to understand what that phrase means. She senses it is important to fractional reasoning but has not yet constructed the reasoning behind it (the multiplicative relation between the whole and the fractional unit). She further questions what fractions are if they cannot be part of a whole. After asking that question, she assimilates that to mean it is perhaps, "Part of a part." This leads to Marsha focusing on getting it "wrong" by not saying what she believes the researchers want to hear, rather than an understanding of what the phrase means.

From here, the teachers each explain what a fraction is, using the fraction 1/4 in their explanations. In Excerpt 4.15, Marsha provides an explanation that has a mix of part-of-whole reasoning and the terminology the teachers learned in SI2.

Excerpt 4.15 – *Workshop #4: Marsha's Explanation of Fractions*

26::00 minutes into the workshop

R: Are we ready to share?

Marsha: I have how many times, and then I left it blank.

R: Why don't you start first then?

Marsha: So, *I have part of a whole, and then how many times the piece can fit into the whole, or something like that* [italics added].

R: And what was the example you had with fourths?

Marsha: A candy bar.

R: Did you draw something?

Marsha: Yeah.

R: Can you show me?

Marsha: (Holds up her drawing) *One-fourth is shaded* [italics added].

Marsha seems to have first assimilated the task into her part-of-whole reasoning, and then switched to the phrasing she had heard throughout SI2. Marsha's spontaneous explanation of fractions being "part-of-a-whole," and her explanation of the drawing as "one-fourth is shaded," illustrate her current reasoning of fractions being x pieces shaded out of x equal pieces. I believe she brings in the other terminology – "how many times the piece can fit into the whole, or something like that," – because she believes this is what the researchers are "looking" for in her explanation. She has pulled that terminology from her notes and adds it to her explanation, but does not seem to understand what it means, which I explained in Excerpt 4.14. If she had understood the terminology at this point, then her drawing would have represented it, rather than the part-of-whole drawing she created. For example, she might have drawn a disembedded 1/4 fractional piece and explained that it was 1/4 because the whole was four times larger than that piece.

Next, the researcher gave teachers a fraction task to discuss (Figure 4.3). After being shown the fractional triangle task, they were asked the following questions: 1) What fraction is the yellow part of Rectangle A? What fraction is the blue part of Rectangle B? Which fraction is larger, 1/6 or 1/4? What fraction is the yellow part of Rectangle B? In Excerpt 4.16, Marsha gives her answers and explains her reasoning to the rest of the group.

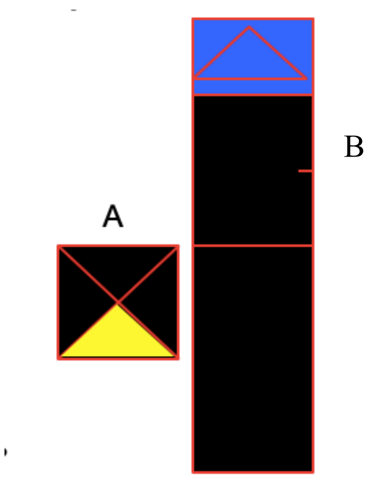

Figure 4.3 - Fractional Triangle Task

Excerpt 4.16 – *Workshop #4: Marsha's Explanation of the Fractional Triangle Task*

34:00 minutes into the workshop

R: Want to start with number one? What fraction is the yellow part of Rectangle A?

Nancy: I got 1/4.

R: Why is that 1/4?

Marsha: *Why is it 1/4? Because there's four pieces, and one of the four pieces is yellow* [italics added]. (Looks at her notes from SI2) And if you duplicate it four more times, then what you wrote in your notebook, that's four times (shakes her head). I don't know.

R: So, Marsha you're saying if I duplicate this yellow piece…

Marsha: Oh no, actually, no. It's not 1/4. I change my mind. Those triangles aren't even on the sides. It's a rectangle. You can't evenly break up a rectangle into triangles like that.

R: If it was a square, would that change your mind?

Marsha: Yes.

R: Okay, for the purpose of this activity, let's pretend it's a square.

Marsha: Then it's still 1/4.

R: Why?

Marsha: *Because then you can repeat that piece four times to get the whole* [italics added].

R: So, what would the repeat look like for you?

Marsha: *1/4, 1/4, 1/4, 1/4* [italics added].

R: Duplicate meaning, I take that yellow one, I do it again, I do it again, I do it again, and there would be a total of four of them?

Marsha: (Nods her head for yes) Each piece is 1/4.

40:00 minutes into the workshop

R: What about question three?

Kelly: Fourth?

R: Why?

Marsha: (Reading from her notes) The smaller the denominator, the larger the piece.

The discussion in Excerpt 4.16 illustrates Marsha's reliance on her notes and the constraints of her part-of-whole reasoning. Marsha's initial explanation is that the fraction 1/4 is one piece out of four (equal) pieces shaded. After looking through her notes, she brings in the other terminology and says, "If you duplicate it four more times, then what you wrote in your notebook, that's four times," but then becomes confused and does not finish the statement. Again, she understands that the terminology is important, but she seems to have yet fully assimilate the meaning of the terminology. On the other hand, when prompted by the researcher to explain what it means for the piece to be repeated, she explains it as, "1/4, 1/4, 1/4, 1/4," suggesting that she is beginning to grapple with the meaning of the phrase. When asked why 1/4 is larger than 1/6, Marsha reads from her notes once again and reads, “The smaller the denominator, the larger the piece.” This leads me to infer that she does not yet understand the inverse relationship between the denominators and the size of the fractional piece, because she is not yet able to explain this without her notes.

At this point in the workshop, the researcher shifts gears into playing the French fry game. This is where I believe Marsha begins to construct the equi-partitioning scheme through perturbations, Ref*AER, and prompting from the researchers and her peers. The teachers are asked to share their French fry strip among three people, but the other teachers hold off while Marsha works through the activity. The other teachers have already done this activity several times and have taught it to their students. They understand that it is the first time Marsha is doing the activity, so they decide to watch her actions and help her through the task, as shown in Excerpt 4.17.

Excerpt 4.17 – *Workshop #4: Marsha Sharing a French Fry Among Three People*

44:48 minutes into the workshop

R1: I would like you to share your French fry among three people.

Nancy: (Guesses her three pieces and marks them on her French fry).

Kelly: How do you know those are equal?

Marsha: They look equal to the best of my knowledge.

Nancy: I think this piece is bigger (points to one of the pieces Marsha drew).

Marsha: Maybe.

Nancy: How could you prove me wrong?

Marsha: By cutting the French fry.

R2: No cutting. Could you use the white piece to help you?

Marsha: (Uses the white strip to check the size of her pieces).

R1: So, what happened with your French fry?

Marsha: These two are even, but this one is bigger by a little bit.

R1: So, could you adjust to make them equal? Because I want to make sure we all get the same size piece of the French fry.

R2: Specifically, do you think you need to make this (points to the first piece) longer or shorter?

Marsha: Longer.

R2: How much longer?

Marsha: A little bit longer.

R2: Try it.

Marsha: (Uses her white strip to mark a new piece and uses that piece to iterate her equal shares on the French fry strip. She gets to the end, and the piece was too long). Are you serious? (Tries again with a new piece) Now I'm this much short (Points to the overage).

Kelly: So, now do you need to make this piece longer or shorter?

Marsha: I need to make my white one shorter.

R1: How much shorter do you think you need to make it?

Marsha: (Makes a circular motion over the left-over piece) This much.

R2: Try it.

Marsha: (Iterates a new piece) It's too short now.

R2: So, do you need to make it longer or shorter?

Marsha: Longer? Because I keep going shorter.

R2: Okay, so you started with this mark here, and you said you needed to make it longer. So, you made it longer. Then you went here, and you said, I need to make it shorter. This is too much. It didn't go right? So, you made it here. And here you said you needed to make it longer. Knowing that it needed to be longer (points at first attempt), it needed to be longer (points at second attempt), it needed to be shorter (points at third attempt), does that help you maybe determine where you can put your next mark?

Marsha: Does it have anything to do with what I'm doing here at the end? Every time I measure here, how much shorter I am, and this seems to be what's throwing me off.

R2: If you want to think of it in terms of this left-over piece rather than what I asked you, then you can.

Marsha: This piece needs to be bigger.

R2: Do you know how much?

Marsha: Somewhere right here (makes a mark on the white strip).

R2: Is that just a guess for you?

Marsha: *I'm thinking, if I do 1, 2, 3, then it needs to be here. I'm measuring how much stuff is left over (Is partitioning the left-over piece into three pieces)* [italics added].

R2: And what are you trying to do with this here (Points to the left-over piece).

Marsha: *Split it up equally between…* [italics added]

R2: And why would that work?

Marsha: *Because I have left-over to give away. Because these pieces are all the same, so if I give that (points to a partition of the left-over piece) to that piece (points to the first iteration), and that (points to a partition of the left-over piece) to that piece (points* *to the second iteration), and that (points to a partition of the left-over piece), to that piece (points to the third iteration), then it would be even* [italics added].

R2: Try it.

Marsha: (Tries and is successful).

Excerpt 4.17 gives compelling evidence of a change in Marsha's reasoning about unit fractions. She seems to be constructing the equi-partitioning scheme, along with understanding the direction and magnitude of change when her partitions are too long or too short (Hunt, Tzur & Westenskow, 2016). At first, when asked to share the fry equally among three people, Marsha tries to split her French fry up visually. She does not assimilate the task as including the possible use of the auxiliary white strip. When asked how to check that the pieces are equal, she says she could cut the fry, but she is reminded that the task disallows such an activity. One of the researchers prompts her to use the white strip to help her check the sizing of her pieces. When she does, Marsha finds that the size of one piece is not the same as the other two. The researcher asks her if she needs to make her first piece larger or smaller. Marsha says that it needs to be longer, indicating an anticipation of the direction of change. Yet, when asked how much longer she should make it, Marsha says a little bit longer, suggesting that she cannot yet anticipate the magnitude of change. Marsha adjusts her first piece, translates this to her white strip, and uses the white strip to again attempt to share the French fry among three people. When she gets to the end of the French fry, she finds that her piece was too long this time. She gets a little frustrated with her results. She tries again, and the piece is again too long. When asked how much she should adjust her next piece by, she circles all of the overage and says, "This much." Again, Marsha is providing evidence that she can anticipate the direction of change but not yet the magnitude of change. She tries it again, and this time her piece is too short, because she took the entirety of the overage and subtracted it from her piece. One of the researchers prompts her reflection by orienting her attention to the different sized pieces she has already created. The researcher asks Marsha if looking at these different pieces may help her determine the correct-size piece.

Marsha's perturbation throughout this whole process seems to yield an a-ha moment. She suddenly realizes that what she is doing with the end of the French fry is affecting her results. She was not able to assimilate this before but has a moment of realization. Her Ref*AER Type II (across a few instances of her activity-effect outcomes) seems to lead to a change in her thinking. She begins explaining that she needs to take the left-over piece and partition it into three pieces in order to adjust her original piece. She fully explains that she can take each of the partitioned left-over pieces and give them to the original pieces. She believes that if she does that, she will create three equal shares of her French fry. Marsha explains how she can distribute the partitioned pieces of the left-over piece to the original iterations. She assimilated the left-over piece into a now accommodated scheme. Marsha then does exactly what she stated and has a successful result. It suggested that Marsha began constructing a multiplicative link between the initially estimated piece, the overage, and the amount needed for adjusting the initial estimate. Eventually, such a realization would be part of a three-level units coordination (*three* iterations of the *estimated piece and its extension* fit within the *whole*).

To further promote Marsha's reasoning, the researcher decides to model sharing a French fry among three people, but her piece is too long, causing an overage. She then asks Marsha what should be done with the overage. Marsha asks for the strips and works through this task. Excerpt 4.18 outlines Marsha's activity sequence in this task.

Excerpt 4.18 – *Workshop #4: Marsha Figuring Out What to do with an Overage*

1:04:00 minutes into the workshop

R: What should you do with this overage?

Marsha: Can I have it? (Referring to the French fry strip and the white strip).

R: Can you use a strategy like you used here to help? (Referring to the last activity Marsha did).

Marsha: (Thinks for a moment) I don't know if I can use that (Refers to her strips from the last activity), because this needs to be a lot shorter (Begins making marks on the white strip).

R: Can I ask what you're thinking about as you do that?

Marsha: This is the extra piece, and I was wondering if I could split it up into threes to make a new piece.

R: What would you split into threes?

Marsha: This extra piece (Pointing to the overage).

R: So, you're trying to split this (points to overage piece) into three?

Marsha: Yeah. (Slides the white strip under the second iteration) Because it needs to be, this is how much is there (Points to the iterated piece on the white strip). *It needs to be that much longer or shorter to fit* [italics added].

R: So, if you did split this into three, let's just eyeball it (partitions the overage into three pieces), what would you then do with those three pieces?

Marsha: Make a new shape.

R: Which would be what?

Marsha: (Iterates a new piece. It doesn't work like she was expecting). Anyone else have an idea?

R: So, this didn't work. This was my next piece. Marsha took this overage and cut it into three.

Marsha: *Need to make each of these (Points to the original iterations) one of these (Points to one of the overage partitions) shorter* [italics added].

R: Why would that work?

Marsha: *Because these two pieces (Points to first two iterations) are too long, and this one (Points to the last iteration) needs to be longer. So, you have to take away from these two (Points to first two iterations) to get more on that one (Points to third iteration)* [italics added].

R: Try it.

Marsha: (Creates a new iterable unit) So there's the pieces that I took away (Iterates her new piece and successfully completes the activity).

1:09:56 minutes into the workshop

R: (Referring to the overage and shortage) Which one do you think is harder?

Kelly: (Points to the strips from the overage).

R: Why?

Marsha: I feel like this one (Points to the strips from the shortage) you had something to break up into three. This one (Points to the strips from the overage), you had to find where to get it. Like here (Points to the underage), I knew what I had to give away. This (Points to the overage), I didn't know how much and how to separate it to distribute back to the other pieces.

R: Why would that work?

Marsha: Because these two pieces (Points to first two iterations) are too long, and this one (Points to the last iteration) needs to be longer. So, you have to take away from these two (Points to first two iterations) to get more on that one (Points to third iteration).

Initially, Marsha is not sure what to do with the overage. Then, through Reflection on her prior activity of splitting the shortage piece into three, she comes to realize that she can subtract one of the overage partitions from the original piece and then iterate that piece three times to correctly share the French fry among three people. In doing so, she is further constructing an understanding of equi-partitioning, especially what should be done with an overage or shortage. Marsha asks for the strips, because she seems to need the concrete materials to work through this (in activity). However, as she works through the task, Marsha is able to determine that she needs to split her overage into thirds as well (without and before actually doing it). She takes one of those partitions and subtracts it from each piece. She does not yet understand how to do this in order to get the correct-size piece. She finds that this strategy did not work. She knows that each of the original pieces needs to be shorter by the size of the overage partitions, but she does not yet know that when she does this, she needs to subtract more than one partition from pieces two and three. Through prompting and her own reflection on effects of her activities, Marsha successfully partitions her overage and distributes one partition piece to one original iteration to create a new iterable unit.

Eventually, the conversation turns to the fact that as long as the first iterable piece is the correct size, it does not matter if the other pieces are equal sizes for that original piece to still be one-third. Marsha becomes quite confused by this realization, which seems rooted in her part-of-whole reasoning constraints that are still present in her reasoning. This perturbation is shown in Excerpt 4.19.

Excerpt 4.19 – *Workshop #4: Marsha's Fractional Perturbation*

1:04:00 minutes into the workshop

R: It doesn't matter the size of the parts. I can have this piece (Points to a piece on the French fry), and I can have this piece, which is much bigger (Shows another piece with her fingers that is bigger than the last piece). What size is this piece (Points to the original piece she pointed out) compared to the whole?

Nancy: A third.

R: Because it doesn't matter about this piece (Points to the bigger piece) here. If this next piece is bigger, it doesn't matter. It doesn't have to be equal.

Marsha: Why? Because you're just looking at the one part?

R: You're looking at the relationship between how the one part fits in so many times.

Marsha: Because you just said that you want them (referring to the students) to see that they're equal.

R: You want them to see that it fits in an equal number of times. (Erases the second line on the French fry, so the only thing represented on the French fry is the one-third piece). So, if this line's not here, how big is this (Points to the one-third piece)?

Marsha: It's one-third.

R: Why?

Marsha: Because I already knew that. OH! But why are you avoiding the word equal? How many times it can fit into the whole, then you will have equal parts.

R: For that particular piece, the way I marked it. But, if I'm just thinking about this piece (Points to the original one-third piece) right here, what size is this?

Marsha: One-third.

R: Why?

Marsha: *Because it fits three times in there* [italics added].

Excerpt 4.19 indicates that in the shift from part-of-whole to multiplicative relation Marsha is still at the participatory stage of the equi-partitioning scheme. Marsha is assimilating this situation, with unequal parts, into her part-of-whole reasoning, in which all pieces must be the same size for the fractions to make sense. The researcher erases all of the lines except the first line on the French fry in an attempt to prompt Marsha. She asks Marsha what fraction that piece represents, and Marsha says one-third. When she is asked why it is one-third, Marsha says because she already knew it was one-third. Then, Marsha seems to have another a-ha moment. She says, "Oh!" as though she has had a realization, but again reverts right back to her part-of-whole reasoning that there needs to be equal pieces. At this point in the exchanges, her prior reasoning (part-of-whole) seems to constrain her ability to understand that the pieces do not need to be equal in order for the relationship to exist, as she also considers the number of times a piece fits within the whole (last sentence in Excerpt 4.19).

At the very end of the workshop, the researchers revisit the original fractional triangle task. One of the researchers asks Marsha if she would answer the question differently. Marsha gives a robust explanation that illuminates the change in her reasoning from the beginning of the workshop to the end. This explanation is presented in Excerpt 4.20.

Excerpt 4.20 – *Workshop #4: Marsha's Construction of the Equi-Partitioning Scheme*

1:28:26 minutes into the workshop

R: I'm going to go back to the math question we started with (Referring to the fractional triangle task). I want to know if you feel like these questions that we talked about were part of equi-partitioning? Why? And would you answer them differently now than you did before?

Marsha: Because you're doing the same thing with B (Referring to Rectangle B) that you were with the French fry, *having the blue piece fit how many times into the black triangle* [italics added].

R: Do you still feel like it's one-sixth?

Marsha: Yes.

R: Why?

Marsha: Because *if you move the blue rectangle down, or you repeat the blue rectangle, it will repeat itself six times. I get what you're saying about the equal parts, because the blue piece is not equal to the black pieces, but it's still 1/6, because if you fill up the black pieces with the blue piece, all those blue pieces are going to be the same size* [italics added].

Excerpt 4.20 indicates the shift in Marsha's reasoning from the beginning of the workshop. In a context highly different than the French fry context in which she worked initially, she reasons about the unit fraction (1/6) in terms of the number of times it fits in the whole. That is, she seems to have constructed the equi-partitioning scheme at least at the participatory stage. I cannot claim that she has constructed it at the anticipatory stage, since she needed quite a bit of prompting throughout this process. However, by the end she went from needing the pieces to be of equal size in order to determine the fractional piece of the fractional triangle task, to realizing that the other piece sizes do not matter. She realizes and makes explicit that the blue rectangle is 1/6 even if the pieces are not all equal size, which just moments earlier presented a constraint to this very way of thinking. Later in the school year, Marsha and the other teachers participated in the fifth workshop, which went back to the multiplicative reasoning schemes, specifically the Mixed-Unit Coordination (MUC) scheme.

Workshop #5. The fifth workshop occurred in January of the second year. The focus of the workshop was on the mixed unit coordination (MUC) scheme. The workshop began with the teachers being asked to work through the following MUC task:

> Schools get single apples, bags of apples, and boxes of apples. Each bag has 10 apples. Each box has 10 bags.
>
> School A has 4 boxes + 6 bags + 19 single apples.
>
> School B has 3 boxes + 15 bags + 11 single apples.
>
> School A has ____ apples in all.
>
> School B has ____ apples in all.
>
> Which school has more apples?
>
> How many more apples does that school have?

Marsha seems to struggle through this task as she gets confused by the units involved, which she seems to assimilate with two units as given and a third in activity. In Excerpt 4.21, Marsha is explaining how she got her answer for school A. She gets the correct answer while struggling to explain how she got the answer.

Excerpt 4.21 – *Workshop #5: Marsha's Explanation for School A*

09:05 minutes into the workshop

R1: Marsha, what did you do?

Marsha: I multiplied 40 by 10, because 4 boxes, each bag has 10 apples, so that's 40 apples in the box, times 10 boxes, [it] is 400 apples.

R1: So, say it again?

Marsha: So, 4 boxes, there were 40 in each box. Wait, I don't know.

R2: There are 40 in a box, but 40 what?

R1: Each box does not have 40.

Marsha: *Each box, there's 4 boxes, so in the 4 boxes, there's 40 apples, and in the bag in the box there's 10 bags. So, 40 boxes times 10 each is 400 apples altogether* [italics added].

R1: 4 boxes have 40 apples, but then you have 40 in all boxes? I'm just trying to understand what you're saying.

Marsha: I don't know.

R1: Can you talk about what you did?

Marsha: That's what I have. I have 40 plus 10, plus 60, plus 10. I have 400 plus 60 plus 19.

R1: And how did you get to the 400? Or the 60? Or the 19?

Marsha: 19. One-by-one, 6 bags times 10 apples in each bag, that's 60 apples and then 4 boxes of 10.

In Excerpt 4.21, while Marsha gets the correct answer, she seems confused by the units on which she is operating. I infer that Marsha is assimilating two units as given (e.g., the bags and single apples, or the boxes and some units of 10 in them) but assimilates the third unit only in activity. Since she is assimilating just two units at a time, she has to work through 40 times 10 sequentially. This is evident when she says, "I multiplied 40 by 10, because 4 boxes, each bag has 10 apples, so that's 40 apples in the box, times 10 boxes, [it] is 400 apples." That is, she seems to assimilate the four boxes and the 10 apples in each bag as given. However, she seems not to assimilate the intermediate unit (10 bags) in each box as given. First, she figures out how many bags are in the 4 boxes, then she can figure out how many apples are in the 40 bags. Marsha seems to conflate the bags and the boxes at the end of her explanation, which seems reflective of her confusion of the units involved in the task and how she was assimilating the third unit.

In Excerpt 4.22, Marsha experiences an a-ha moment after prompting from one of the researchers and a teammate. In this excerpt, the teachers are explaining how they determined the difference in apples between school A and school B. Marsha originally used a subtraction algorithm to find the difference, but after the prompting, she realizes that the units can be exchanged for other units to find the difference.

Excerpt 4.22 – *Workshop #5: Marsha's A-Ha Moment*

18:11 minutes into the workshop

R: What about the next question?

Marsha: Which one has more? A

R: How many more?

Marsha: 18

R: How did you get it?

Marsha: I subtracted 479 and 461.

R: With an algorithm, like the subtraction algorithm?

Marsha: Yeah.

19:42 minutes into the workshop

R1: (Pointing out the number for school B) There's 3 boxes and they each have 10 bags, each has 10 apples, each box has 100. So, I have 300. This (points to the 15 bags) is like another box, right? The 10 of the 15 is another box, so I have 4 boxes. 4, 50, and 11 more. Actually the 11, the 1 here (points to the tens place) is actually like another bag. Can you use what I just presented to solve which school has more?

Tracy: So, school B has 3 boxes, 15 bags, and 11 single apples. Well, I know that each box has 10 bags in it, so if I take 10 from the 15 and put it over to a box, that's 4 boxes and then I still have 5 bags left, but my single apples I can put 10 of those into a bag, so then I have 6 bags. So, I have 4 boxes, 6 bags, and 1 single apples, so then I have 461.

R2: And if we are trying to answer this last question here (referring to How many more apples does that school have?), how can you use what you're explaining to find the difference?

Tracy: I guess just converting the first one too. So, like, I'd still have my 6 bags and 4 boxes, but then changing the 10 (referring to the tens place in the 19 single apples) or the 7 bags, so 79 so 479 and then, well I can see it now. 4 boxes minus 4 boxes.

Marsha: Oh!

Tracy: So, that's zero, and then I would have 7 bags minus 6 bags is 1 bag, and then the single apples left over, which would be 9 minus 1 is 8, so it would be 18 left over, the difference.

R1: Marsha, what was the "oh?"

Marsha: Because when you were talking, I was thinking, I don't know another way to get 18 without subtracting the two big numbers, but as she was doing it then I was seeing what she was doing.

R1: What is it you now know? Walk me through how you would use it. Let me suggest a specific thing I heard and ask you to explain it. Initially what I heard Tracy saying, I can represent this as 4 boxes, 6 bags, and 1 apple. That's 461, I'm back to working on 1's. Then Tracy talked to me about, oh but I could just work on the boxes and the bags and the single without turning them into ones. I don't need a common denominator to make a comparison (referring to a previous discussion about how they had to originally change all of the numbers into a common unit of single apples). Then I heard Oh. So, take us from there.

Marsha: So, now, I feel like I'm basically doing the same thing except I didn't make a standard algorithm. *I converted all my singles, and bags, and boxes and combined them to smaller numbers. Then A and B boxes cancel out, and 7 minus 6 bags from A and B is 1, and 9 singles minus 1 single is 8 and you get 18* [italics added]. So, to me it looks like the basic standard.

The explanation from Tracy and the researcher seemed to cause a perturbation and then an a-ha moment for Marsha. This need for prompting from Tracy and the researcher, leads me to infer that Marsha has constructed the MUC scheme at the participatory level. Originally, she relied on the standard algorithm to find the difference in apples, because she was assimilating the task with a goal of operating only on the single apples. However, after the prompting, Marsha was able to assimilate the task as involving three different units and use those different units in order to find the difference. Once she was able to assimilate the three units, she was able to operate on each of them to determine the difference in apples between school A and school B. This excerpt illustrates the moment at which this a-ha occurred and gives us a glimpse into a transition in Marsha's reasoning – from operating on 1s only or on 1s and the largest-size unit (e.g., boxes) to coordinating operations on three levels of units (i.e., boxes, bags, and single apples).

Later in the workshop, the discussion turns to explicitly explaining the three levels of units involved in the Same Unit Coordination (SUC) scheme. In Excerpt 4.23, one of the researchers explains what the third level of units is, but Marsha seems to again struggle with assimilating this unit.

Excerpt 4.23 – *Workshop #5: Marsha's Struggle with a Third Unit*

60:23 minutes into the workshop

R: In same unit coordination, we have a third level of unit. It's not just ones and composite units anymore. And we actually ask them (referring to students) to remember the ones are there, but not operate on them. That makes it really difficult. They have to add a third unit. Remember the first one, but not bring it in to the operation.

Marsha: What's the third one you're talking about?

R: The collection of all the composite units.

Marsha: Got it.

R: That's a different unit.

Marsha: *Why is it different? It's still towers* [italics added].

R: Why are the 4 boxes that were 40 bags different than the 6 bags? In school A? In what way are the 6 bags and 4 boxes, which are 40 bags different?

Marsha: Well, these are still just towers in this question.

R: In what way are 4 boxes different from 1 box?

Marsha: Oh. It's bigger.

R: Why?

Marsha: There's more in there.

R: So, in what way is a group of 6 towers different from a group of 4 towers?

Marsha: *It's bigger. There are two more* [italics added].

Excerpt 4.23 indicates that Marsha is assimilating two levels of units as given, with the third level in activity (of discussing the situation). At first, Marsha struggles to assimilate the third unit (collection of composite units), because she does not understand how the third unit is different from a composite unit; she seems to conceive of towers and the collection of towers as the same unit. After prompting, however, she assimilates the third unit when she recognizes that a collection of six towers is larger than a collection of four towers with a difference of two towers. Back in Year 1, while working on mDC tasks, Marsha could assimilate three units without prompting, but when she is working through higher multiplicative reasoning schemes, she requires the prompting in order to assimilate the unit. The final set of data collected from Marsha was her second AOP session, which provides a culminating summary of her progression through the project and shows how much growth took place in Marsha's reasoning over the two-year long project.

AOP #2. At the conclusion of the project, a final AOP was conducted with Marsha. This AOP session is extraordinary in that it illustrates how much Marsha's reasoning grew over the course of the project. In this AOP, Marsha teaches a lesson in which the students are learning to draw the figurative version of the PGBM game. Throughout the AOP, Marsha is giving evidence into her own multiplicative reasoning as the distribution of units within other units and the simultaneous tracking of multiple units. She also diagnoses her students' reasoning in terms of the units they are working with in the lesson. The first excerpt (Excerpt 4.24) comes from the pre-interview conducted before Marsha taught the lesson. In this interview, the researcher asked Marsha to explain what she was going to teach and elaborate on the importance of the towers and cubes for students' learning.

Excerpt 4.24 – *AOP #2: Marsha Explaining the Importance of the Units*

03:43 minutes into the pre-interview

R: Towers and cubes, can you explain to me the towers and cubes and the purpose of the towers and cubes for their (referring to the students) learning?

Marsha: To represent what 3 x 6 is. That *it's units within something* [italics added]. That they're not just memorizing that. So, if we ask them next year, what is 3 x 6, they can say it's, and I don't know if I've been doing it backwards, but we've been doing 3 towers of 6 cubes. That they can figure that out, and that I see some of them doing *1, 2, 3, 4, 5, 6 (counts six times on index finger) 7, 8, 9, 10, 11, 12 (counts six more times on middle finger)* [italics added]. When I tell them to start covering up, I'm encouraging them to use their fingers as other ways, because they're not always going to be able to look at their cubes. So, that's what I want to do today, that if they're stuck on here (mimics counting on her fingers), because they can skip count too, then when the numbers get bigger then they start to get confused. With this (mimics drawing the towers) they have a visual representation of how to do it without drawing out all their towers and all their cubes.

R: So, what would be the difference if they did draw it out versus if they didn't draw it out?

Marsha: I feel like it would take forever. Maybe they're not understanding that, I don't know, because I feel like some of them are. *Because they know that this finger (points to index finger) has six cubes in it. Whether they're saying 6, because when the numbers get too high, they start to lose count or they will count 1, 2, 3, 4, 5, 6 (counts six times on her index finger)* [italics added].

R: When you're doing this (mimics counting six times on her index finger), what does this, can you tell me what this is representing?

Marsha: *Towers (swipes up her index finger), and then the number of cubes (mimics counting six times on her index finger)* [italics added].

07:03 minutes into the pre-interview

R: So, they've been physically working with towers and cubes, and in the past you've noticed that they're counting all the cubes?

Marsha: Some of them will resort back to counting all of the cubes. If they've made a mistake, they're like one or two off, they will resort to counting them all. Some of them.

R: How do they know if they're one or two off?

Marsha: I tell them. Or they'll come up to me and say, who's right? They'll say, is 6 x 8 forty-eight? I'll say, what did you get? He got 46. I said, how did you get 46? One boy in particular, he grouped them, liked doubled, doubled, doubled, doubled, and then counted on. Then realized he made a mistake.

11:24 minutes into the pre-interview

R: Can you talk about why this (referring to PGBM) would be representative of multiplication?

Marsha: Because it's kind of repeated addition. It's kind of skip-counting. Those are all the strategies that they use when they're trying to figure that out. When you're drawing it out, that's what you're doing. *You're skip-counting all the way at the bottom, but you're keeping track of two things. You're keeping track of towers and you're keeping track of cubes to figure out the total* [italics added] and apply it to your math work.

From Excerpt 4.24 I infer that Marsha has constructed the mDC scheme at the anticipatory level and is assimilating at least two levels of units as given. She is able to model the tracking of two units using her fingers, and overtly explains that she is looking for her students to do the same with their fingers or through their figurative drawing. She is looking for her students' reasoning to include the tracking of at least two units, instead of using additive reasoning, which she considered to be multiplicative reasoning at the beginning of the project. Although she mentions "repeated addition," Marsha seems to focus on more from her students than getting the correct answer. She wants them to track single units and composite units simultaneously.

Excerpt 4.25 comes from the post-interview, after Marsha taught the lesson. The researcher asked Marsha what she noticed her students doing mathematically throughout the lesson. At one point, the discussion turns to the reasoning Marsha thought one of the students has been using. This student was attempting to solve for the total amount of cubes in four towers of four cubes each. The student seemed to struggle with the problem for a while and eventually got the answer of thirty-two. The researcher asked Marsha what the student was doing when trying to solve the task. Marsha's explanation shows that she was focused on the units the student was working with and how he was operating on those units.

Excerpt 4.25 – *AOP #2: Marsha Explaining the Student's Reasoning*

08:49 minutes into the post-interview

R: What was he doing? I couldn't see it.

Marsha: So, he had the 4 cubes (in one tower). He was doing (turns around to draw on her whiteboard and draws one tower with 4 cubes in it. She puts a 4 in the first cube, an 8 in the second cube, a 12 in the third cube, and a 16 in the fourth cube). He was going 4, 8, he was counting each individual cube as 4. So, he did 4 times 8, which is correct, 32. So it's not like he flip-flopped towers and cubes, I don't know.

R: How was he getting to 32? Did he show you?

Marsha: Yeah, that's what he did. He was doing 4, 8, he was counting. But it's not backwards. I don't even know what that is. He was doing like cubes within cubes.

Marsha's own mathematical reasoning seems to have led to an increased ability to analyze her students' reasoning. She is no longer just looking for a correct answer from her students but is focusing on the units the students are working. In this last excerpt, Marsha does not seem worried about the fact that her student got an incorrect answer. Instead, she focuses on what the student was doing and how he was reasoning with units. She concludes that he was possibly assimilating the cubes as composite units made up of other single units. This indicates a critical shift in Marsha's ability to identify her students' units and make inferences into their reasoning.

In conclusion, this AOP illustrates the growth in Marsha's own reasoning and her ability to analyze student reasoning. Marsha is attempting to diagnose her students' reasoning in terms of units and operations, because she now understands the importance of them tracking multiple units. She seems to have constructed a stronger understanding of multiplication as the distribution of items of one composite unit over items of another unit. She is able to verbalize this importance and is able to identify it in her students. This is a big jump from the first AOP, conducted at the end of Year 1. In the first AOP, Marsha seemed to assimilate composite units as "holding units" for the single units, and not as a unit of its own right. At the end of Year 2, Marsha is assimilating the composite unit as a unit itself, that is also made up of the single units; she is able to assimilate both units as given. Accordingly, in the first AOP Marsha's ability to analyze her students' reasoning seemed limited to identifying whether or not her students had the correct answer (preferably using memorized facts); she was just beginning to look at other ways her students were reasoning. In the last AOP of Year 2, Marsha was able to go much further than that and analyze the units her students are operating with and how they are operating with them.

Summary of Year 2

In Year 2, the largest amount of growth in Marsha's own reasoning occurs, which also leads to a better ability to analyze her students' reasoning and their levels of units coordination. Marsha seems to have moved from the participatory level to the anticipatory level of reasoning within the mDC scheme (Excerpt 4.10). She also seems to have moved from pre-fractional reasoning to the participatory level in the equi-partitioning scheme (Excerpts 4.17 – 4.20). She experiences several perturbations in the workshops of Year 2, which lead her to a-ha moments and the construction of new reasoning through (Excerpts 4.17 – 4.20 and Excerpt 4.22). The growth in her own reasoning afforded her the ability to recognize the three levels of units her students were working with and how they were operating on those units (AOP #2). This was a major shift in her mathematical knowledge for teaching, which made it possible for her to move beyond looking for correct answers, to really digging into her students' reasoning and making inferences based on what her students were saying and doing.

Summary of Marsha's Transitions

Marsha began the project with common knowledge of content, meaning she was able to determine her students' computational work for accuracy. As the project continued, Marsha began to develop specialized knowledge of content, because she shifted to identifying the strategies her students were using, specifically the units students were using. By the end of the project, Marsha's specialized knowledge of content afforded her the ability to not only recognize her students' assimilatory units but also to begin making inferences about their reasoning. These shifts in her MKT came about as her own mathematical knowledge and assimilatory units transitioned.

When Marsha entered into the project, she had a procedural understanding of multiplication, and referred to it as repeated addition and the building of arrays (Excerpt 4.1). This allowed her to focus in on whether or not students were getting the correct answer and if they had memorized their multiplication facts (common knowledge of content). Through Year 1, she began to bring in her specialized knowledge of content as she noticed and analyzed her students' reasoning in terms of additive counting methods (Buddy-Pair #4). By the 5th and 6th Buddy-Pair sessions, she was beginning to reason multiplicatively at the participatory stage and seemed to understand that multiplicative reasoning involved the tracking of multiple units. This realization came with her own assimilation of two levels of units as given. Her assimilation of these units allowed her to begin constructing the multiplicative reasoning schemes, as well as begin to identify the units her students were assimilating and operating on. By the end of Year 1, Marsha's assimilation of two levels of units afforded her the ability to analyze her students' assimilatory units and whether or not they are tracking those units (Excerpt 4.9). That is, in Year 1 there were major transitions in Marsha's mathematical knowledge which positively affected her MKT, specifically her specialized knowledge of content.

Marsha's transitions continued into Year 2. During that year, Marsha continued to construct her levels of units – adding a third level at least in activity, as well as her multiplicative reasoning. She also began to work through the first fractional reasoning scheme, moving from pre-fractional reasoning to equi-partitioning at the participatory level. By the end of the project, Marsha was assimilating at least two and a half levels of units into her multiplicative reasoning schemes. Her MKT transitioned from looking for correct answers in her students' reasoning (beginning of the project) to identifying her students' assimilatory units and how they operated on those units (Excerpt 4.25). This was a significant shift in her own reasoning, which seemed related to significant shifts in her MKT.

In an attempt to connect my three research questions, Table 4.1 provides an outline of the shifts that occurred in Marsha's levels of units coordination, her multiplicative and fractional reasoning, and her MKT. Blank boxes were used when there was no change from the box above it. I used the N/A notation when that specific reasoning type was not analyzed. Based on this table, Figure 4.4 then illustrates shifts (over time) in Marsha's assimilatory units and where her multiplicative and/or fractional reasoning was at that same moment.

Table 4.1 - Conceptually Clustered Matrix of Marsha's Conceptual Progression (Miles & Huberman, 1994)

	Levels of Units Coordination	Multiplicative Reasoning	Fractional Reasoning	Pedagogical Reasoning
		Year 1		
Buddy-Pairs #1 and #2	At least 1.5 units (1 unit as given; 1 in activity)	Pre-mDC Procedural Repeated addition and building of arrays	N/A	Focuses on correct answers and memorized facts
Buddy-Pair #4		Pre-mDC Procedural	N/A	Can analyze students' additive counting methods Can differentiate between concrete and figural representations
Buddy-Pair #5	At least 2 levels of units as given	mDC and QD at the participatory level Beginning to understand the tracking of units	N/A	Analyzes two levels of units in students' assimilatory scheme Analyzes how students operate on their units
Buddy-Pair #6		mDC at mid to high-participatory level	N/A	Notices students keeping track of units and how they operate on their units
Workshop #2	1.5 levels of units (1 levels as given; 2nd unit in activity)	N/A	Pre-fractional Part-of-whole	N/A
AOP #1	At least 2 levels of units as given		N/A	Struggles to transfer her own reasoning to her teaching explanations Focuses on how students are tracking units
		Year 2		
Workshop #3		mDC at high-participatory level Moves from pre-UDS to participatory level of UDS	N/A	N/A

Table 4.1 Cont'd

Workshop #4		N/A	Moves from pre-fractional to equi-partitioning at the participatory level	N/A
Workshop #5	2 to 2.5 levels of units	SUC and MUC at the participatory level	N/A	N/A
AOP #2	At least 2.5 levels of units	mDC at the anticipatory level	N/A	Analyzes units students are assimilating and how they are tracking those units

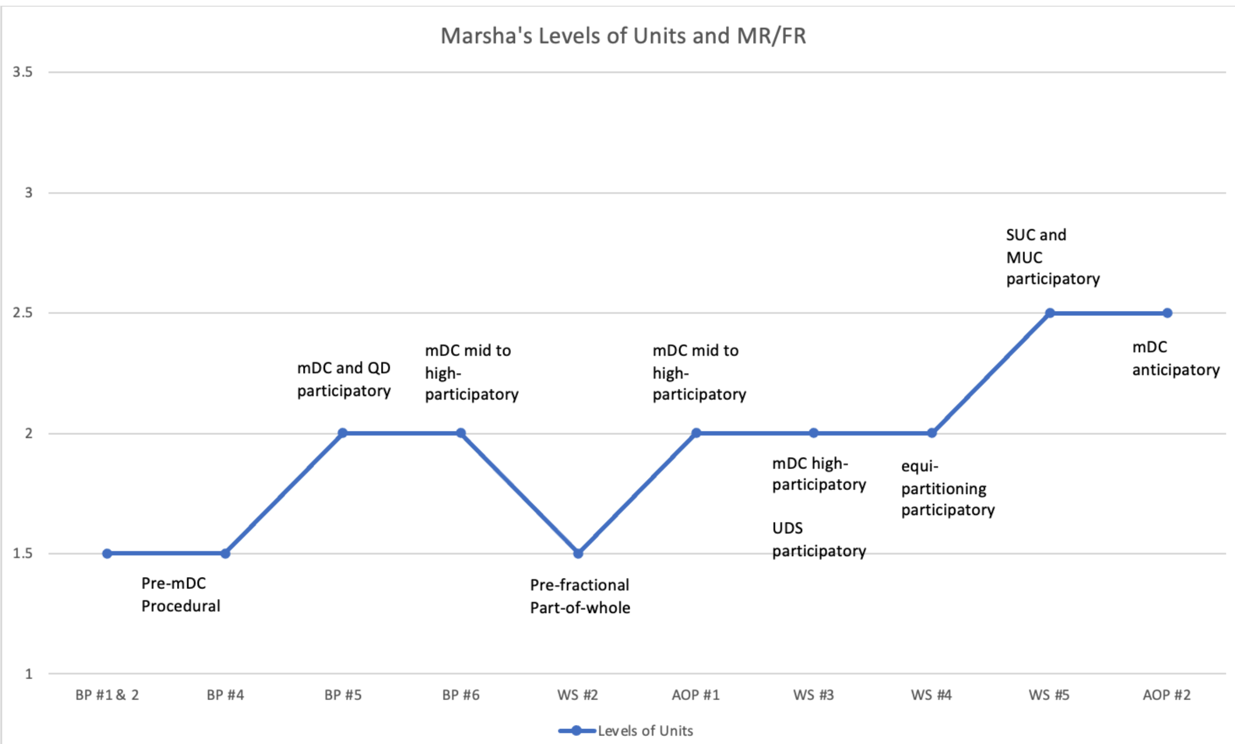

Figure 4.4 - Growth Gradient for Marsha's Levels of Units Coordination and Multiplicative and Fractional Reasoning (Miles & Huberman, 1994)

The Case of Nancy

The analysis of Nancy's reasoning begins with a pre-intervention AOP. That data set provides baseline collected before the start of the AdPed project. The rest of the analysis follows the progression of the project, through its Year 1 and Year 2.

Pre-Intervention

In this section, I explore Nancy's mathematical reasoning in regard to her understanding of multiplication, particularly her levels of units coordination, when she first entered the AdPed PD program in late spring of 2016. At that time, the researchers conducted an AOP before any intervention or coaching took place. The purpose of my analysis is twofold: articulate (a) Nancy's reasoning before she received any intervention and (b) how her reasoning seemed to guide what she was able to notice in her students' reasoning at that specific point in time.

To provide the reader with a sense of an advanced organizer of my inferences, through my analysis I found that Nancy had a procedural understanding of multiplication, predominantly thinking of it as repeated addition of equal groups. Importantly, the data indicated she also understood multiplication as the distribution of one composite unit's items over items of another composite unit. Specifically, I inferred that Nancy's assimilatory scheme of multiplication involved two levels of units as given. Further, I conjecture that, in part, she considered multiplication as repeated addition of equal groups due to this notion being prevalent in her school's mathematics curriculum (and likely also her own schooling experience).

Excerpt 4.26 demonstrates Nancy's initial meaning for multiplication as the repeated addition of equal groups. It includes three segments, all from an interview conducted before she taught her lesson. In Excerpt 4.26, the researcher asks Nancy what the focus of her upcoming mathematics lesson will be. She explains to the researcher that the class has been working on learning different multiplication strategies. Later in the excerpt, Nancy goes further into explaining what she will be doing with a small group of students who are already using repeated addition as their multiplication strategy, as well as her goal for them to move past drawing pictures of equal groups in order to think more abstractly about the multiplication task. The final segment of the excerpt illustrates Nancy's understanding of multiplication as repeated addition, while also giving a glimpse into a plausible way of conceptualizing multiplication as units made up of units. I present all three segments of the excerpt with indications of the time within the interview when they occurred, followed by analysis of the data in them.

Excerpt 4.26 – *Pre-Interview*: *Nancy's Twofold Meaning for Multiplication*

2:50 into pre-interview

R: What will be the focus of your lesson today?

Nancy: …It was at that point (talking about the previous week), and expected, equal groups and pictures and drawing things out to make an understanding of it. So, what I was doing now was to kind of push them from pictures to maybe even numbers, *or repeating that addition and seeing that connection* [italics added]. Then, for some of my higher kids, to then take that onto multiplication in the form of, like, 2 times 4 or 6 times 8… For a small group (referring to a group of students who needed extra help), I'm going to pull the students who are still continuing, I believe they have the sense that it's 4 plus 4 plus 4 plus 4, four groups of 4, that are still kind of relying on drawing these pictures and seeing if they can put it into number and repeating that addition instead of drawing all of these.

18:00 into pre-interview

Nancy: To get them (referring to the lower students) out of, well we first just started drawing groups, we've kind of moved to now, okay if I can draw the groups and *I'm adding them up anyway, because I can add 4 plus 4 plus 4 plus 4, how can I push myself to just see those groups in my head and just repeat that addition?* [Italics added.] Once they can do that then how can we make it, this is my understanding, this is how I teach, but how can I now even say, well if I know that two groups of 4 are 8, do I have to continue to do the repeated addition and start putting things together, or could I start to break it up by known facts that I know? You know, 2 groups of 4 are 8, another 2 groups of 4 are 8, and how can I put those together?

20:00 into pre-interview

R: How will the students know when to stop their repeated addition sequence if their manipulatives are covered?

Nancy: So, if they've got 6 groups of 3, they understand that it's 6 groups of 3. So, if they're going to do 3 plus 3 plus 3 plus 3, that they're going to have 6 threes, because each one is a group of 3.

I infer from Excerpt 4.26 that Nancy is looking for her students to be able to transfer their figural representations into more abstract reasoning by using repeated addition to solve the multiplication tasks. Her assimilatory multiplicative reasoning scheme at this time seems to allow her to notice whether her students are reasoning figuratively versus abstractly. This is inferred based on the criterion she used to distinguish three sub-groups within her class, with the lowest still working on drawings or tangible ojects whereas the highest could move to working with numbers. Nancy explains the use of repeated addition of equal groups in order to solve multiplication tasks. For example, Nancy describes a possible solution for four groups of four as adding 4 + 4 + 4 + 4. Then again, Nancy describes six groups of three as 3 + 3 + 3 + 3 + 3 + 3. Nancy seems to also link the drawing of pictures to her students' ability to "see" repetition of the same unit. This is evident when she explains that if her students can draw the groups, then they can push themselves into seeing those groups in their head, in order to add the groups together. She also goes further to discuss the link between repeated doubling of known facts (i.e., knowing that two groups of 4 is 8 and repeating that known fact twice gives you 16). This gives further evidence into how Nancy reasons about multiplication and its connection to repeated addition, recursive doubling, and known facts.

The second segment in Excerpt 4.26, twenty minutes into the interview, gives further glimpse into Nancy's own understanding of multiplication beyond repeated addition. Specifically, I infer that Nancy's interpretations of the task involved operating with two levels of units, e.g., six composite units, each of which is made up of three single units. Nancy gives a different explanation of six groups of three by stating that, "They're going to have six threes', because each one is a group of 3." To further demonstrate that Nancy is operating with two levels of units, next I explore more data segments from the Pre-Interview.

Excerpt 4.27 focuses on Nancy's plausible conceptual understanding of multiplication as the distribution of items of one composite unit over items of another composite unit. Nancy seems to assimilate at least two units within her multiplicative reasoning schema. The first three segments of Excerpt 4.27 illustrate the levels of units that Nancy assimilates into her current schemes. The last segment of Excerpt 4.27 comes from the post-interview conducted after she taught her lesson. In that final segment, Nancy has some focus on units the students needed to assimilate in order to engage in the task they were given.

Excerpt 4.27 – *Pre-Interview and Post-Interview: A Glimpse into Nancy's Conceptual Understanding of Multiplication*

6:20 into pre-interview

Nancy: What do I know that multiplication involves? Well, it involves, I've got groups of things, I've got numbers within those groups, and what am I looking for?

11:00 into pre-interview

R: How do you see the link between multiplication and division? In what way do you understand the link?

Nancy: My understanding of multiplication, we have groups, we have numbers within, and we're searching for that total. Once we have an understanding of that, when it comes to division, we're starting with that total group, and we're either dividing it up, dividing it into groups, or dividing it based on how many we know are in each group.

26:00 into pre-interview

R: For you, is 5 x 3 the same as 3 + 3 + 3 + 3 + 3?

Nancy: I think of this (points to the multiplication problem) as the next phase. I think of this (points to the addition problem) as them seeing groups and adding them together based on their mathematical experience to this point. This (still pointing to the addition problem), they see the groups (moves her fingers like she is counting the five groups). They can go back and say, "I have 1, 2, 3, 4, 5; I have five groups of 3" (points to each 3 in the addition problem as she is counting). But this (points to the multiplication problem), to me is the next level of understanding because they don't even have this (points to the threes in the addition problem).

15:00 into post-interview

R: What was going into your mind in terms of the math they were dealing with and how they were working that apparently did not take them to being able to pose a problem?

Nancy: Being able to see groups, for example, we're at the fair. Then being able to think of something that legitimately is groups, and would legitimately have numbers within those groups, became tricky. So, we need to think about, okay if we're going on rollercoasters and I need tickets for them and how many rides does each ticket need, I think it was *just too many things to think about* [italics added], to plug into these story problems.

R: So how do you think about that? You did model one example, and they gave some context, and you modeled one problem, said here's I think about it, so on and so forth. Then they moved to the [work in small] groups and apparently there are too many pieces for them to hold together and they can't do it. What do you think about that move from you presenting to them doing it on their own? What was not working there?

Nancy: Well, I think I modeled it, but I think it would have been a good idea to have done that, exposed them to that a few more times before I sent them off to do it in a sort of We Do situation.

R: So, you give more examples?

Nancy: Give more examples, have them be a working part of the examples and really brainstorming what do groups look like and what could be inside of those groups and then ultimately, what is that total? Because we are not looking for the total of carts, because that's the number of groups we have. What is the total that we're looking for? We're looking for the number of people that are riding that ride. I think that can get a little bit tricky.

R: With all of this in mind, you actually had some students that I think you felt were getting some of it, right? Can you give me an example of a child that you thought actually got some of it?

Nancy: One little girl said, if their setting was Chuck E. Cheese, and she, like, we're having a party and we are going to buy four pizzas and there's going to be nine slices in each pizza, how many pieces of pizza will we have for the party? So that told me she understood that the group and the number in each had to be related, they had to be about the same thing, and that we were searching for a total.

In Excerpt 4.27, Nancy is expressing multiplication as more than repeated addition of equal groups. She seems to understand that there is something more to multiplication besides repeated addition. I infer that she assimilates at least two levels of units as given. It is not evident to me that she has a third unit as given, as I am not convinced that she sees the total as a unit as well, but instead as just the result of the calculations within the multiplication task. This is evident when she states, “My understanding of multiplication, we have groups, we have numbers within, and we’re searching for that total.” Her understanding of groups with numbers within in each group, suggests that she assimilates single units within composite units.

Further, the researcher asked Nancy if, for her, 5 x 3 is the same as 3 + 3 + 3 + 3 + 3. Her explanation suggests that she understands them both as multiplication, but with 5 x 3 being higher reasoning than the repeated addition, because the students have to abstract the five groups of 3 from the multiplication problem, rather than seeing the five 3’s in the addition problem. The way she counted the five 3’s and then said, “I have 5 groups of 3,” leads me to infer that she understands that there are five composite units, each made up of three single units; therefore, I again infer that she assimilated at least two units as given.

The final section of Excerpt 4.27 occurred after Nancy taught her lesson to the students. The researcher’s focus in the post-interview was on how Nancy thought about the lesson she had taught. It is interesting that this excerpt shows a slight shift in Nancy’s focus in her students’ reasoning. Originally, in the pre-interview (Excerpt 4.26), Nancy focused on her students being able to move from figural drawings to using abstracted repeated addition to solve the multiplication tasks. Here, however, Nancy is using a real-world context (number of carts and riders on a roller coaster ride) to get the students to distinguish the two types of units similarly to what she uses in her reasoning. Nancy states that she would have liked to have given the students more examples in order to get them, “really brainstorming what do groups look like and what could be inside of those groups and then ultimately, what is that total? Because we are not looking for the total of carts, because that's the number of groups we have. What is the total that we're looking for? We're looking for the number of people that are riding that ride.” Her goal for the students seemed to have shifted during the lesson, from using repeated addition while solving multiplication problems to helping the students engage in some real-world context for units made up of units (e.g., rollercoaster carts consisting of riders). I am not sure what caused this shift in Nancy’s goal for her students, but it is an interesting shift that further illustrates Nancy’s own conceptual understanding of multiplication as something more than repeated addition, that is, a situation calling for operating on two levels of units. She seems to understand that this is important for her students’ own understanding of multiplication.

The last part of Excerpt 4.27 also illustrates Nancy’s ability to identify her students’ ways of reasoning in-regards-to the units they are working with. Nancy is able to pay attention to units her students recognize and can work with. This is evident when she describes the student who was working with pizzas at Chuck E. Cheese. Nancy states, “One little girl said, if their setting was Chuck E. Cheese, and she like we're having a party and we are going to buy 4 pizzas and there's going to be 9 slices in each pizza, how many pieces of pizza will we have for the party? So that told me she understood that the group and the number in each had to be related, they had to be about the same thing, and that we were searching for a total." This data segment indicates that Nancy analyzes how this student is thinking about the units, and she sees this as important for the student's reasoning.

Excerpts 4.26 and 4.27, from the pre-intervention AOP session, suggest that Nancy came into the AdPed project with an assimilatory scheme that allows her to not only understand multiplication as the mental distribution of one unit (single units) over a second unit (composite units) to create a third unit (compilation of composite units), but also to identify units her students are using. She does begin the AdPed project with a procedural understanding of multiplication as repeated addition of equal groups, but I claim that this comes from deep, embedded past learning and teaching from mathematics curricula that focuses on multiplication as repeated addition. In addition to this, Nancy had constructed some understanding of the distribution of units on her own, which is further evidenced in Excerpt 4.27. My take-away from this AOP session is that Nancy explained multiplication as repeated addition from her past exposure to that sort of reasoning, while her own conceptual understanding of multiplication involves the distribution of units – due to her having an assimilatory scheme that includes at least two levels of units as given. With this understanding of Nancy's own mathematical reasoning and its linkage to reasoning about her students' thinking, I move on to her reasoning after participating in one year of the AdPed PD program.

Nancy's Progression in Year 1

The first year (Year 1) of the AdPed project began with a Summer Institute (SI1) between the spring and fall of 2016. Nancy attended this week-long professional development in full. The focus of SI1 was on helping teachers' construct their own multiplicative reasoning, as well as learning to identify their students' ways of reasoning multiplicatively. Following SI1, Year 1 continued into the 2016/2017 school year. During that year, Nancy participated in multiple Buddy-Pair sessions (partnered with Marsha), professional development workshops, and an AOP. All data I present in this section come from the 2016/2017 school year (not including SI1). This is due to a lack of evidence from the SI1 data, because the videorecording included all participants of the workshop, but not close-ups of individual teachers.

Upon delving into Year 1 data, it became clear that Nancy had a shift in her own reasoning between the pre-intervention AOP and her first sessions during the school year, resulting in a shift in her ability to identify her students' reasoning as well. In the pre-intervention AOP, Nancy had a mixture of understanding multiplication as repeated addition of equal groups *and* some initial understanding of multiplication as the distribution of one unit over another unit. This led me to infer that Nancy was working with at least two units as given in her assimilatory scheme. As a result of Nancy's reasoning at the time, she was able to identify units her students were working with, but she was not yet able to analyze how they were working with those units, nor what it meant about her students' reasoning.

At the beginning of Year 1 (after SI1), Nancy seemed to let go of any language relating to repeated addition of equal groups. She also had begun to analyze how students were operating on their units and whether or not they were tracking the units as they worked through multiplication tasks. As the year progressed, Nancy's language around multiplicative reasoning continued to grow as she became more aware of the units students were working with and the importance of students' abilities to track those units simultaneously. By late spring of that year, the first evidence of Nancy assimilating three levels of units as given is observed, which in turn seemed to also afford her the ability to identify three levels of units in her students' reasoning, as well as how operated on those units. The very last session of that year was an AOP data set, which gave evidence as to Nancy's growth in that year. In the post-interview for that AOP data set, Nancy immediately grouped her students based on the units she saw them operating on, and how they were operating on these units. She identified students who were operating with three levels of units and what that meant about their multiplicative reasoning.

With the above paragraph as an advanced organizer, in this section I analyze data from Year 1 in the order the data occurred, in order to paint a picture of Nancy's progression in reasoning and ability to analyze her students' reasoning throughout the year.

Buddy-Pair 1. Year 1 began with a buddy-pair session in which Nancy and Marsha were paired up. The lessons for the session consisted of Nancy teaching a lesson designed to help students construct the multiplicative double counting scheme, and then another mDC lesson was taught in Marsha's classroom. Both of the lessons were debriefed later that day. In this session, it seems that Nancy has grew in her own understanding and reasoning about multiplication, as a result of the week-long SI1. When discussing multiplicative reasoning, Nancy no longer refers to equal groups, and the only time she mentions repeated addition is when she is analyzing what students actually did to solve a problem during the lesson. Instead, Nancy's explanations and analyses of her students' reasoning is more focused on the units students are working with, especially their use of composite units, and their ability to track the units they are working with.

In Excerpt 4.28 (from the post-lesson debrief), Nancy is focused on how her students are operating with composite units during the lesson. She explains how many of them are doubling their composite units in order to solve the tasks. Nancy's own understanding of composite units seems to afford her the ability to recognize that her students are working with composite units rather than only ones, and also how they are operating on those composite units.

Excerpt 4.28 - *Lesson Debrief: Nancy's Noticings of Student Reasoning*

0:10 minutes into the debrief

R: Tell me about how the lesson went.

Nancy: I noticed similarities in a lot of kids putting together a couple, or it's like, for instance, if it was 3 towers with 6 cubes in each, then it was like, well, I took 6 and 6 and I put them together and it was 12 and then I added on another 6 and that was 18.

R: When they did that – did they go 13, 14, 15, and counted on?

Nancy: Yes.

R: They doubled then counted on.

Nancy: Right. So, they're doubling, and maybe they know their doubles, maybe that's something they've just memorized. Then counting on. I had one girl even say, I put these two sixes together and then I counted on from there the last group, the last tower. And I noticed that happened again with another student where she said that 'I know that 7 plus 7 is 14 and then I took the 14 plus another 14 is 28.' So, I guess this is where I start to break down in my understanding, because I see that they are putting the groups together in some sort of either doubling in putting them together or some form of repeated addition where they are saying 7 plus 7 plus 7 more, whether they are counting on they know that more fluidly from things that they've memorized, or actually know. Then they are putting those groups together. That's kind of what I say was a trend.

R: So, one of the things that you are paying attention to is really important, are they operating, the doubling suggests that they are operating on some level as a composite unit.

Nancy: Yes.

R: As a thing as opposed to ones versus sometimes they operate on the ones. That's a very important distinction to make. All the time look for what are they operating on? Or are they operating on both, which will be the direction of where the gain is leading.

N: That’s my question. When she said, I said could you have done this another way and I think she did, my interpretation was, she wasn't quite sure how to do it another way. She knew she could add it another way, but she didn't know she could stick with units in, *units within units* [italics added], another way. That when she said I could break off, and she's breaking off groups and so in my head I'm saying, I'm thinking, okay if someone initially started to do that is it because they are more comfortable working with 9 and 9. Are they making groups that are more comfortable for them to work with, or are they just trying to add it another way?

In Excerpt 4.28, Nancy brings in the term units of units of units, illustrating her shift towards understanding that multiplication involves operations on different levels of units. This indicates a shift from her pre-intervention explanations of multiplication being the repeated addition of equal groups. Another shift shown in this excerpt was her addition of a third unit into her reasoning. In the pre-intervention AOP, Nancy does talk about units made up of other units, however it is never clear whether she assimilates the product of a multiplication problem as a unit in itself or just the resulting total of 1s of the multiplication task. Here she gives some evidence that she does in fact understand the product to be a third unit and not just the result of the task. This is seen in her statement, “…she didn’t know she could stick with units in units within units.” In this statement, Nancy is expressing how her student was recursively doubling her composite units while keeping track of her total cubes. Nancy was aware that the student was tracking the third unit, which suggests that Nancy was able to assimilate that third unit as well. It is not clear to me whether, for her, this third unit is taken as given or is in activity, so I do not attempt to make that claim here. I only claim that there is in fact a third unit being assimilated into Nancy's current scheme.

In Excerpt 4.29, the researcher and Nancy are discussing a conversation Nancy had with one of Marsha's students, who appeared to have mixed up his towers and cubes while engaging in an mDC task. As Nancy was working with this student, she began asking him questions about what the answer would be if the towers and cubes were switched. The researcher presses her to find out why it was so important for her to figure this out.

Excerpt 4.29 – *Lesson Debrief: A Shift in Nancy's Pedagogy*

37:00 minutes into the debrief

R: He was not doing what you had asked for. In what way is this mathematically important? Why, for him, to work on 5 towers of 7 and not 7 towers of 5? We all know the commutative property would kick in and they know 35. So why is that important?

Nancy: And he said that to me, he even said to me wouldn't it be the same thing? Wouldn't it just be the same equation? And I kind of like sat there, and he looked at me, and was frustrated and started to break them apart. And I wanted to see if he would just say the same answer. Do you know what I mean? Would he just say 35 because he knew it was the same number of cubes and nothing had changed?

R: Why was it important for you? Why is that an issue?

Nancy: Just trying to figure [out] if he knew what the units were, and then what the amounts within the units were ... I picked that one in particular before I knew whether he would mix up the units and the number within the units. Because it was 7, because it was more difficult. I wanted to see what he could do if he couldn't just count by those numbers or hadn't memorized that.

Nancy's response is telling. She explains that she wanted to know whether the student would use memorized facts and his knowledge of the commutative property, or could he differentiate between the kinds of units with which he was engaging in the task. For Nancy, it was important to analyze what units the student was working with. Excerpt 4.29 illustrates a shift in Nancy's MKT, because she is now actively asking questions of her students in order to determine what units they are assimilating and how they are operating on those units. Nancy's own progression seems to underlie her awareness of students' reasoning, and she is becoming curious as to units on which they are operating.

A little later after the above exchanges took place, Excerpt 4.30 consists of Nancy's explanation as to why she thinks it is important for students to be required to go across the room to build the towers their partners are asking for while playing the mDC game, Please Go and Bring Me (PGBM). To recall, in this game one student asks their partner to bring them one tower at a time, made up of a certain number of cubes. For example, if the student wanted 4 towers of 3 cubes each, they would send their partner to build a tower of three cubes, four different times.

Excerpt 4.30 – *Lesson Debrief: Nancy's First Explanation of Unit Tracking*

44:00 minutes into debrief

R: You're the sender, right? And you need to know that if she tells you it's 37, no it's 35. Why is this important?

Nancy: So then, by having the student do that, I think it's pushing them to think about that while the other student, to be attempting that while the student is getting the towers. It gives them a chance to practice that *holding, working with both units at the same time* [italics added], and then it gives them a chance to compare their answers.

Nancy's explanation is the first time in the data where she brings forth the idea of simultaneous unit coordination. She explains that this activity is necessary, because it gives the students a chance to practice holding multiple units in their head at the same time. By this, she means that both students (sender and bringer) are holding onto the number of cubes in each tower, the number of towers they have built so far, and what their total amount of cubes is at any given point in the activity. Nancy sees this as important for helping the students assimilate and coordinate those multiple units simultaneously. I infer that this focus is enabled by Nancy's own understanding of the need to track units simultaneously in order to reason multiplicatively.

Taking all data from this buddy-pair session together shows quite a jump (from the pre-intervention) in Nancy's own reasoning, as well as her ability to identify her students' reasoning. At this point in the AdPed project, Nancy is already reasoning about multiplication as involving simultaneous coordination of units within units, to create a third unit. She thus seems to pay close attention to how her students are operating on units during multiplication tasks, and she is beginning to show evidence that she understands the importance of students tracking those units simultaneously. I further explore this shift in her reasoning with data from a Buddy-Pair session (#3), which took place roughly in the middle of Year 1.

Buddy-Pair 3. The third buddy-pair session for Nancy and Marsha took place in January of 2017. For this buddy-pair, one of the project GRA's taught another lesson in which she used PGBM game in Marsha's room. The debrief for this buddy-pair began with Nancy continuing to refine her understanding of multiplicative reasoning and the importance of tracking multiple units. Halfway through the debrief, however, Nancy seemed to experience a perturbation that caused her some confusion about the meaning of multiplicative double counting. After some further prompting by one of the researchers, Nancy was able to reequilibrate from the perturbation and her reasoning was back on track.

Excerpt 4.31 presents data from the very beginning of the debrief, when the researcher asked Nancy what she thought about the lesson. Nancy has a strong understanding of composite units and the ability to analyze how students were keeping track of their units.

Excerpt 4.31 – *Lesson Debrief: Nancy's Focus on Units Coordination*

2:50 minutes into debrief

R: How do you think the lesson went?

Nancy: I was impressed by the way that they (referring to the students) were able to *use their fingers and keep track of the units* [italics added] and know when to stop. Like, Felipe (pseudonym) knew that three towers of three were nine, and then he just added one more tower of three. So, seeing them either *keeping track of the towers* [italics added] (swipes a finger on her right hand over each finger of her left hand to "show" the composite units) and counting one tower at a time. Or even doubling and having them start from the three groups of three and then adding on one more was a lot more complex than I thought. I didn't think they would do that. So, I was happy to see that. I thought they might do one or two (referring to the towers) and then count on.

Nancy's motioning to mimic how she saw the students keeping track of their composite units (the swiping the finger on her right hand over each finger of her left hand) illustrates her own understanding of composite units being tracked, and how she was looking for the students to be keeping track of their composite units as well. Nancy explicitly explains that she had observed the students keeping track of their units and knew when to stop while they were counting. Implicitly, this swiping motion may also indicate her own reasoning about the distribution of items of one composite unit (cubes that make up each tower) over items of the other composite unit (number of towers). This last inference seems further supported by data in the following excerpt (about 13:40 minutes into the debrief session).

In Excerpt 4.32, Nancy takes her reasoning further by explaining to one of the researchers how, if children were skip-counting or using doubling, she would know if they were tracking multiple units. In the discussion, the researcher is asking Nancy which type of reasoning she would consider to be higher when solving an mDC task involving 6 towers of 5 cubes each. The researcher exemplifies the first way by referring to a child who holds up one finger, which they know represents a composite unit made up of five single units, then counts by fives to get the answer (5, 10, 15, 20, 25, 30). The other way of reasoning is exemplified with a child who counts the first two fives, then continues on counting by tens (5, 10; 20; 30). Nancy doesn't explicitly say which way of reasoning is higher, but she does give evidence of deeper reasoning by explaining that she would be looking for whether or not the second student knew how many towers made up the tens they were counting. That is, Nancy seems to differentiate between a child who is simply skip-counting by tens, without tracking any units, versus a child who is skip-counting while also tracking the units they are counting.

Excerpt 4.32 – *Lesson Debrief: Nancy's Noticing of Different Ways of Student Reasoning*

13:40 minutes into debrief

R: Let's say that the child now knows that this is 5 (holds up one finger). Now she has 5, 10, 15, 20, 25, 30; or she has 5, 10, 20, 30. So she goes from 5 to 10, but then she operates on the 10's continuing as opposed to operating on all of them. If she operates on the 5's or she operates on the 10's, in what way are these similar or different, and if they're different in what way is one more advanced than the other?

Nancy: I think the key was that she could still differentiate how much was in a tower, and that the 10 was 2 towers, and then she knew where to stop. In the past, when you see kids doing this, they either keep going or they get discombobulated when you ask them that type of question.

Excerpt 4.32 above indicates that, for Nancy, accrual of 1s in a multiplicative situation requires a child is also aware of the number of accruing composite units. This seems related to her own reasoning – understanding of multiplicative reasoning that involves differentiating between and simultaneous coordinating two types of units. Her response to the researcher's two ways of reasoning indicated she focused on how a student is not just finding the answer but also reasoning as they are working through the task (skip-counting without tracking units and skip-counting while tracking units). Nancy gives evidence that she sees the second way of reasoning as higher and important for student reasoning due to her own understanding of multiplicative reasoning requiring the simultaneous tracking of multiple units.

Excerpt 4.33 further illustrates Nancy's focus on the units students are operating on, rather than a focus on students getting the correct answer. This is shown in her response to one of the researcher's questions of what she and Marsha would think about a student who was tracking units, while counting by ones, but got the incorrect answer. Nancy explains that, for her, it is more important that a child be keeping track of single units and composite units, rather than getting the correct answer.

Excerpt 4.33 – *Lesson Debrief: Nancy's Noticings of Students' Units*

32:53 minutes into debrief

R1: I have a question. Let's say you have two kids, and you have 6 towers of 3. One does 1, 2, 3 (counts on one finger) 4, 5, 6 (counts on another finger) all the way to 6 towers but ends up with the answer of 17. The other one says I know that 3 times 6 is 18, and I have 6 towers, that's 18. Which one would be more, in terms of reasoning in multiplicative double counting?

Nancy: Well, I would say (to the child), how did you know?

R1: To the last kid?

Nancy: Yeah, and I would have to hear what they said.

R2: I just know, 6 times 3 is 18.

Marsha: Six towers, well what about the 6 towers?

Nancy: I think it's 19. Prove it to me.

R1: So, what about the 17?

Nancy: I mean, he's still (swipes index finger on left hand over each finger of the right hand to show the student keeping track of the towers), he could have missed one, he could have been going quick, he could have just left one off on accident. He still knew to stop; he knew how many towers to stop at. Could it have been just like a clerical error? Or did he not know how many to put in the last one?

Her own reasoning has afforded her the ability to look beyond a student's correctness when engaging in math tasks, and instead she is able to analyze how a student is operating on particular units. Nancy is not as concerned with the student's answer as she is with *how* the student is reasoning. Nancy sees the error of 17 as a possible careless error that cannot be taken as evidence of the students' reasoning. Instead, she seems more concerned with whether or not the student is tracking their composite units. This is evident when she explains that she would ask the student further questions to see if the student was tracking their composite units (she mimics this on her fingers) and knows where to stop counting, because they have reached the end of their composite unit count. For Nancy, the reasoning of the student lies in their operation on units (e.g., tracking of those units), not in the correctness of their answers. This pedagogical focus seems to reflect her own multiplicative understanding of unit distribution and the simultaneous coordination of multiple units.

Data in Excerpt 4.34 continue on from Excerpt 4.33. This is the point in the conversation where Nancy experiences a slight perturbation that causes her to temporarily question the reasoning involved in the task. In this part of the discussion, Marsha asks what if the child is counting all, which causes Nancy to accommodate the question into her understanding of counting methods, in which counting all (3 + 4 is found by starting at 1 and counting all the way up to 7) is considered lower than counting on (3 + 4 is found by starting at one of the addends and counting on from that number; start from 3 and count 4, 5, 6, 7). Nancy is very aware of this, and the question throws her for a loop, since the student in the situation was counting by ones, she began to question if this was lower reasoning or not.

Excerpt 4.34 – *Lesson Debrief: Nancy Folds-Back*

34:00 minutes into debrief

Marsha: Did he count all?

Nancy: He counted all.

R1: He did 1, 2, 3 (counts on one finger) 4, 5, 6 (counts on another finger) 7, 8, 9 (counts on a third finger) all the way to the last tower and got 17.

Nancy: Yeah, he's counting all. He's keeping track of the units, but he's counting all. He's not putting anything together, he's not starting with putting units together that he knows, and he didn't get the right answer. So why didn't he know? Was it because he was counting too quick? Or was it because he didn't know?

R2: So, let me bring a distinction, because you're really struggling with something that is really important to notice. We call it levels of units coordination. How many units is the child operating on at the same time? So, if the child did, the towers are here 1, 2, 3 (counts one tower on the table) 4, 5, 6 (counts second tower on the table) 7, 8, 9 (counts third tower on the table) 10, 11, 12 (counts fourth tower on the table) 13, 14, 15 (counts fifth tower on the table) 16, 17 (counts sixth tower on the table). Okay, the child is done counting all the ones on the cubes. Whereas there's a child who counts 1, 2, 3 (counts on one finger) 4, 5, 6 (counts on another finger) 7, 8, 9 (counts on a third finger) 10, 11, 12 (counts on a fourth finger) 13, 14, 15 (counts on fifth finger) 16, 17 (counts on sixth finger). Is there a difference between these two ways? Or are they just the same? In both instances the child is counting all the ones and making a mistake. It could be a child that counts all the way to 18 and doesn't make a mistake.

Nancy: Hold on, the first one you were doing 1, 2, 3 (counts on her index finger)

R2: They counted the cubes.

R1: They had the towers in front of them.

R2: In the other one, they counted, and they pointed to the fingers and they stopped when they were at six.

Nancy: Yes, that's more advanced.

R2: In what way?

Marsha: Because he's using his fingers to represent the cubes, instead of using the cubes themselves to count.

Nancy: It wasn't in front of him. It wasn't something that he could just look at.

R2: So, it was figural. Is there any other reason why he's higher? It is figural, and obviously by default that's a better way than just doing cubes themselves.

Nancy: Because they know how many are in each tower. They're able to say 1, 2, 3 (counts on index finger) and that's one tower. 4, 5, 6 (counts on second finger) that's another tower. 7, 8, 9 (counts on third finger) that's another tower.

Marsha's question about counting-all vs. counting-on orients Nancy to accommodate it into her (Nancy's) current scheme and briefly question the reasoning level of the student. However, Nancy's ability to still focus on the units the student is working with and reequilibrate her analysis suggests to me a Ref*AER Type II might have taken place (comparing across two instances of reasoning. With that reflection, I consider her reflection as a plausible source for moving into the anticipatory stage of mDC. This temporary lapse in Nancy's reasoning suggests that she may have briefly confounded her understanding of additive counting methods with her understanding of multiplicative reasoning but was aided by the researcher's prompt. Originally, Nancy knew the student who was counting by ones while tracking the single units and the composite units on their fingers was reasoning multiplicatively. However, when Marsha brings in the fact that the child was counting all the single units, Nancy temporarily questioned the student's reasoning, because she went back to her understanding of additive counting methods and was not thinking about multiplicative reasoning. This is apparent when she states that the child is not putting units together, and she questions why he got the incorrect answer. Marsha's question created a perturbation for Nancy. However, researcher two brings in a prompt in which he shows two different ways of counting the single units (counting all of the cubes one-by-one and counting single units within composite units, represented by the fingers). This prompt seems to help Nancy get through the perturbation, because she once again explains that the child counting by ones on their finger is still reasoning multiplicatively because they are keeping track of two units, which she mimics them doing in the final part of the excerpt. I now present for analysis the fourth workshop the teachers participated in, which occurred near the end of the first year.

Workshop #4. In Year 1, the project team engaged teachers in several workshops aimed at helping the teachers construct better multiplicative and fractional reasoning. The first three workshops did not contain much evidence on Nancy's reasoning, but the fourth workshop (held in April of 2017) gave evidence that Nancy was assimilating at least two levels of units as given, and perhaps also a third level assimilated as given. This workshop focused on a deep-dive into Tzur et al.'s (2013) multiplicative reasoning schemes (mDC, SUC, UDS, and MUC).

In Excerpt 4.35, Nancy is describing what she thought to be a big shift in her ability to analyze her students' reasoning. She explains that the big shift occurred when she was able to see if, and how, her students were keeping track of their units figuratively, either through drawings or using their fingers. She further describes this by mimicking how she would expect to see her students keeping track of the units.

Excerpt 4.35 – *Workshop: Nancy's Shift in Analyzing Student Reasoning in Her Own Words*

18:00 minutes into workshop

Nancy: I think using the hands and kind of modeling that and then pushing them into using that strategy is a really good tool to be able to see if they are being able to *keep track of both units (puts up her fingers to show how a child would keep track of both composite and single units) to be able to monitor it more effectively* [italics added]. And really see what's going on in their heads. I saw my biggest jump when they actually started to show like *one tower (points at a finger on her left hand. Counts four fingers on right hand and puts up another finger on her left hand), two towers of four is eight* [italics added]. You know? So, me learning that and pushing them to try that is where I saw the biggest jump in understanding.

18:45 minutes into workshop

Nancy: I also really liked visually the way you've showed us how to represent the towers and keep track of all the units as we're figuratively representing it has been very helpful. That's when I've seen kids really, okay, *I've got one tower of 4 cubes (mimics drawing a tower in the air), I've got 4 cubes total. I've got another tower, a second tower with four cubes (mimics drawing another tower in the air), and now I have eight* [italics added]. That has really concretely, explicitly helped them to solidify that understanding.

I infer from Excerpt 4.35 that Nancy can assimilate at least two units as given. She mimics how she would know if a child was operating on at least two levels of units using their fingers or drawings. She shows the tracking of single units on one hand and composite units on the other hand. Nancy thus also seems to see this as an important way to analyze her students' reasoning. In order for her to identify those units in her students, she would have to be assimilating at least those same units (two units) as given herself.

Excerpt 4.36 provides further evidence that Nancy knows the conceptual difference between the multiplicative reasoning schemes SUC and UDS (Tzur et al., 2013). The teachers were given a UDS problem in which they were asked to find the difference in eggs between two compilations of eggs cartons (9 cartons of 12 eggs each and 5 cartons of 12 eggs each). Nancy is able to explain how a child has to first find the difference in the egg cartons using the SUC scheme, and then find the difference in single eggs using the mDC scheme. Nancy's ability to express how UDS builds off of SUC and mDC in terms of how to operate on the composite and single units gives further evidence that Nancy is working with at least two levels of units as given.

Excerpt 4.36 – *Workshop: Nancy's Levels of Units as Given*

55:00 minutes into workshop

R: What is similar and different between SUC and UDS?

Nancy: We were talking about knowing which, depending on which unit you're talking about, which operation you're using to answer those questions...like, in the first question, who has more bags, right?... If I am strictly going to look at this from the composite unit [aspect], then I'm going to start there, and I'm going to say, well, Joe bought 9 egg cartons and then I bought 5. I know the difference between them is 4 [cartons]. So am I able to tell, am I able to use that operation, because I know composite unit and I know I can...I'm subtracting the composite units...so, *I'm relying* *on my same unit coordination to help me compare the amount of composite units here. From there I need to find the total number of eggs, so I'm working on my single or my multiplicative double counting* [italics added].

Excerpt 4.36 leads me to further infer that Nancy's own reasoning allows her to assimilate at least two levels of units as given. Nancy understands how to use SUC to find the difference in composite units, and then use mDC to find how many single units make up that difference in composite units. In both, she explicitly differentiates and coordinates composite units and 1s in them (e.g., difference in cartons and difference in eggs only within the 4 different cartons).

It is possible that Nancy also has the third level as given, into which there is a brief glimpse when Nancy quickly explains her understanding of the MUC scheme (Excerpt 4.37). In this excerpt, Nancy is explaining what she would look for in her students' reasoning if they were solving the following MUC task: "Maria has 17 bags, with 7 candies in each bag. Eric gives Maria 56 more candies. If Maria puts those candies into bags of 7 candies each, how many bags will she have altogether?"

Excerpt 4.37 – *Workshop: Nancy's Third Level of Units*

1:21:00 into workshop

R: How is MUC similar to and different from the first three schemes?

Nancy: I would look for students to be using knowledge of *keeping track of both units to take those 7 candies and spread them across, to make composite units with them until they reach 56* [italics added].

MUC requires the assimilation of three levels of units as given, because it involves a given compilation of composite units and another collection of 1s that must be organized into another compilation of composite units as a first step to then add the two compilations. The fact that Nancy is beginning to tackle this multiplicative reasoning scheme, may be evidence of her already assimilating a third level as given, or possibly in the construction of a third level as given. This is indicated in Nancy's distribution of the single units across the composite units being built as part of one compilation based on the unit rate extracted from the other compilation. I infer that for Nancy to not only operate on this task herself but also to anticipate what her students would need to do, she would need to already assimilate all three levels of units as given. Nancy understands that the students need to take the 56 single candies, distribute them into composite units of 7, and stop when they have used up the 56 candies. This type of reasoning requires three levels of units as given, because one must track the single candies, the composite units they are creating, and the sum of composite units when considering the compilation with which they began the task. To this end, they must be able to operate additively on the original composite units and the new composite units (made from the given 1s) in order to find the total amount of composite units. To be caution, I am making that claim tentatively, as this is the first piece of evidence (first time in our data) that suggests she assimilate a third unit as given. Next, I present Nancy's AOP session at the end of Year 1 for further analysis of her progression from pre-intervention to that point in time.

AOP #2. Shortly after the workshop described above, Year 1 culminated with Nancy's second AOP session (the first one being the pre-intervention AOP), which occurred in April of 2017. For the lesson part of the AOP, Nancy taught a partitive division lesson, where the students were given story problems to solve. This was a powerful session in that it provided rich evidence of Nancy's own reasoning, her levels of units coordination, and her ability to identify her students' reasoning and levels of units coordination. Furthermore, in the post-interview, Nancy immediately groups her students based on the units she saw them operating on and how they were operating on them. She also clearly articulates exactly what she saw in her students' reasoning and what that meant about their multiplicative reasoning levels. One of the most powerful parts of the post-interview occurred when she described how one of her students was operating at three levels of units while manipulating the units back and forth without losing the meaning of the units.

Excerpt 4.38 comes from the very beginning of the post-interview, in which Nancy was asked by the researcher how she thought the lesson went. Nancy immediately begins describing what she saw her students doing based on the units they were operating with and how they were operating on them (as they solved the problem of building towers of 6 cubes each from 42 cubes).

Excerpt 4.38 – *AOP Post-Interview: Nancy's Noticings of Student Reasoning*

0:10 minutes into interview

R: Talk to me a little bit about how you think the lesson went.

Nancy: I saw kids when I gave them a chance to go back to their seats and work with the manipulatives and represent their thinking. I kind of, just working at the one table that I did, I saw three things happen. I saw one student working with ones [1s], breaking up the pile of cubes into ones, the 42 cubes. Then I saw another student start with 5 in each, and then just add on what they had by ones, the remainder that they had. Then I saw Daniel [pseudonym], who went back to drawing towers and cubes in each. He originally started with 6 in each, because it was 6 towers, but he drew out, he drew 6 in each, and then he got to 7 towers and a total of 42. He knew that even though that got him to his answer, that he had sort of like represented the

> units incorrectly and then went back and flipped it. So, when I spoke to the student who put five in each [Rhianna – pseudonym], she said, "Well I knew there would be at least 5 in each, because I knew that 5 groups of 7 was 35," and then just put the last 7 into the groups, like one at a time. Then Brandi [pseudonym] was operating on just one at a time.

Excerpt 4.38 led me to infer that Nancy was assimilating three levels of units as given, because she would have to be assimilating at least the same number of units that she identified in her student. Nancy clearly describes three different ways of student reasoning that she observed within her lesson. She saw a student who operated on single units only by using her concrete manipulatives to fairly share out her cubes (places one cube at a time into each of the 6 towers). She observed another student who created composite units of 5 each, then added on the additional two cubes to each tower when she had some cubes left over. The third student she described, used a figural drawing to draw out his towers and cubes, but initially mixed up his units in the drawing. For that student, she seemed to understand that his solution of 6 towers of 7 each, or 7 towers of 6 each, represented a way of partitioning the global unit (42). That is, it seemed that for her 42 could be a unit of six units of seven and vice versa. Overall, Nancy was paying close attention to how her students were operating on their units, because she then had a clear distinction on what these each mean for her students' reasoning. In Excerpt 4.39, the researcher pushes Nancy further and asks her what these observations mean mathematically about her students.

Excerpt 4.39 – *AOP Post-Interview: Nancy's Understanding of Her Students' Mathematical Abilities*

6:10 minutes into interview

R: How do you categorize all of that thinking? What do you think, like mathematically, were they successful with?

Nancy: If I were to rank them in their understanding, I would rank Brandi [least advanced], and then Rhianna, and then Daniel. Brandi, and even when we were working on the carpet down here, was having trouble thinking outside the box of ones. She pushed herself to do three in each when it was supposed to be 6 in each down here (pointing to the floor). So, she's pushing herself to think, "Oh, I could put more into each group, instead of ones." So, she was operating on ones. Then by thinking about the fact that I knew that 5 groups of 7 is 35, so I can start there and then figure out how many I have left and divide those up, would be a next level of understanding to me...So Daniel knew he could multiply by 6 and that would get him, then he stopped at 42 and he realized that he had 7 towers. Then I see a lot of kids stop there and say, "Okay it's 7," but then my question to them is like, "Okay, but the way you represented it, you have 6 in each and then you have 7 towers. But the information we were given was that there were 6 towers." So, having them think about that, where he, that would normally be where I would push them to think about, "Okay, maybe I'm on to something, but is what I'm doing really showing the units where they are, is it really showing the correct units where they should be?" He got to that 7 towers and then he flipped it, and he knew then to go back and do one tower of 7 was 7 and go up to 6.

For Nancy, a student operating on ones is less advanced than a student who can create composite units to operate on, even if the units have been confused by the latter student. Nancy understands the importance of the composite unit in multiplicative reasoning and, because of that, is able to differentiate between her students' ways of reasoning multiplicatively. Nancy attempts to make sense of what she observed her students do in the lesson, and she is trying to decide which ways of reasoning are more advanced than others. Nancy explains that Brandi's reasoning was lower than Daniel's reasoning, because Brandi's initial, spontaneous operation on the units was with single units, while Daniel's was with composite units. Nancy explains that Brandi was struggling to move past working with ones, while Daniel was attempting to use his composite units of six to solve the task. For Nancy, her focus was not on the fact that he switched his units, but that he was able to interpret the task in a way that called for the construction of composite units of six (or seven), rather than needing to rely on single units as Brandi did. Nancy thus seems to understand that the construction of composite units is more advanced, because it requires the coordination of two levels of units (single units and composite units) rather than just one level of units (single units only).

In Excerpt 4.40, Nancy delves further into her analysis of Daniel's reasoning and how he was operating on his units.

Excerpt 4.40 – *AOP Post-Interview: Nancy's Analysis of Daniel's Reasoning*

12:56 minutes into interview

Nancy: (Referring to Daniel) I saw that he was already able to manipulate the units and see that they were all made up of the same thing, and that he used multiplication and he was able to say that "If I know that 42 divided by 6 equals 7, then I know that 6 times 7 equals 42."

22:53 minutes into interview

Nancy: Daniel showed me that he could be flexible in his thinking of all of the units and be able to see that inverse before I said it, and that's really powerful. Also, too, with his drawing, he was able to put his initial thoughts down and then switch the units and put the correct numeric label with the unit. And that's huge.

I infer that Nancy is possibly assimilating three levels of units as given, which afforded her the ability to identify those same units in her student's reasoning. Nancy described the way in which Daniel was able to manipulate his units without losing the meaning of those units (the single units of 6 distributed over 7 composite units to create a third unit of 42, and vice versa), which she believes is powerful in-regards-to his reasoning. For Nancy, this ability means that he understands the units well enough to be flexible as he is operating on them. He can switch them back and forth, but not lose the meaning of the composite unit nor the single unit within the problem. This is evidence of three levels of units as given, which Nancy would also need in order to identify it in her student, because she sees within his reasoning the single units of 6 (or 7) distributed over the composite units of 7 (or 6) to create the third unit of 42. She also seems to understand that the ability to think of these units flexibly adds to the higher-level reasoning, because Daniel is not losing sight of the three units as he switches between them. Nancy seemed to infer this when she saw that he could accurately label his units no matter how he switched them. The ability to flexibly coordinate and switch between structures while taking all three levels as given is a sign that an individual is operating within Stage 3 of units coordination (Norton & Boyce, 2015; Norton et al., 2015). The focus on my claim is not on Daniel's ways of reasoning, but on Nancy's ability to identify the flexibility within Daniel's assimilation of the units. For her to notice that flexibility would require her to have that same flexibility and to be able to switch between the structures herself.

Summary of Year 1

Throughout Year 1, Nancy progressed in her own reasoning, as well as her ability to analyze her students' reasoning. She began the year having already made a transition from describing multiplication as the repeated addition of equal groups, in the pre-intervention AOP (Excerpt 4.26), to multiplication being the distribution of units within other units (Excerpt 4.28). As she progresses, she adds onto this understanding, eventually being able to explain the differences between Tzur et al.'s multiplicative reasoning schemes (Excerpt 4.36). This growth in her understanding affords her the ability to identify her students' reasoning as they are constructing each of the schemes as well. She knows what her goals are for student learning throughout these schemes, and she is able to make determinations about how they are reasoning in each multiplicative task they are given.

Nancy's levels of units coordination also seem to be developing throughout the year. She begins the year giving evidence that she may be assimilating two levels of units as given (Excerpt 4.27), with the third unit possibly being coordinated in activity. By the end of the year, she was beginning to assimilate that third unit as given (Excerpt 4.38). It is possible that she had that third unit as given from the beginning of the year, but there is not sufficient evidence to make that claim. It is towards the end of the year that the evidence for this claim comes into play. This progression in Nancy's levels of units coordination seemed to underlie her focus on identifying the units her students are working with and how they are operating on them.

As the year progresses, so does Nancy's ability to identify her students' levels of units. At the beginning of the year, she is able to identify when students are operating with composite units, but she does not yet differentiate between those students and students who are operating only on single units (1s) (Excerpts 4.29 and 4.30). She also does not seem to focus on whether or not students are keeping track of their units. By January of the first year, Nancy shifts into analyzing how students are keeping track of both their single units and their composite units as they engage in their multiplication tasks (Excerpt 4.31). By the end of the year, Nancy is really focusing in on her students' levels of units, how they are operating on those units, and what that means about their mathematical reasoning (Excerpts 4.38 - 4.40). In conclusion, as Nancy's multiplicative reasoning and levels of units progressed, so did her ability to analyze her students' ways of reasoning multiplicatively and their levels of units being operated on.

Growth in Nancy's Reasoning: Year 2

In Year 2, Nancy shifted into fractional reasoning as her teaching also shifted to helping her students construct fractional reasoning schemes. Nancy "looped" with her grade-3 students from Year 1 and taught them again in Year 2 as fourth graders. She had spent all of Year 1 teaching lessons with tasks that promoted her students' construction of the multiplicative reasoning schemes. She wanted to continue moving along this progression with the students, so her focus for Year 2 was on fractional reasoning. In order to effectively help her students, Nancy also needed to work on her own fractional reasoning. Therefore, in addition to the project's work on fractions during Summer Institute 2, one of the AdPed researchers gave her extra coaching sessions in Year 2. During these coaching sessions, Nancy and the researcher would co-teach lessons that involved fractional reasoning tasks. Nancy also began joining the fifth-grade team for some of the grade-level workshops, since the fifth-grade team was getting professional development specifically focused on the fractional reasoning schemes.

In Year 2, there is more evidence of Nancy assimilating three levels of units as given, while she works through her own and her students' fractional reasoning. She also continues to focus her analyses of students on the units they are working with and how they are operating on those units. However, with her teaching shift to fractional reasoning schemes, her analyses also shift to the fractional units students are operating on. Nancy's progression through Year 2 begins with one last workshop associated with multiplicative reasoning, then moves onto her grappling with the first two fractional reasoning schemes of equi-partitioning and partitive fractions. She continues on with the construction of the more advanced fractional reasoning schemes and ends the year at the Unit Composition Fraction scheme. I begin the analysis of Year 2 with a workshop that occurred in the middle of the first semester of the school year.

Workshop #6. In October of Year 2 (2017), the AdPed team conducted another grade-level workshop for the participating teachers. Based on teachers' specific request, the workshop's focus for Nancy's team was on using UDS to solve multi-digit multiplication problems through the distributive property. The following excerpt illustrates Nancy's levels of units coordination as she worked through the first task presented to the teachers. In Excerpt 4.41, the teachers have been asked how they solved a task in which they were asked to build three towers of six cubes each (four of the cubes in each tower were one color, and the other two cubes were another color) (Figure 4.5). They were then asked to find the total number of cubes. The purpose of the task was to prompt the teachers to decompose the towers of 6 into new composite units for multiplying. Excerpt 4.41 presents Nancy's explanations of how she solved this task.

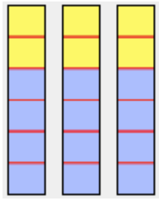

Figure 4.5 - Three towers of six cubes, each composed of 4 blue and 2 yellow cubes.

Excerpt 4.41 – *Workshop: Nancy's Reasoning with Three Levels of Units*

37:00 minutes into workshop

R: Were the two white ones on top of the four any hint for you? Any reason it was helpful?

Nancy: The colors allowed me to look at this and see 3 groups of 4 and then 3 groups of 2, 3 towers of 4 and 3 towers of 2. So, I didn't decompose anything, but I looked at it differently. I saw that 3 groups of 4 were 12 and 3 groups of 2 were 6. So, I put those together and that's 18.

R1: Did you see it as 12 plus 6 at that point, or you went back and saw it as 3 towers of 6 to get to 18?

Nancy: I would say that I saw it both ways. *I could see that there would be 3 towers of 4, and another 3 towers of 2, but then in the end I would also see that it is still 3 towers of 6 (makes a circling motion around the 3 towers)* [italics added].

Tracy: Would she have thought about it if the colors wouldn't have been separated like that? (Asking about Nancy's thinking)

Nancy: But I do think about that when numbers get larger. Just in my head.

R2: Can you give an example?

Nancy: So, I'm thinking of like 3 towers of 12, right? So, *if these were 12, I would [do the] same kind of thing, like 3 towers of 10 and then 3 towers of 2. But I know that is 3 towers of 10 plus 3 towers of 2, but I also know that is 3 towers of 12* [italics added].

Excerpt 4.41 leads me to infer that Nancy is assimilating three levels of units as given and has constructed the UDS scheme at the anticipatory level. Nancy's spontaneous activity sequence for this task provides evidence that she is fluent in her ability to decompose the initial composite units (three composite units of 6) and create new composite units (three composite units of 4 and three composite units of 2) in order to multiply the new composite units of 6 and 12 to find the total number of cubes (18). She does not need to do this in activity, and her ability to do this without losing sight of the original composite units (three composite units of 6) suggests that she is assimilating three levels of units as given, while solving a UDS task without any prompting. The ability to create these recompositions in her head without losing sight of that initial composition is inferred to involve assimilation of units at Stage 3, because the units are being assimilated as given, rather than in activity, and she is able to switch between the assimilated structures (moving from composite unit of 6 to multiple composite units of 4 and 2). Furthermore, she initiates a transfer of the given situation to one involving a unit of 10 and two units of 1 (i.e., 12) taken three times – further indicating her anticipatory stage of coordinating three levels of units. Similarly, Excerpts 4.42 and 4.43 give two more examples of Nancy assimilating three levels of units as given, but with a new situation. In this task, the teachers were asked to build five towers of 13 cubes each, with 10 of the cubes being one color, and the remaining 3 cubes being another color (Figure 4.6).

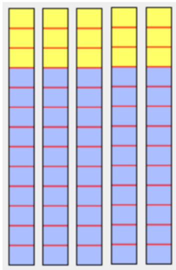

Figure 4.6 - Five towers of thirteen cubes, each composed of 10 blue and 3 yellow cubes.

Excerpt 4.42 – *Workshop: Further Evidence of Nancy's Three Levels of Units as Given*

58:00 minutes into workshop

R: How did you get your answer?

Nancy: I did the 5 groups of 10 is 50 or 5 towers of 10 is 50. And 5 towers of 3 is 15, and then put them together. And then I thought, 5 towers of 5 is 25 and 5 towers of 5 is 25 and 5 towers of 3 is 15 and I put those together.

Excerpt 4.43

1:40:00 into workshop

R: How many cubes do you have altogether?

Marsha: I don't know why I didn't do 10 times 12. I did 10 times 6 is 60 and 10 times 6 is 60. I did 120 and then I did 3 times 12 is 36. I added the 120 to the 36.

Nancy: I thought of it as 10 towers of 12 and then I thought of it as 5 towers of 3 is 15, 5 towers of 3 is another 15 and then 2 towers of 3 was 6. So, 2 groups of 15 are 30 and 2 groups of 3 are 6.

I again claim that Nancy is assimilating three levels of units as given while working through a UDS task at the anticipatory level. When teachers were asked how they solved this task, Nancy's response again shows her ability to decompose the initial composite units (five composite units of 13) in order to make new composite units (five composite units of 10 and five composite units of 3) that are easier to multiply in her head, and she does this without losing sight of the original five composite units of 13. In this sense, Excerpts 4.42 and 4.43 give further evidence as to how Nancy is assimilating the units as given. She is able, at will and in anticipation prior to any activity, to decompose the initial composite units in multiple ways to create new composite units, which she can multiply in her head. She seems to take the units as given, while readily switching between different compositions of units (composite units of 13 to composite units of 10 and 3). This way of reasoning with whole numbers seemed to also be present in the following workshop, in which there was evidence into Nancy's transition from multiplicative reasoning to fractional reasoning.

Workshop #7. A month later (November 2017), Nancy participated with the grade-5 teachers in another grade-level workshop. The focus of this one was on the equi-partitioning fraction scheme and playing the French Fry game (Tzur & Hunt, 2015). At this point, Nancy had already taught the French Fry game in her classroom, and played it herself during Summer Institute 2, so she had already begun to construct the equi-partitioning scheme. Excerpt 4.44 occurred towards the beginning of the workshop, when the teachers were asked to explain what a fraction is while using 1/4 as an example. In this excerpt, Nancy struggles with the language and the use of the term "whole," but her explanation does give evidence that she understands fractions as a multiplicative relationship, rather than part-of-a-whole.

Excerpt 4.44 – *Workshop: A Glimpse into Nancy's Fractional Reasoning*

26:22 minutes into workshop

R: What is a fraction, and can you give an example with 1/4? Write it down. We are going to share. (Gives the teachers time to work on this) Are we ready to share?

Nancy: I don't know how to get away, *I don't remember how we get away from this idea of whole* [italics added]. But like I drew the fraction bar with four pieces, and I said that *the whole is four times as large or that that is how many times that can be like iterated into the whole (Uses her fingers to show a piece being iterated in the air), or fits into the whole, or repeated into the whole* [italics added]. But I keep saying whole.

R: Okay, did you draw something?

Nancy: I just drew like, if this is 1/4 (points at a piece she drew) then *the whole is 4 times as large* [italics added], I don't even know if that's right, but I know that this (points to the whole she drew) is how many times that this (points to the piece she drew) can be iterated into it or *I would do that (move her fingers four times in a row to show the iteration) four times* [italics added].

There is evidence from this excerpt that Nancy understands fractions as a multiplicative relationship. However, there are moments in the excerpt where Nancy struggles with the language to use when describing her reasoning. Throughout the workshop, Nancy tries to avoid using the term "whole," due to her assimilation of fractions not being part-of-a-whole. Nancy had been told multiple times that fractions are not part-of-a-whole, but instead a multiplicative relationship between the whole and the fractional piece. Interestingly, Nancy seemed to assimilate this to mean that she could not use the term "whole" when describing fractions or her reasoning. However, despite that setback in language use, Nancy's descriptions of her reasoning do show that she conceptually understood fractions as a multiplicative relationship.
Her explanation of what a fraction is, using the 1/4-unit piece, gives evidence that she does conceptually understand the multiplicative relationship involved in the task. Nancy explains that

the fractional piece is 1/4, because that piece can be iterated four times to create the whole. She also mimics this with her fingers to accurately show what iterating looks like in this context. The mimicking of the iteration of the 1/4 piece suggests that she "sees" (in her mind's eye) the iteration of the piece as the piece being multiplied four times to recreate (or fit within) the whole. This requires her to understand the relationship of the whole (composite unit) being made up of fractional unit pieces (single units) of 1/4.

Moving along to the following task, Excerpt 4.45, the teachers were shown a picture (Figure 4.7). The researcher told them that Rectangle A is ½ in size of the larger, Rectangle B, that Rectangle A is partitioned into four equal parts (by diagonals), and that the triangle within the blue part of Rectangle B is equal in size and shape (a copy of) the yellow triangle within Rectangle A. The researcher then asked: What fraction is the yellow part of Rectangle A? What fraction is the blue part of Rectangle B? Which fraction is larger, 1/6 or 1/4? What fraction is the yellow part of Rectangle B? The researchers provided the teachers with this task, because in the past, they had teachers who struggled with this task if their fractional reasoning was based on a part-of-whole understanding. The teachers who struggled with this in the past had not been able to say what fraction the yellow part was of Rectangle B since the yellow part was not inside Rectangle B, or not a part of that whole.

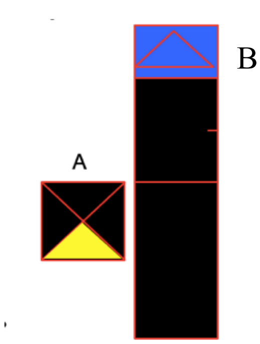

Figure 4.7 - Fraction Task

Excerpt 4.45 – *Further Evidence of Nancy's Fractional Reasoning*

39:51 minutes into workshop

R: Nancy, did you get 1/6? (Referring to the second question)

Nancy: (Shakes head yes)

R: Why?

Kelly: For the same reason. You can take that piece 1, 2, 3, 4, 5, 6 (Moves her hand down the rectangle as though counting the six iterations of the blue rectangle). Six times.

Nancy: *That's how many times it will fit inside the whole, and you can iterate that six times and it will make the, and I know the word whole, I know that I'm using that* [italics added], but…

Marsha: Can we say total?

Kelly: Into the compared piece?

Nancy: Of the piece.

R1: You can say whole. It's fine.

R2: You can say whole, yeah.

Nancy: Okay. *So that's how many times the piece will fit into the whole. Or that's how many times I can iterate that piece to give me that whole rectangle B* [italics added].

From these excerpts, I infer that Nancy had already constructed the equi-partitioning scheme at the anticipatory stage from the work that was done in Summer Institute 2 and the teaching she had already done on the French Fry game. Here in Excerpt 4.45, Nancy struggles with the term "whole," while still being able to accurately express fractions as a multiplicative relationship. Her explanation that the 1/6 piece can be iterated six times to "fit into the whole," suggests that she understands the relationship between the whole and the fractional unit piece, rather than it just being one piece out six pieces (part-of-whole reasoning). The mimicking again gives evidence into her reasoning about the fractional unit and its ability to be iterated within (or to recreate) the whole. The fractional piece (single unit) can be used to create the whole (composite unit) and that this is done through the iteration of any fractional unit piece. Accordingly, I infer that, at this point in her growth, Nancy has constructed an anticipatory stage of the equipartitioning scheme.

Workshop #8. The eighth workshop was held in January of the second year and focused on the Mixed Unit Coordination (MUC) multiplicative reasoning scheme. The teachers were given the following MUC task to solve in a place value, base ten (PVB10) context:

> Schools get single apples, bags of apples, and boxes of apples. Each bag has 10 apples. Each box has 10 bags. School A has 4 boxes + 6 bags + 19 single apples. School B has 3 boxes + 15 bags + 11 apples. School A has ___ apples in all. School B has ___ apples in all. Which school has more apples? How many more apples does that school have?

This task requires the assimilation of three levels of units as given, in order to determine the total amount of apples. The boxes of apples require one to assimilate the ten bags of ten apples each (100 apples total), which is three levels of units – 10 single apples (level 1) make up a bag (level 2) and 10 bags make up a box (level 3). In Excerpt 4.46, Nancy's reasoning as she worked through the task presented to her and the other teachers is observed.

Excerpt 4.46 – *Workshop: Nancy's Reasoning Within the Mixed Unit Coordination Scheme*

10:00 minutes into workshop

R1: What did you do Nancy?

Nancy: Well, I first knew I needed to figure out how many were in each box. So, *if there were 10 bags of 10 each, I knew that 1 box had 100* [italics added]. So, 4 boxes with a hundred in each would be 400 boxes. And then I knew there were 10 in each bag, and I had 6 bags, so I had 60 within the bags. And then 19 single apples. So, I combined them.

R2: I heard you say 400 boxes.

Nancy: 400 apples.

R2: And 60 what?

Nancy: So, *I had 400 apples that were inside the boxes. I had a hundred in each, and I had 4 boxes. So, 4 boxes of 100 is 400* [italics added].

R2: And you said 60. What were the 60?

Nancy: So, I had 6 bags and I had 10 in each, so I had 60 apples. And I had 19 singles. So, when you combine all of that it's 479.

In Excerpt 4.46, the evidence provides a glimpse into Nancy's ability to assimilate all three levels of these units as given, because when solving the task, she works it out in her head and explained all three levels in relation to one another. She explains that each box had 100 apples (total amount of single units found through the compilation of composite units), because each box had 10 bags (composite units), and each bag had 10 apples (single units). Nancy did not need to work through the three levels in activity, as she was able to assimilate all three as given. From there, she was able to combine the total of 400 apples in the boxes with the apples in the remaining bags and the single apples.

Nancy's assimilation of three units as given, as well as her ability to solve the task without any prompting also leads me to infer that she has constructed the MUC scheme at the anticipatory level. She does make a simple, "slip-of-the-tongue" mistake in her explanation, but I infer that it was due to her becoming a little confused with all of the numbers in her explanation, rather than not having an understanding of the task nor the inability to assimilate the three levels of units. This inference is due to the fact that when researcher 2 repeats what she said, she quickly corrects herself. The next segment of analysis (buddy-pair #8) again examines Nancy's fractional reasoning, specifically her construction of the reversible fraction scheme – in which three levels of units coordination is expected.

Buddy-Pair #8. In January of Year 2, Nancy had a buddy-pair session in which her teammate (Tracy) was supposed to be present. However, Tracy was sick that day, so the session consisted of just Nancy and one of the researchers. In this session, Nancy and the research co-taught an iterative fraction lesson. The debrief of the session was interesting, because Nancy struggles with a reversible fraction scheme task that the researcher asked her to solve. The researcher asks Nancy to recreate the whole from the non-unit fraction of 5/7 (Excerpt 4.47).

Excerpt 4.47 – *Nancy's Reversible Fractions Scheme*

27:21 minutes into workshop

R: I'll show you an example of a reversible problem. This is 5/7 of a chocolate bar (draws an unpartitioned bar on a sheet of paper). Can you show me what the whole chocolate bar was? What would you do?

Nancy: I would break this up. So, I know that I need to have 7 total pieces. I have 5 of the, well I know that the whole would be 7 total pieces, and that I have 5 of them. So, I would create a whole that had 2 more pieces.

R: How would you determine the size of those 2 more pieces?

Nancy: I would first decide the size of the pieces in here (points to the unpartitioned bar) and then add on two more.

R: You would draw a whole new piece, or you would add them onto this one?

Nancy: Was the question, "Draw the whole??

R: (Shakes head yes).

Nancy: I would draw a new one.

R: By doing what?

N: By (thinks for a moment), how would I do that? (Thinks for a long time). *How do you determine exactly how big? (Partitions the bar into five pieces). I don't know. How do you determine exactly how big it would be?* [italics added] That much bigger?

R: So, what did you just do here with those lines?

Nancy: I guess I made, well I would need to make 7.

R: So, what do you have here?

Nancy: *Fifths* [italics added].

R: You have fifths?

Nancy: Yeah. *Well no, I have 5 pieces of sevenths* [italics added].

R: You have 5/7?

Nancy: Yeah.

R: So, if you take one of these pieces…

N: Right, and I would add two more. One more seventh.

R: Just one of them would be?

Nancy: 1/7

R: Okay, so then how would you create the whole?

Nancy: I would add two more sevenths onto the end.

R: Or you could do? Is there a different way?

Nancy: I could add two more over here (points to the other end of the bar).

R: Okay, is there a different way?

Nancy: What? I don't know.

R: I could take one of these sevenths and repeat it seven times.

Nancy: Oh sure! Yeah.

R: The unit fraction.

Nancy: Right!

The need for prompting causes me to infer that Nancy is at the participatory level of understanding within the reversible fraction scheme. At first, Nancy knows that she needs to partition the given bar into five pieces and create a new whole with two more pieces. However, when asked how big those pieces should be, she gets stuck in how to determine that. Initially, Nancy erroneously thinks to partition the given composite fraction (5/7) into seven equal parts. After some more thinking, Nancy decides and the physically partition the bar in order to begin thinking about what to do next. She seems to fall back on two levels of units as given and has to partition the bar (in activity) in order to assimilate the third unit. However, Nancy is still not able to determine the size of the unit fraction, even after trying to find it in activity. So, the researcher attempts to prompt Nancy's reasoning further.

Nancy briefly explains that she would create an entirely new whole, but when she could not figure out how to do that, she changes her thinking and states that she would simply add two more pieces onto the given bar to create the whole. She is not able to assimilate the task in a way that would allow her to disembed the unit fraction piece, which she could iterate 7 times to create the whole. Once the researcher gives her some prompts, she is able to agree with those prompts, but never assimilates that into the task herself. I infer that the complexity of this task caused

Nancy to fall back on two levels of units as given and the third level only in activity (the need to physically partition the bar). Even through the assimilation of those units, Nancy required prompting in order to figure out what to do with those units. To further analyze Nancy's construction of fractional reasoning, I present Nancy's third AOP session, which was focused on the teaching of the reversible fraction scheme.

AOP #3. In the spring of Year 2, the researchers conducted a full set AOP with Nancy. The full set was comprised of a pre-interview, an observation of Nancy teaching, an interim interview, another observation of Nancy teaching, and a final post-interview. For the lessons, Nancy chose to teach the reversible fraction scheme. Her goal for the students was for them to recreate a whole pizza when given a proper (composite) fraction of the pizza. Unlike in Excerpt 4.47, throughout the interviews Nancy seems able to accurately explain what the reversible fraction scheme is. In Excerpt 4.48, Nancy accurately explains what a child should do if they are reasoning within the reversible fraction scheme – which indicates her own fractional reasoning at three levels of units coordination.

Excerpt 4.48 – *AOP Pre-Interview: Nancy's Reversible Fraction Scheme*

9:30 minutes into interview

R: I want to go back to the sequencing of the lesson and talk a little bit more about today. You stated that the goal was to create the whole pizza from a portion of the pizza. Go through what you kind of see the sequencing is, maybe that help me better understand the goal for the students.

Nancy: So, like this is a scaffold to me in my understanding to get them to the point where you could give them just a piece of the fraction and say that this is 3/8 (draws a bar that is partitioned into 3 pieces) and then they could use that to create the whole.

R: What would you expect them to do, someone who was successful at this, what would you expect them to do?

Nancy: The goal is for them to then *take this portion and from there create the unit fraction and iterate it again to have the eighths* [italics added].

R: Show me exactly what a successful kid would do. So, you drew two lines.

Nancy: So, *I drew three eighths, and so I know that each one of these is one-eighth, so I would continue to draw that (iterates one of the pieces eight times on her paper) to create the whole* [italics added].

R: And you would just draw them any size?

Nancy: No, I would want them to be the size of the unit fraction.

R: So, the size of the unit fraction is determined by them splitting the 3/8?

N: Yes

R: And then they would use that to…

Nancy: They would use this (points at one piece of the 3/8) to iterate that unit fraction until they had the whole, 8/8.

R: Any other way they might do it? Or that's your goal, this is how you want them to do it?

Nancy: That's how I would hope they would do it, because *they have to maintain the fact that this size (points to one piece of the 3/8) never changes. They're using the size of this piece to demonstrate that they know that that is the size of the piece that will fit into the whole 8 times* [italics added].

R: Walk me through how you want to get them there.

Nancy: (Points to her computer screen, which has the Fraction Bars software open. At the top is a long bar that represents the whole. Below that is a 1/10 fraction of the bar, a 2/10 fraction of the bar, a 3/10 fraction of the bar, and a 5/10 fraction of the bar. Each bar is already partitioned for the students[1]). We start out by having the whole pizza here, so providing the whole for them. Then asking them if they can use the portions provided here to create, so the idea right there is the pizza store, and that we can buy pizzas in 1/10, 2/10, 3/10, and 5/10. But we have to buy the same portion size to create the whole. How could we use these portion sizes to create the whole?

R: What do you expect they will do for this?

Nancy: I think they would look at the 5/10 and know that, because we've been working with the partitive and they're been able to add the fractions together or multiply the fractions to get the whole, I think they'll look at the 5/10 and say, if I do that times 2 or if I do that two times, that I'll get the whole. Or I could equally see them saying the 1/10, if I do that times 10, I'll have the whole.

R: Any of them you think would say the 2/10, or no?

Nancy: Yeah.

R: Of the 3/10?

Nancy: I don't think they would go there first. I think they'll be wigged out, because it's split already. But I think they might go there and be like, okay I could do that. Or I don't know what they'll do. But I think they'll gravitate towards these two (points at 1/10 and 5/10) because it's easy to just double that one and we've been working with the unit fraction so much. And then I think they would go, I would say, are there any

[1] This task was introduced to the teachers in Summer Institute 2, as part of learning to reason and teach the reversible fraction scheme. It was created by Tzur (1996, 2004).

> other portions we could use to create the whole? I don't know what they would do, but I think they would go to the 2/10 before they would go to the 3/10. They could say we can't with the 3/10, or they might say because we've been doing this (points back to the board from yesterday's lesson on iterative fractions) they might say, because we've been going over the whole, they might say, well I could use 3/10 three times and then buy one more and split the 3/10 and use one-tenth to create the whole and have 2/10 left over. That would be awesome.

Nancy's ability to fully explain the reversible fraction scheme, leads me to believe that Nancy is possibly at the anticipatory stage (with three levels of units coordination). Her explanation on how to use a 3/8 fraction piece to recreate the whole is evidence that she herself has the conceptual understanding of the scheme and is able to assimilate three levels of units within the task. She explains that the 3/8 piece (composite unit) would need to be partitioned into three pieces (single units) in order to disembed one piece that can be iterated eight times to create the whole (larger composite unit). Furthermore, she has the ability to understand the importance of examining how the students are working with the units within the task. Nancy is also able to predict how her students might reason through this task, based on her own understanding of fractional reasoning and her ability to assimilate three levels of units as given.

Excerpt 4.49 comes from the interim interview that was done between lesson one and lesson two of the AOP set. In this excerpt, Nancy is explaining to the researcher how she thought her lesson went (in teaching the reversible fraction scheme) and what she saw some of her students do during the lesson.

Excerpt 4.49 – *AOP Interim Interview: Nancy's Explanation of Unit Tracking*

0:30 minutes into interview

R: How do you think the lesson went?

Nancy: There were a good number of students who were able to see that the 5/10, by doubling that two times, I think it was Amy [pseudonym] who did that, created the whole. And using the 2/10 they could also create the whole by doing that five times, times five. So, *for them to be keeping track of all of those units* [italics added], I kind of was trying to get them to, they were able to see, they knew how many they needed to make the whole. Then they knew how many of that portion they needed to make that whole.

R: When you say how many they needed to make the whole?

Nancy: They knew that 10/10 made the whole, and they knew that if they doubled the 5/10 that they would make the whole. So, *they were able to keep track of all of those units and not lose sight of what made the whole* [italics added]. So, I felt like that was pretty fluid for the ones I spoke to.

For Nancy, students' ability to track multiple units within the task seems an important criterion needed to solve the task. Nancy is expressing how important it was for the students to be able to keep track of their units and not lose sight of the whole, which reflects her own understanding of the scheme and the importance of tracking the units involved (e.g., the tracking of the composite fractional units, like 2/10, as they are iterated to recreate the whole). She explains that the students knew the whole was made up of 10 one-tenth pieces, and from there the students were able to determine how many of each portion (1/10, 2/10, 3/10, 5/10) they needed in order to create the whole. Her ability to notice this reasoning in her students suggests she, herself, understands that this task involved three levels of units that needed to be tracked in order to recreate the whole, which was made up of composite fractional units, which are made up of single fractional units. Specifically, one level of units here is the 10/10 whole, the other level is 1/10 of that whole (a multiplicative relation for her), and the third level is the 2/10 as a unit composed of two 1/10ths units.

The interview continued into Nancy's reasoning about the unit fraction of 1/10. Again, in Excerpt 4.50, she explains the importance of the students' ability to track three levels of units simultaneously. In the lesson, Nancy never brought the students' attention to the unit fraction of 1/10. When the researcher asked why she did not, Nancy explains that she actually should have done so.

Excerpt 4.50 – *AOP Interim Interview: Nancy's Further Explanation of Unit Tracking*

21:56 minutes into interview

R: Why did you feel you should have?

Nancy: *To come back to the unit fraction so that they're tracking all three, or that they're tracking the whole, the unit fraction, and then the non-unit fraction* [italics added].

R: Why would that matter for the ultimate goal that you are getting to?

Nancy: I feel like that helps them think about this (points to the exit ticket – a bar partitioned into 7 equal parts and designated as "7/10" with the task of using it to reproduce the whole).

R: How?

Nancy: Because they're having to track all of those units while doing a task like this.

R: What units are they tracking?

Nancy: Everything goes back to the unit fraction, and then the whole. Right?

R: For you. For them? That's what you wanted them to do?

Nancy: Well in my mind, when I think about what I want them to do, like you need to be aware of all of those things simultaneously to do something like this. Right?

R: Do you think so?

Nancy: I think so. *Because if they don't understand the unit fraction, they don't understand how to reach the whole. If they don't understand the whole, then they don't understand how they can break things up and make the whole, and what they have left over* [italics added].

Here again, there is evidence that Nancy's own assimilation of three levels of units as given affords her the ability to also understand the importance for her students doing this as well when engaging in fractional reasoning at this level. She understands that they need to track all three units simultaneously (n/n-whole, 1/n, and m/n composite) in order to be successful with this task and other tasks like it. This is evident when she states she needed to go back to the unit fraction, so the students were "tracking all three, or that they're tracking the whole, the unit fraction, and then the non-unit fraction." She further explains the relationships between the units when she states, "if they don't understand the unit fraction, they don't understand how to reach the whole. If they don't understand the whole, then they don't understand how they can break things up and make the whole, and what they have leftover." She seems to believe that the students need to know how the unit fraction fits within the whole (it can be iterated x number of times to create the whole), in order to then understand how the proper, composite fractions (made up of the unit fraction) can also be used to create the whole (iteration of composite units).

Nancy realizes that by not bringing the students' attention to the unit fraction (1/10), they may have been able to solve the task she presented to them without the unit fraction, thereby only needing to operate on two levels of units (the composite units of 2/10 or 5/10 and the whole), because they just needed to know how many times 2 or 5 went into 10. They did not necessarily need to assimilate the single units within the proper fractions and could simply add them using whole number reasoning. This may have led the students to engage in the task while not meeting her ultimate goal for them – the tracking of three levels of units simultaneously. This is an important distinction that Nancy makes. She not only noticed how students were operating on the units, but she realized the limitation in the task if only operating on the composite fractions (2/10, 5/10) and its inability to elicit the reasoning she wanted from the students. In turn, her clarity of this missing link indicates her own reasoning of all three levels of units involved in a reversible fraction task.

Excerpt 4.51 comes from the post-interview that occurred after the second lesson of that same AOP data set. It was not part of the actual interview but was an extra coaching session the researcher did with Nancy after the interview was over. The researcher and Nancy began chatting about the lesson from the day before (the first lesson of the AOP set), specifically discussing the exit ticket Nancy had given her students. This exit ticket had a bar partitioned into 7 pieces, which was 7/10 of the whole, and the students were asked to recreate the whole.

Nancy and the researcher decided to examine two of the students' solutions. One student, Alan (pseudonym), took the 7/10 and added three more pieces onto the end of the bar. Another student, Dana (pseudonym), disembedded one of the pieces from the 7/10 and created the whole by iterating the 1/10 piece ten times. The previous day, Nancy had thought that Alan was higher in his reasoning, because she thought he was using a counting-on strategy, while Dana was using a counting-all strategy. Essentially, Nancy had reverted back to her understanding of whole number reasoning and determined that Alan was higher due to counting-all being a higher conceptual level than counting-all. However, later that day, Nancy discussed this with one of her fellow teachers, which led Nancy to have a transition in her understanding of her students' reasoning.

Excerpt 4.51 – *AOP Post-Interview Coaching: Nancy's A-Ha Moment*

43:00 minutes into interview

R: Alan drew three more. Dana draws an additional 10. The question was if there is a difference or whether one was higher than the other. What do you think?

Nancy: I went to her (referring to the other teacher) and I said, I think there's a difference. I think that, we just kind of hashed it out a little...Thinking about it more last night, I think you're thinking about just adding on, to me that says, I think the unit fraction's one piece of the whole. So, if I add on another 3 I'll have the whole. *Doesn't tell me that they know, that they have the concept of the unit fraction. It tells me that they know how to add 7/10 plus 3/10, or that 7 plus 3 is 10* [italics added].

R: In your conversation with Mona (the other teacher), then why was this (points to Dana's work) an indication that they had the unit fraction? Because they could pull it out and repeat it?

Nancy: Because I was just thinking about the fact that *the end goal is for them to track all three units. So, they're not necessarily doing that by just adding on 3 more* [italics added].

R: If we compare Alan to what Dana did yesterday, I would say that I agree with what you and Mona both determined. Dana is higher, because Alan is just seeing it as I have 7, I just need to add 3 more and I have 10. Whereas Dana can see this as 7/10. It is already marked for her, but she can also see it as a [reversibly] decomposed 7/10, take one of those tenths, and compose a [10/10] whole from it.

Nancy: Yeah, *she's showing all three units there* [italics added].

This change in thinking about Dana's work was quite an "aha" moment for Nancy and gives further evidence into her understanding about the importance of tracking three levels of units in a reversible fraction scheme task. She herself noted that, originally, she thought Alan's solution indicated a higher level of reasoning, because in whole number reasoning, counting-on is higher reasoning than counting-all. However, after discussing it with her colleague Mona and doing her own reflection, Nancy comes to the realization that whole number reasoning cannot be used in this situation to determine which student is reasoning at a higher level. She explains that simply adding three more pieces onto the end of the 7/10 bar does not tell her enough about their reasoning, "that they have the concept of the unit fraction. It tells me that they know how to add 7/10 plus 3/10, or that 7 plus 3 is 10." Instead, she realizes that Dana is actually reasoning at the higher level because her way of reasoning requires an assimilation of all three levels of fractional units. In particular, Nancy seems to understand that this way of reasoning required Dana to assimilate 7/10 as a composite fractional unit of seven 1/10 units, which could be partitioned and one 1/10 piece disembedded in order to iterate it ten times to create the whole. For Nancy to come to this realization suggests that she is also assimilating the three levels of units within the task and can use her own reasoning to analyze how Dana and Alan were reasoning.

In all, this full AOP data set provided ample evidence into Nancy's anticipatory stages of the partitive and reversible fraction schemes, as well as her ability to examine her students' ways of reasoning, including the three levels of units they are working with and how they keep track of those units. Her assimilation of three levels of units does afford her the ability to analyze her students' levels of units coordination and make determinations, which she can also reorganize, about their level of reasoning based on the evidence she gets from the students. Not long after this AOP occurred, Nancy participated in her final buddy-pair session (buddy-pair #9), which further delves into her fractional reasoning.

Buddy Pair #9. Nancy's final buddy pair session occurred in March of Year 2. For this buddy pair, Nancy was paired up with her teammate Tracy once again. Nancy and one of the researchers co-taught a lesson on recursive partitioning to Nancy's class. During the debrief, the researcher ended up leading Nancy down a discussion about equivalent fractions. The researcher asked Nancy to explain why 1/4 and 3/12 are equivalent (Excerpt 4.52). At first, Nancy really struggles with the reasoning, but through prompting from the researcher, she eventually constructs a better understanding and a mathematically sensible explanation.

Excerpt 4.52 – *Debrief: Nancy's Recursive Partitioning Scheme at the Participatory Level*

8:28 minutes into debrief

R: Why are 3/12 and 1/4 equivalent?

Nancy: If I take this 1/4 piece and I divide it into 3, I've now, how do I say this? Like a third smaller? I don't know.

R: Keep going with what you're trying to say. What is a third smaller?

Nancy: The 1/4 piece now, this piece now, oh I don't know. I don't know how to explain it.

R: Why is a fourth a fourth? What makes a fourth a fourth?

Nancy: Because the size of the piece will fit in there four times, or the whole is 4 times larger than that piece

R: So, if I repeat it four times, it's exactly one thing that will allow me to get to the four. It fits four times, nothing else. Why is the 3/12, 3/12?

Nancy: *Because it is the size of the piece that will fit into the whole four times as well, but it's the size of three of the pieces that will fit into the whole 12 times* [italics added].

R: So, what would 3/12, four times get you?

Nancy: The whole.

R: The whole, four times as much as 1/4, which is the 1/4-piece fits in four times exactly. 3/12, the whole is what? The whole is 12 times as much as 1/12.

Nancy: Four times.

R: Four times, right.

Nancy: It fits into the whole four times.

R: So why would that matter when trying to find something equivalent to 1/4?

Nancy: *Because you're comparing, at the end of the day, you want the size of that piece to fit into the whole four times, or the size of the pieces, or you want the same size. You want to create something that will also fit into the whole four times* [italics added].

Nancy struggles with her explanation, because she has not yet constructed an understanding of this scheme at the anticipatory level. This evidence further leads me to infer that when given a task for a scheme she has not yet constructed at an anticipatory level, Nancy is not able to assimilate all three units as given, but instead reverts back to working through at least one level in activity. Eventually, through the step-by-step prompting, Nancy is able to construct a stronger understanding of such an equivalence. That is, the researcher first takes her back to the fractional unit piece of 1/4, which Nancy explains fits into the whole four times. Then the researcher asks her about the 3/12 pieces, which Nancy explains fit "into the whole four times as well, but it's the size of three of the pieces that will fit into the whole 12 times." Nancy eventually accommodates her scheme to include an understanding of why the two fractions are equivalent. This is evident when she states, “at the end of the day, you want the size of that piece to fit into the whole four times…You want to create something that will fit into the whole four times.” She is describing how the 3/12 (composite unit) is equivalent to 1/4, because they both fit into the whole exactly four times, therefore having the same unique relationship to the whole. Importantly for the inference about three levels of units coordination, she seems to understand that one of the three pieces after partitioning ¼, that is, 1/3 of ¼ of the whole, would fit 12 times within the whole – so the whole is composed of four sub-units, each composed of three sub-units.

Near the end of Year 2, a final AOP session was conducted with Nancy. The analysis of Nancy’s AOP is presented next.

AOP #4. The end of Year 2 brought about the end of the AdPed project with the teachers, and one last AOP set was completed with Nancy at the end of the school year. This AOP shows the culmination of Nancy’s growth throughout the project and how much her reasoning had changed in the two years she had been working with the AdPed team.

For the observation portion of the AOP, Nancy taught a lesson on the Unit Composition Fraction Scheme, which two years previously she would not have been able to teach. During the pre-interview Nancy accurately articulates exactly what the scheme entails and what she planned to look for in her students’ reasoning. She also explains many different types of reasoning the students may bring to the activity. Nancy then explains what recursive partitioning is, after the researcher asked her to explain that further as well. This can all be seen in Excerpt 4.53 below.

Excerpt 4.53 – *AOP Pre-Interview: Nancy's Unit Composition Fraction Scheme*

1:05 minutes into interview

R: What are you going to teach today?

Nancy: We are going to get into the Unit Fraction Composition stage of the scheme, where instead of having to, now they're going beyond just changing the unit fraction, but they're looking outside of the unit fraction.

R: Can you say more about that?

Nancy: *So, you have 1/5 of a whole pizza, and you give me 3/7 of your portion of the pizza, what fraction of the whole pizza did you actually give me? So, I'm looking for them to apply the reversible scheme to be able to create the whole. Maybe they don't draw it out, maybe they can do it in their head, and then relying back on the recursive partitioning to be able to create then the new unit fraction* [italics added]. Now we are going beyond the new unit fraction to seeing more than one of the unit fraction, so the 3/7. And *knowing that they are going to equally partition each of those fifths into 7, so now we have changed the unit fraction to thirty-fifths* [italics added]. I kind of thought about some things that they might do. Like they could not know what to do, because it's been a while since we've done this, and they could be like, what? So, we might have to go back. Some of them could say I gave you 3/7 of the pizza. They could say I gave you 3/7 of 1/5. Or some of them I could see drawing it out and partitioning each of the fifths into seven pieces and saying I have thirty-fifths now, or I have 3/35. I could also see some of my kids doing it in their head and just saying, if I take those five pieces and I put seven in each, I will have 35 pieces, so I know my unit fraction is thirty-fifths and if I have three of them, I have 3/35. So, I could see a variety of things kids might do.

R: You said the kids have worked with Recursive Partitioning. Can you talk a little bit about what they've done? What that means?

Nancy: So, we kind of did that whole breakfast, lunch, and dinner thing where we had a Twizzler and we were going to split up the Twizzler for breakfast, lunch, and dinner. We were going to give part of it to myself and part of it to someone else. So, we were taking that whole, we had breakfast, lunch, and dinner, but then we wanted to split up the unit fraction further by, I took part of the breakfast portion, and you took part of the breakfast portion. So now what fraction of the total have we eaten? So, *it would be then repartitioning the unit fraction and creating a new unit fraction. Being able to connect that back to the whole* [italics added]. They did pretty well with it. For a while it was hard for them to see that oh, I could have thirds, but I could also have sixths. So, then we worked with that for a little bit more. Realizing we could take this Twizzler and *we could break it into thirds, but we could also have sixths, and I could eat one-third and you could eat sixths at the same time* [italics added].

R: How will you know if they are struggling?

Nancy: If they're not relying on their previous understandings of how large is the whole. Like if this is 1/5 that's the unit fraction we're beginning with, then what is the whole? If they're not connecting to that. *If they're not knowing to partition each piece into seven then that could tell me that I need to go back to that recursive partitioning* [italics added]. If they don't know to do that five times. If they just do it once and say

I have 3/7 of 1/5, which is technically true, but I would be looking for them to, you know.

R: Do you need to see them physically get to the whole?

Nancy: No, because I'm anticipating that some of them could go (mimics using her fingers to skip count) thirty-fifths, and if I have 3 of them, I have 3/35. If they do that, that would be awesome.

I claim that Nancy is assimilating three levels of units as given and has constructed both recursive partitioning and the Unit Fractions Composition Scheme at the anticipatory levels. She is not likely to make a distinction in her students' reasoning and levels of units coordination if she did not have the ability to assimilate all three units as given herself. So, Nancy's own levels of units coordination has afforded her the ability to "see" these types of reasoning in her students' levels of units coordination. Specifically, in the excerpt Nancy was able to accurately describe what the goal is in the Unit Fractions Composition Scheme. She explains this using the example of finding 3/7 of 1/5 of a whole pizza. She explains, "So, I'm looking for them to apply the reversible scheme to be able to create the whole…and then relying back on the recursive partitioning to be able to create then the new unit fraction. Now we are going beyond the new unit fraction to seeing more than one of the unit fraction, so the 3/7. And knowing that they are going to equally partition each of those fifths into 7, so now we have changed the unit fraction to thirty-fifths." Nancy understands the link between the Unit Fractions Composition Scheme and the two schemes that come before it (Reversible Fraction Scheme and Recursive Fraction Scheme). In anticipating that one of the three sevenths of 1/5 would be 1/35 of the whole pizza Nancy demonstrates she takes those three levels of units as given (without losing sight of there being three such 1/35 units).

Accordingly, Nancy explicitly explains how the students will need to use the Reversible Fraction Scheme in order to partition a 1/5 piece of the pizza into 7 pieces, in order to then disembed three of those pieces to get 3/7 of the one piece. In order to find what fraction of the whole pizza they have, the students would then need to use the Recursive Partitioning Scheme in order to create their new fractional unit of thirty-fifths to find that they have 3/35 of the whole pizza. In order for Nancy to fully explain this as she did, and make the connections between the three schemes, she would need to have constructed all three schemes (Reversible Fraction Scheme, Recursive Fraction Scheme, and Unit Fractions Composition Scheme) at the anticipatory level.

Nancy then goes further into how her students may reason through this task. She believes that some students may struggle with the problem altogether, while others may need to draw out and partition a bar to solve the problem (in activity), and some may be able to solve the problem in their head (as given). Nancy clearly sees a difference in these types of reasoning for her students. She understands that some students may not be ready to engage in a task like this, while some students may need to assimilate the units in activity, and some may be able to assimilate the units as given. Nancy does not use the language of *in activity* and *as given*, but she does seem to understand the difference in these types of reasoning. Nancy explains that she does not need to see her students physically partition the entire whole, and that it would actually be preferable if they did not need to. I infer that this is because Nancy understands that students who are assimilating all three units as given would not need to partition the entire whole in order to reach the solution. Again, Nancy's own assimilation of the units and understanding of the scheme affords her the ability to determine how her students might reason through the task.

During the lesson, Nancy made some interesting real-time decisions and modifications due to the reasoning she observed in her students. Her goal for the students changed twice during the lesson when she realized the students were struggling. She ended up going back and developed on-the-spot prompts that would help the students build up the units in activity. After students worked (and struggled) to solve the original problem, Nancy gave them a new task. She asked them, “If I gave you 1/7 of my 1/5 piece of pizza, what fraction of the whole pizza would you have?” That is, Nancy took them back to recursive partitioning to begin helping them build up the units through activity.

After the students worked on that task, realizing their struggle, she then changed it again to, “If this is 1/5 of the pizza, can you create the whole pizza?” This change meant going back to the equi-partitioning scheme, because the students needed to build their units up even further through activity. Her realization that the students needed to build up the individual units through activity in order to get to the ultimate goal, led Nancy to make real-time adjustments to her task and goal for the students. This was an important piece of data that gave me an insight into how Nancy’s own reasoning strengthened her teaching ability. It also gave evidence into how much Nancy’s reasoning and teaching had grown over the last two years. In Excerpt 4.54, Nancy explains what her thinking was behind the decisions to change her goals for the students throughout the lesson.

Excerpt 4.54 – *AOP Post-Interview: Nancy’s Pedagogical Change*

30:39 minutes into interview

R: Your original goal for the lesson, would you say the students reached that?

Nancy: No.

R: Would you say that your goal changed through the lesson?

Nancy: Yes.

R: Can you talk a little bit about how it changed?

Nancy: *It changed from going back to seeing if they could think about it with just the 1/7 of the 1/5, and then it changed to going back to prompt them more heavily with, like, if this is my 1/5, can they create the whole?* [italics added] By them seeing that whole, I thought we could go forward to saying, if you gave me 1/7 of your portion of the piece, and I highlighted it in yellow, this is the 1/5 portion, this is your piece, you gave me 1/7 of that. I thought that they would at least be able to start by doing that and going okay, so what is this piece of the whole?

This is significant evidence into Nancy's reasoning and levels of units coordination. She clearly understood the fractional reasoning schemes that lead up to the Unit Fractions Composition Scheme herself at an anticipatory level, which seemed to afford her the ability to make quick, real-time adjustments to her instruction. Nancy explicitly states that she realized the students were not reaching her original goal, so she changed that goal twice during the lesson in order to help them build up the required units for the task (first going back to Recursive Partitioning, then all the way back to the Equi-Partitioning Scheme).

Nancy made intentional adjustments when she realized that the students were struggling and needed to go back to each unit and build it up in activity. Her reasoning for this was, "going back to seeing if they could think about it with just the 1/7 of the 1/5, and then it changed to going back to prompt them more heavily with like if this is my 1/5, can they create the whole? By them seeing the whole, I thought we could go forward to saying, if you gave me 1/7 of your portion of the piece…" That is, Nancy's own reasoning at three levels of units seemed to underlie her clear goal in changing the tasks when she realized the students were struggling. She first went back just one scheme in order to help them focus on just one unit piece of another unit piece. However, when that did not work, she decided to take then all the way back to the single unit fraction of 1/5 to help them build that up to the whole. From there she was hoping they would then be able to add additional units.

Summary of Year 2

In Year 2, Nancy made major growth in her mathematical reasoning. She constructed the fractional reasoning schemes up to the Unit Composition Fraction scheme (Excerpts 4.53 and 4.54). While she had moments in which she seemed to be more at a participatory stage of understanding (needed prompting from the researcher) (AOP #3 and Buddy-Pair #9), by the end of the year, she seemed to have constructed the schemes at the anticipatory stage. By her final AOP, Nancy was able to fully articulate the Unit Composition Fraction scheme and what the goals for the scheme entailed (Excerpt 4.53). In doing so, Nancy also provided evidence that she was assimilating three levels of units as given, except the one instance when she seemed to revert back to two levels as given and one in activity – when she was struggling to construct a solid understanding of recursive partitioning (Buddy-Pair #9).

Like Marsha, Nancy's mathematical reasoning also seemed to afford her the ability to analyze her students' reasoning, especially in-regards-to what units they could assimilate and how they worked with those units. She always seemed highly aware of how her students worked with the required units in the tasks she provided them. This was the main focus of her analyses when determining whether or not her students were meeting the goals for the lessons (Excerpt 4.54). Accordingly, by the end of Year 2, Nancy's strong ability to analyze her students' reasoning seemed to enable real-time pedagogical decisions that led to better instruction when her students were struggling with content (Excerpt 4.54).

Summary of Nancy's Transitions

The findings from Nancy's pre-intervention AOP suggested that she entered the project with both types of MKT, with her common knowledge of content somewhat stronger than her specialized knowledge of content. She was able to identify the accuracy of her students' answers, but also had some understanding of how students were reasoning (Excerpts 4.26 and 4.27). In my analysis of that session, I had attributed to her the assimilation of at least two levels of units as given. Her own assimilatory scheme and the levels of units within it made it possible for her to identify and analyze such units within her students' assimilatory schemes. So, before any intervention had taken place, her own assimilatory units afforded her the ability to use her specialized knowledge of content to recognize her students' assimilatory units. During that same pre-intervention AOP session, Nancy provided indications of her multiplicative reasoning as well, suggesting that she understood multiplication not just procedurally (repeated addition, recursive doubling, and memorized facts) but also a beginning knowledge of the mental distribution of units. Her own reasoning, and specialized knowledge of content, seemed to afford the ability to notice her students reasoning beyond procedures.

As the project progressed, Nancy continued to build up more units within her assimilatory scheme, as well as her multiplicative and fractional reasoning. These transitions within her own reasoning led to major shifts in her specialized knowledge of content. By her first Buddy-Pair session at the beginning of Year 1 (not long after Summer Institute 1), Nancy's multiplicative reasoning seemed to have eliminated any references to procedural reasoning. She no longer referred to multiplication as equal groups (or repeated addition) and was explicitly referring to it as units of units of units (Excerpt 4.28). The findings also suggested that she was assimilating at least two and a half levels of units at this point. This early shift in her reasoning seemed related to a shift in her MKT as well. Her focus on student reasoning had completely shifted to identifying the units her students were assimilating, especially their assimilation of composite units and how they tracked their units (Excerpts 4.28 – 4.30). She also had begun to explicitly ask questions of her students to dig deeper into their reasoning in order to identify the units they were operating on (Excerpt 4.29). Nancy's shift from at least two levels of units as given to at least two and half levels of units, as well as her shift in multiplicative reasoning allowed her MKT to shift towards an active focus on her students' assimilation of units and their operations on those units.

By the end of Year 1, Nancy was assimilating three levels of units as given, identifying those same three levels in her students' assimilatory schemes, and was becoming proficient in differentiating between her students' varying levels of reasoning. The findings from her second AOP session provided evidence suggesting that Nancy's own assimilation of three levels of units as given (Excerpt 4.38) afforded her the ability to identify those units within her students' reasoning and make inferences into how they were operating on those units. Within that session (Excerpt 4.39), she explicitly identified three different ways her students were reasoning based on the units she noticed them assimilating (students assimilating single units only, students assimilating single units and composite units, and students assimilating all three levels of units). Not only did she identify the units the students were assimilating but she also differentiated between those differing levels of reasoning based on the assimilation of their units.

In Year 2, Nancy continued to construct new reasoning, specifically fractional reasoning. Her assimilation of three levels of units as given, when matched with higher multiplicative and fractional reasoning, eventually led to a high-level of MKT. Then, Nancy not only could identify her students' assimilatory units but was also able to make real-time modifications in her lessons to help students build up new levels of units in activity (AOP #4). By the end of the project, I had attributed to Nancy all multiplicative reasoning schemes at the anticipatory level, as well as the fractional reasoning schemes up to the Unit Fraction Composition Scheme. Having constructed these schemes at the anticipatory stage afforded her the ability to predict student reasoning within those schemes, notice how they were operating within those schemes, how their unit assimilation affected their reasoning within the schemes, and how to begin modifying lessons to help students construct those schemes and the units necessary within them.

Table 4.2 provides an outline of the shifts that occurred in Nancy's levels of units coordination, her multiplicative and fractional reasoning, and her MKT. Blank boxes were used when there was no change from the box above it. I used the N/A notation when that specific reasoning type was not analyzed. Based on this table, Figure 4.8 then provides an illustration of shifts in Nancy's assimilatory units and where her multiplicative and/or fractional reasoning was at that same moment in time.

Table 4.2 - Conceptually Clustered Matrix of Nancy's Conceptual Progression (Miles & Huberman, 1994)

	Levels of Units Coordination	Multiplicative Reasoning	Fractional Reasoning	MKT
		Year 1		
Pre-Intervention (AOP #1)	At least 2 levels of units as given	Procedural explanations (repeated addition, recursive doubling, and memorized facts) Low-participatory level of mDC	N/A	Notices figurative vs. abstract reasoning Wants students to differentiate between two units (singles and composites) in real-world contexts Can notice units students are assimilating and how they operate on them

Table 4.2 Cont'd

Buddy-Pair #1	At least 2.5 levels of units (2 levels as given; 3rd level in activity)	Mid to high-participatory level of mDC Explicitly refers to units of units of units	N/A	Focuses on units students are assimilating and how they track those units Actively asks questions of students to determine assimilated units
Buddy-Pair #3			N/A	Begins to differentiate between different reasoning levels in her students based on assimilated units and how they operate on those units
Workshop #4	3 levels of units as given	mDC at high-participatory to anticipatory level SUC, UDS, and MUC at the participatory level	N/A	N/A
AOP #2			N/A	Identifies three levels of units in students' assimilatory schemes Proficient in differentiating between students' levels of reasoning
		Year 2		
Workshop #6		Uses distributive property within UDS at the anticipatory level	N/A	N/A
Workshop #7		N/A	Equi-partitioning at the anticipatory level	N/A
Workshop #8		MUC at the anticipatory level	N/A	N/A

Table 4.2 Cont'd

Buddy-Pair #8	Reverts back to 2.5 levels of units (2 levels as given; 3rd level in activity)	N/A	Reversible Fraction Scheme at participatory level	N/A
AOP #3	3 levels of units as given	N/A	Reversible Fraction Scheme at tanticipatory level	Wants students to keep track of their units (unit fractions and proper fractions) without losing sight of the whole Can explain what she is looking for in student reasoning
Buddy-Pair #9	Reverts back to 2 or 2.5 levels of units	N/A	Recursive Partitioning at participatory level	N/A
AOP #4	3 levels of units as given	N/A	Recursive Partitioning and Unit Fractions Composition Scheme at anticipatory level	Can differentiate between students who are assimilating units as given vs. in activity Makes real-time modifications to help students build up their units in activity

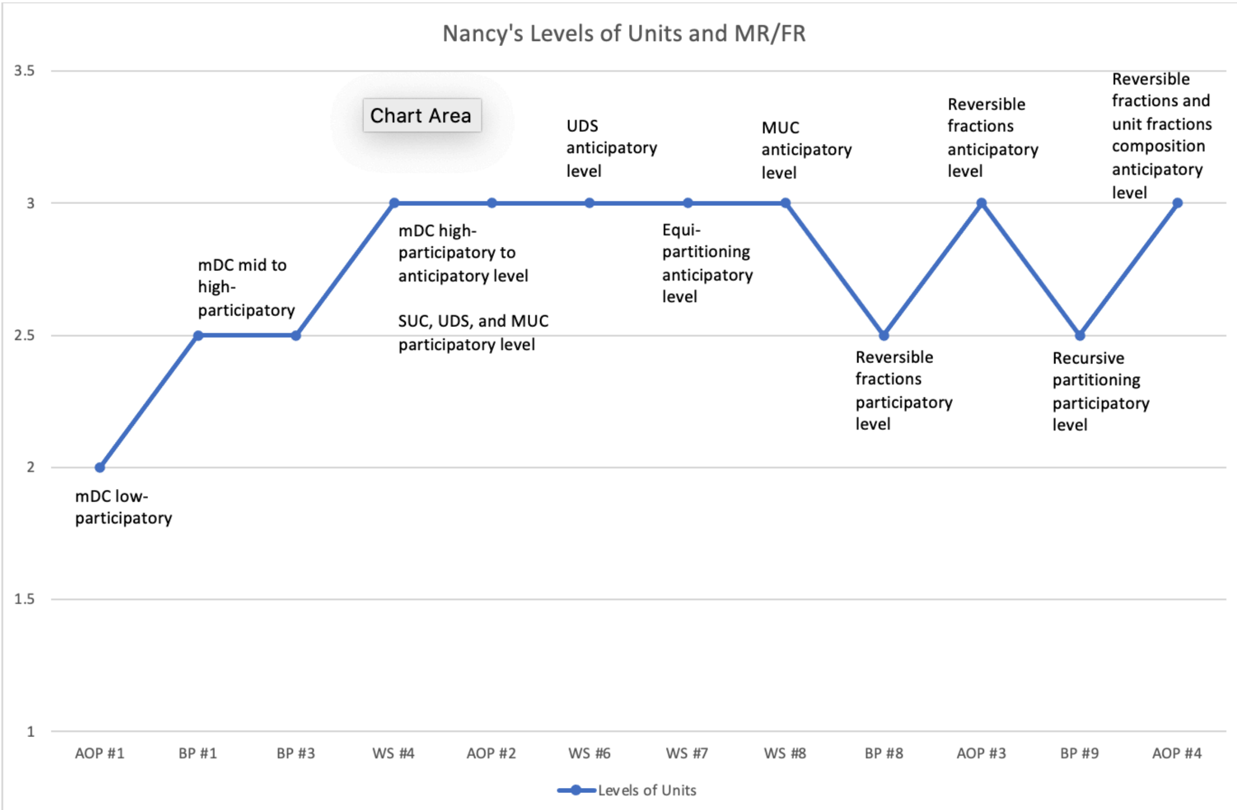

Figure 4.8 - Growth Gradient for Nancy's Levels of Units Coordination and Multiplicative and Fractional Reasoning (Miles & Huberman, 1994)

Cross-Case Analysis

When taking the findings from both Marsha and Nancy's data, a cross-case analysis can be conducted to identify similarities between their progressions. In order to stay away from a deficit approach in the analysis, I have chosen to only focus on the similarities between their progressions instead of looking at differences.

In Figure 4.9, I have compiled Figures 4.4 (Marsha's within-case growth gradient) and 4.8 (Nancy's within-case growth gradient) to create a cross-case growth gradient. This figure provides a look across their cases in terms of transitions throughout the two-year project in both their levels of units coordination and their multiplicative and fractional reasoning schemes.

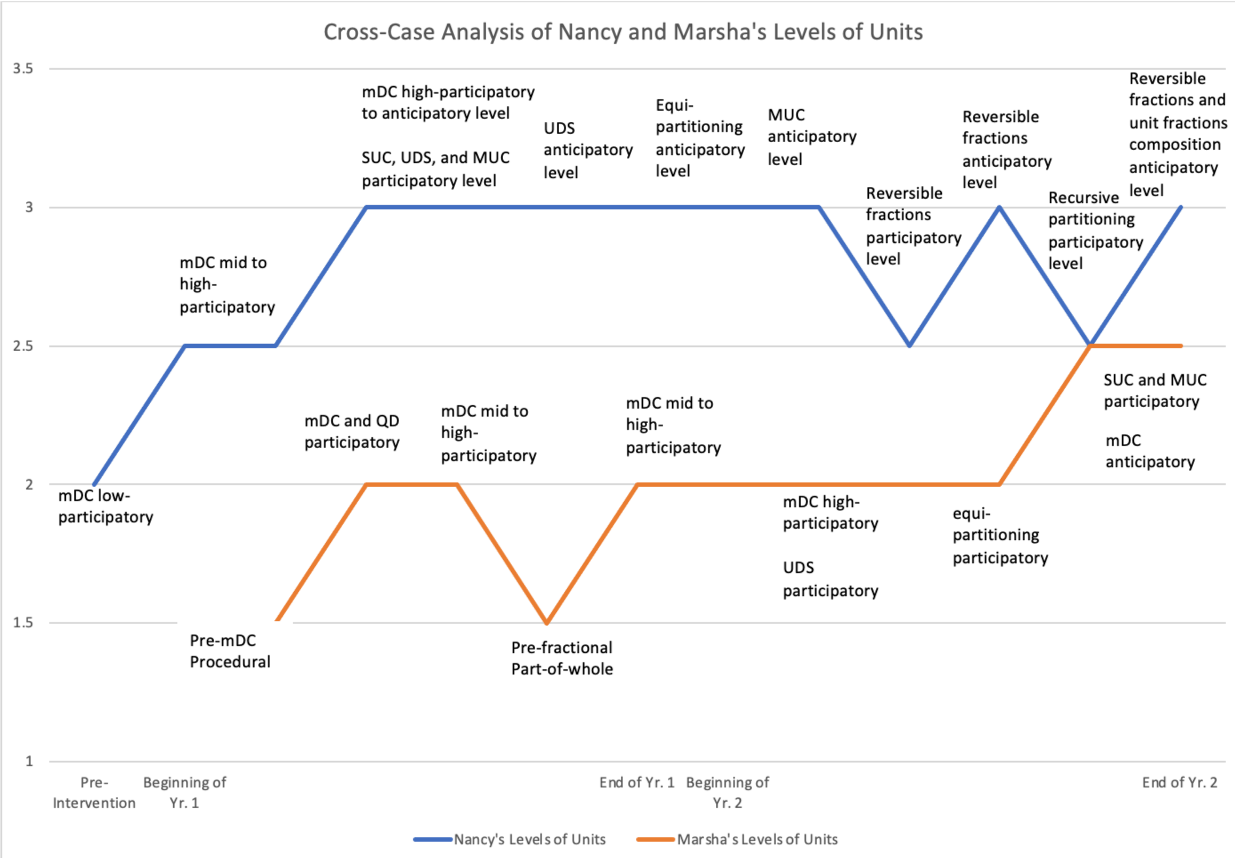

Figure 4.9 - Growth Gradient for Cross-Case Analysis of Nancy and Marsha's Levels of Units Coordination and Multiplicative and Fractional Reasoning (Miles & Huberman, 1994)

The first cross-case aspect that emerges is that, by the end of the project, both Nancy and Marsha seemed to assimilate one more unit than when they entered the project. Marsha transitioned from one and half levels of units at the beginning of the project to two and half levels of units by the end of the project. Nancy transitioned from two levels of units at the beginning of the project to three levels of units by the end of the project. This suggests that within the course of a two-year PD intervention, similar to the AdPed project, a mathematics educator may expect a teacher to build up one more level of units than at the start of the intervention.

This finding is significant, in that it gives researchers an idea of how long it may take a teacher to build up a particular number of units over time. It is a slow process, suggesting that a teacher may not be able to transition much in their assimilatory units from only a few professional developments, let alone a single professional development. This is similar to findings of studies in which students built up units over many intervention sessions (Norton & Boyce, 2015). Helping teachers build up new assimilatory units that they may operate on will take time and in-depth interventions similar to what Nancy and Marsha experienced (Buddy-Pair sessions, coaching, and workshops). The good news is that over time teachers can assimilate new units and learn to simultaneously track them as they work through mathematical tasks for schemes they are constructing.

A second aspect of the cross-case analysis pertains to levels of units coordination required for constructing a multiplicative or fractional scheme at the anticipatory stage. When I further analyzed the connection between the schemes and the teachers' levels of units coordination, I found they both were assimilating two levels of units as given when they had constructed the mDC scheme at a low-participatory stage. This suggests that two levels of units as given may be sufficient for teachers to construct the mDC scheme at the participatory stage. In contrast, the construction of a fractional reasoning scheme at the anticipatory stage requires the assimilation of three levels of units as given, which is evident in Nancy's construction of the fractional reasoning schemes, specifically the reversible fraction scheme and higher. As she was constructing them at the participatory stage, she was assimilating two and half levels of units (Marsha was able to construct equi-partitioning at the participatory stage while assimilating two levels of units). Once Nancy had constructed those fractional reasoning schemes at the anticipatory level, she was assimilating three levels of units as given. These findings are in alignment with previous studies linking the multiplicative and fractional reasoning schemes to levels of units coordination (Hackenberg, 2010; Steffe & Olive, 2010; Steffe et al., 2013; Ulrich, 2015; Ulrich, 2016).

In addition, the analysis also indicates that when both teachers were presented with tasks outside of their assimilatory schemes, their level of units coordination dropped by half, requiring one of the units to be coordinated in activity. For example, to Marsha this happened when she was given equi-partitioning tasks while still in a pre-fractional stage of reasoning (Excerpt 4.7). She went from two levels of units as given to one and half levels of units. Similarly, Nancy experienced this unit "drop" when presented with tasks for the reversible and recursive fraction schemes. In both situations she went from three levels of units as given to two and half levels of units.

These findings indicate that when teachers are put in an unfamiliar mathematical situation, they may revert back to coordinating a higher-level unit in activity even though they had previously assimilated it as given. This aligns with previous research done on students' levels of units coordination, in which students' levels of units may change when presented with unfamiliar mathematical tasks (Boyce & Norton, 2019; Norton & Boyce, 2015; Boyce, 2014). This suggests that levels of units one assimilates into their reasoning schemes can be somewhat fluid. In other words, just because an individual assimilates three levels of units in one context, does not mean those three levels of units are automatically transferrable to new contexts. They will more than likely experience a dip similar to Nancy and Marsha, in which they have to rebuild units in activity. Luckily, the findings for both Marsha and Nancy indicate that this "drop" in levels of units coordination is temporary. In both cases, that level jumped back up to their previous unit assimilation sometime later. In turn, that jump-back seemed to open the way for both teachers' construction of the scheme for that task at either the participatory or anticipatory stage. For example, once Marsha was at the participatory stage of equi-partitioning, she went back to assimilating two levels of units as given. Similarly, once Nancy had constructed the reversible and recursive fraction schemes, she went back to assimilating three levels of units as given.

Conclusion

In conclusion, Nancy and Marsha came into the AdPed project with different levels of reasoning, and both made growth in their reasoning throughout. Even though their progressions varied, they both demonstrated critical changes in their reasoning. These changes also led to very important changes in their ability to analyze student reasoning and, hence, their pedagogical practices. Therefore, it can be concluded that as teachers' multiplicative and fractional reasoning change, along with the number of levels of units they can coordinate, their teaching abilities change as well.

CHAPTER V

DISCUSSION

This study used within-case and cross-case analyses of two case studies – Nancy and Marsha – to address three research questions:

1. What pathways of reasoning, markers and transitions, may teachers go through? That is, what changes in their multiplicative and fractional reasoning schemes could be inferred?
2. To what extent, and in what ways, do teachers' levels of units coordination affect their learning pathway?
3. How do teachers' levels of units coordination affect their ability to recognize levels of units in their students?

The data set for this study spans a two-year period, in which the two case study teachers (Nancy and Marsha) participated in a larger research project called *Adaptive Pedagogy for Elementary Teachers (AdPed)*. The AdPed project's focus was on implementing professional development (PD) interventions to help the participating teachers (a) construct new mathematical ways of reasoning, specifically multiplicative and fractional reasoning, and (b) move toward a conception-based pedagogical perspective (as explained in Chapter II). Being a graduate research assistant on the project, I was present for almost all data collection and intervention sessions that occurred over the two years. As my analysis and summaries of the two cases has shown (Chapter IV), both Nancy and Marsha were exhibiting shifts in their own mathematical knowledge, as well as their MKT. In particular, this analysis pointed to their assimilation of units and how they operated on those units, as well as their ability to identify and analyze the units their students assimilated in classroom tasks.

Building on the work of Simon (2006), Silverman and Thompson (2008), and Hackenberg (2010), this study contributes to the field by articulating, through a constructivist lens, three new sub-dimensions of MKT: 1) Teachers' pathways in the construction of multiplicative and fractional reasoning, 2) Teachers' assimilation of levels of units and their transitions to higher levels of unit assimilation over time, and 3) How teachers' assimilatory units affect their ability to identify and make inferences into their students' assimilatory units and multiplicative/fractional reasoning. The findings presented in the previous chapter give insight into how levels of units coordination afford or constrain teacher knowledge and their ability to analyze their students' reasoning. Having that knowledge, researchers may begin to identify appropriate interventions and PD for helping teachers construct new assimilatory schemes with higher levels of units coordination. My analysis demonstrated that as teachers construct more levels of units coordination, they may strengthen their MKT in the sense of better analyzing their students' reasoning.

In this chapter, I begin with a brief summary of the main findings from my analysis in Chapter IV. Then, I discuss main contributions that studies about teacher reasoning (levels of units coordination; multiplicative and fractional reasoning) and its effects on their mathematical pedagogy can make to the field. I conclude by outlining the limitations of the study and its implications for the field of education and educational research.

Analysis Summary

While addressing the three research questions, my analysis in Chapter IV had two main foci. For each of the data segments, my first focus was on the nature of Nancy and Marsha's schemes for multiplicative and fractional reasoning and their levels of units coordination. That focus was followed by how their levels of units coordination linked to their ability to identify and analyze their students' assimilation of, and operation on, units. My key findings suggest that not only did Nancy and Marsha experience significant shifts in their mathematical knowledge but as that knowledge deepened conceptually, they assimilated more units themselves and their MKT shifted significantly. In other words, as they assimilated more units they progressed in the identification of units in their students' reasoning. Thus, they were able to make inferences into their students' reasoning based on what they noticed about their students' use of units. In this sense, both Nancy and Marsha provided a rather telling story of how teachers' levels of units coordination and mathematical reasoning may be tied to their MKT.

Although both Nancy and Marsha joined the project at different levels of reasoning, they both built up their assimilatory schemes for units coordination over the course of the project, with Marsha transitioning from one and a half levels of units to two and a half levels of units, and Nancy transitioning from two levels of units to three levels of units. As they both built up more units in their assimilatory schemes, they also constructed new multiplicative and fractional ways of reasoning. Marsha transitioned from a procedural understanding of multiplication to constructing the mDC scheme at the anticipatory stage. She also ended the project having constructed the SUC, UDS, and MUC schemes at least at the participatory stage. In terms of fractional reasoning, Marsha transitioned from pre-fractional, part-of-whole reasoning to constructing the equi-partitioning scheme at the participatory stage. When Nancy entered into the project, she already had some conceptual understandings of multiplication as distribution of units and seemed to have constructed the mDC scheme at the low participatory stage on her own. Over time, she constructed the mDC, SUC, UDS, and MUC schemes at the anticipatory level. Her fractional reasoning transitioned from pre-fractional, part-of-whole reasoning to having constructed all eight fractional reasoning schemes (see Tzur, 2019), up to the unit fractions composition scheme, at the anticipatory stage.

These shifts in Marsha and Nancy's assimilatory schemes also led to significant shifts in their MKT. As they built up more units and constructed new reasoning schemes, they began to notice new conceptions in their students' reasoning. At the start of the project, Marsha was focused on the accurateness of her students' answers and on the importance of memorizing math facts. As the project progressed, her analyses of student reasoning shifted toward a focus on the units they were assimilating and on the simultaneous tracking of their units. When Nancy entered into the project, she already had a focus on students' ways of reasoning. However, over time, that ability became stronger, leading to a shift towards analyzing students' assimilation of units as given versus in activity. By the end of the project, Nancy was able to make real-time analyses of students' assimilatory units and modify her lessons based on the data she was collecting.

Contributions to the Field

Based on the analysis and summaries of the two case studies, I have identified four main contributions that my dissertation study can make to the field of mathematics education in relation to the three sub-dimensions of MKT I have outlined throughout the study. The first contribution is an expansion of MKT, which includes foregrounding and linking teachers' levels of units coordination with their specialized knowledge of content. In other words, it examines the sub-dimensions of MKT dealing with teachers' assimilatory units and reasoning transitions over time, including how their levels of units coordination affect their ability to identify and make inferences into their students' assimilatory units and mathematical reasoning. My study points to teachers' assimilation of units as an important factor that affords their ability to analyze their students' assimilation of units in mathematical tasks. The second contribution is a new lens on the AOP methodology (Simon et al., 2004) as a means for articulating teachers' MKT. By utilizing the AOP methodology researchers can go beyond testing teachers' mathematics through an assessment involving correct/incorrect answers to focus on the teachers' reasoning. This new lens involves using the AOP data sets (interviews, observations) in order to analyze (a) how teachers use their specialized knowledge of content to not only notice but also make inferences into their students' reasoning and (b) transitions that occur as teachers move through conceptually distinct stages. Implied by the second contribution, the third contribution is in using a constructivist lens to conceive of MKT, specifically tying MKT to Simon's notion of hypothetical learning trajectories (HLT). The fourth contribution focuses on implications of the study for teacher educators and researchers; specifically, how teacher educators and researchers can use teachers' levels of units coordination to determine goals and interventions for future teacher learning. Next, I further discuss each of these contributions, while interweaving the last contribution (implications for teacher educators and researchers) into the three other contributions.

Expanding the Notion of MKT: Teachers' Levels of Units Coordination

Units coordination is a mental operation involved in people's assimilation of mathematical tasks; it pertains to their noticing and keeping track of different types of units simultaneously (Norton et al., 2015; Steffe, 1992). Units coordination thus entails people must understand that smaller units (e.g., units of one) are embedded (or nested) within larger units (e.g., composite units), which in turn could be embedded within larger units (e.g., compilations of composite units). Multiplicative and fractional reasoning requires the assimilation of three levels of units (Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsak et al., 2012; Norton & Boyce, 2013; Norton et al., 2015; Olive & Caglayan, 2008; Steffe & Olive, 2010). For example, in fractions, a unit fraction of another unit fraction of the whole involves three levels of units coordination (e.g., 1/7 of 1/5 of the whole is 1/35; see Excerpt XYZ). Past research on units coordination focused on students, and only recent research has focused also on teachers' levels of units coordination (e.g., Lovin et al., 2018). This dissertation study attempted to contribute to the focus on teachers, by examining the levels of units coordination in Nancy and Marsha's assimilatory schemes. Furthermore, my study also focused on linking their assimilation of units to their MKT, including inferences into students' units and operations. My analysis showed that as Nancy and Marsha assimilated more levels of units into their reasoning schemes, they not only transitioned into higher levels of reasoning themselves but were also able to apply those same units into their MKT and inferences into their students' reasoning.

The analysis of both cases in my study sheds light on the role a teacher's level of units coordination can serve in reconceptualizing MKT. For example, when the evidence from Nancy's case is examined, it is found that she entered into the project already assimilating two levels of units as given. This afforded her MKT with an ability to notice those same two units in her students' assimilatory schemes, even before participating in any PD intervention from the AdPed research team (Excerpt 4.27). Over time, her assimilatory scheme advanced to assimilating two and half levels of units (two levels as given and one in activity), and quickly into three levels of units as given (Excerpt 4.37). Once she reached three levels of units as given, she was constructing the mDC scheme at the anticipatory level and the SUC, UDS, and MUC schemes at the participatory levels (Excerpts 4.35 - 4.37). With this high level of multiplicative reasoning, Nancy's MKT then supported identification of three levels of units in her students' assimilatory schemes, including needed differentiation among students' varying levels of reasoning (4.38 – 4.40).

In Year Two of the project, Nancy's MKT continued to strengthen her assimilation of three levels of units as given, thus supporting her construction of multiplicative schemes at the anticipatory stage. Similarly, Nancy's reasoning also took a shift in terms of constructing the fractional reasoning schemes. By the time we collected the first evidence of her fractional reasoning, her MKT has already included the equi-partitioning scheme at the anticipatory stage, while assimilating three levels of units as given in unit-fraction situations (Excerpts 4.44 and 4.45). As she worked through constructing the reversible fraction scheme at the participatory level, her MKT seemed to take a "step back" to the assimilation of two and half levels of units, needing to build up one unit in activity (Excerpt 4.47). However, she quickly shifted back to three levels of units as given and constructed this advanced fractional scheme at the anticipatory stage. Once that happened, her MKT seemed to afford a focus on how her students assimilated those same three units when working through the reversible fraction scheme themselves (Excerpts 4.48 – 4.51). When Nancy's MKT advanced to the next fraction scheme – recursive partitioning (first at the participatory stage), she again reverted back to two and half levels of units (Excerpt 4.52). Eventually, by the end of the project, Nancy had constructed both the advanced, recursive partitioning and unit fractions composition scheme at the anticipatory stage, with the assimilation of three levels of units as given (Excerpts 4.53 and 4.54). Endowing her MKT with this high level of reasoning and the assimilation of three levels of units not only afforded her the ability to differentiate her students' assimilatory units but she was also able to infer whether the students were assimilating those units in activity or as given. This led to making real-time modifications to her lessons in order to help her students build up units in activity when she realized that they weren't assimilating the units as given (Excerpt 4.54).

Like Nancy, Marsha's MKT also revealed significant shifts in her levels of units coordination, which led to shifts in her mathematical reasoning and use of it to infer into her students' reasoning. Marsha's MKT upon entering the project seemed to include assimilating one and a half levels of units (one level as given and her second and third levels in activity). During that time, my analysis indicated she was at a pre-multiplicative reasoning (Excerpt 4.1). Her MKT focused on common knowledge of content, so she was able to identify the correctness of her students' answers. By her fourth Buddy-Pair, Marsha's assimilation of one and half levels of units, and a slight shift in her own reasoning, afforded her the ability to begin differentiating between her students' additive counting strategies (which require the assimilation of only one level of units) and whether or not they required concrete of figural representations while working through mathematical tasks (Excerpt 4.2). Not long after that, Marsha advanced to assimilating at least two levels of units while constructing the mDC and QD schemes at the participatory stage. This led to a shift in her ability to recognize her students' assimilatory units. At that point, her MKT seemed to thus afford identifying the same two units in her students' reasoning and analyzing how they were tracking and operating on those units (Excerpts 4.3, 4.5, and 4.8 – 4.9). Eventually, Marsha advanced to assimilating two and half levels of units (Excerpts 4.21 – 4.23), enriching her MKT with an anticipatory stage of the mDC scheme (Excerpts 4.24 and 4.25) and the SUC, UDS, and MUC schemes at the participatory stages (Excerpts 4.21 - 4.23). This transferred into her pedagogical ability to not only identify the units her students were assimilating but also make inferences into their respective reasoning based (Excerpts 4.24 and 4.25).

In the first glimpse into Marsha's MKT in terms of her assimilatory schemes for fractions, it was seen that she was pre-fractional, causing her to revert back to one and half levels of unit (Excerpt 4.6 – 4.7). She eventually advanced to the participatory stage of equi-partitioning, with the assimilation of two levels of units as given (Excerpts 4.17 – 4.20). Because we did not get to observe her teaching of fractional reasoning schemes – I can only conjecture a similar affordance to infer into her students reasoning and levels of units coordination has been obtained.

These findings provide interesting implications for mathematics educators working with teachers to promote their MKT. If mathematics educators can identify teachers' levels of units coordination, they may be able to guide (perhaps even predict) their transitions and growth over time and identify appropriate interventions for helping the teachers build up their assimilatory units. This could lead to transitions in teachers' MKT - particularly the ability to identify their students' assimilatory units, thus strengthening the teachers' specialized knowledge of content.

To this end, mathematics educators and researchers may begin any PD intervention with an analysis of teachers' levels of units coordination and schemes for reasoning (e.g., multiplicatively or with fractions). For teachers who do not yet have an assimilatory scheme for three levels of units as given, the mathematics educator could focus on identifying potential goals and interventions for helping them build up their units. As my analysis indicated, the mathematics educator should be prepared for teachers to temporarily revert back to lower levels of units when encountering new mathematical reasoning. In the data, both Nancy and Marsha experience this when they were working through difficult multiplicative and fractional reasoning tasks that they had not yet constructed at the participatory stage (Figure 5.1). For example, when Marsha was at the pre-fractional reasoning stage and was attempting to engage in equi-partitioning tasks, she reverted back to one and a half levels of units. However, this was a temporary change, and she was soon back at two levels of units as given once she was working through reasoning tasks that were more appropriate for her at that time.

In all, my findings of Nancy and Marsha's transitions in their assimilatory schemes (multiplicative reasoning, fractional reasoning, and levels of units coordination) suggested an expansion to the notion of MKT. In the next section, I turn to a discussion of how data collected through qualitative approaches (AOP's, Buddy-Pair sessions, and professional development workshops) provided a new lens onto the teachers' MKT.

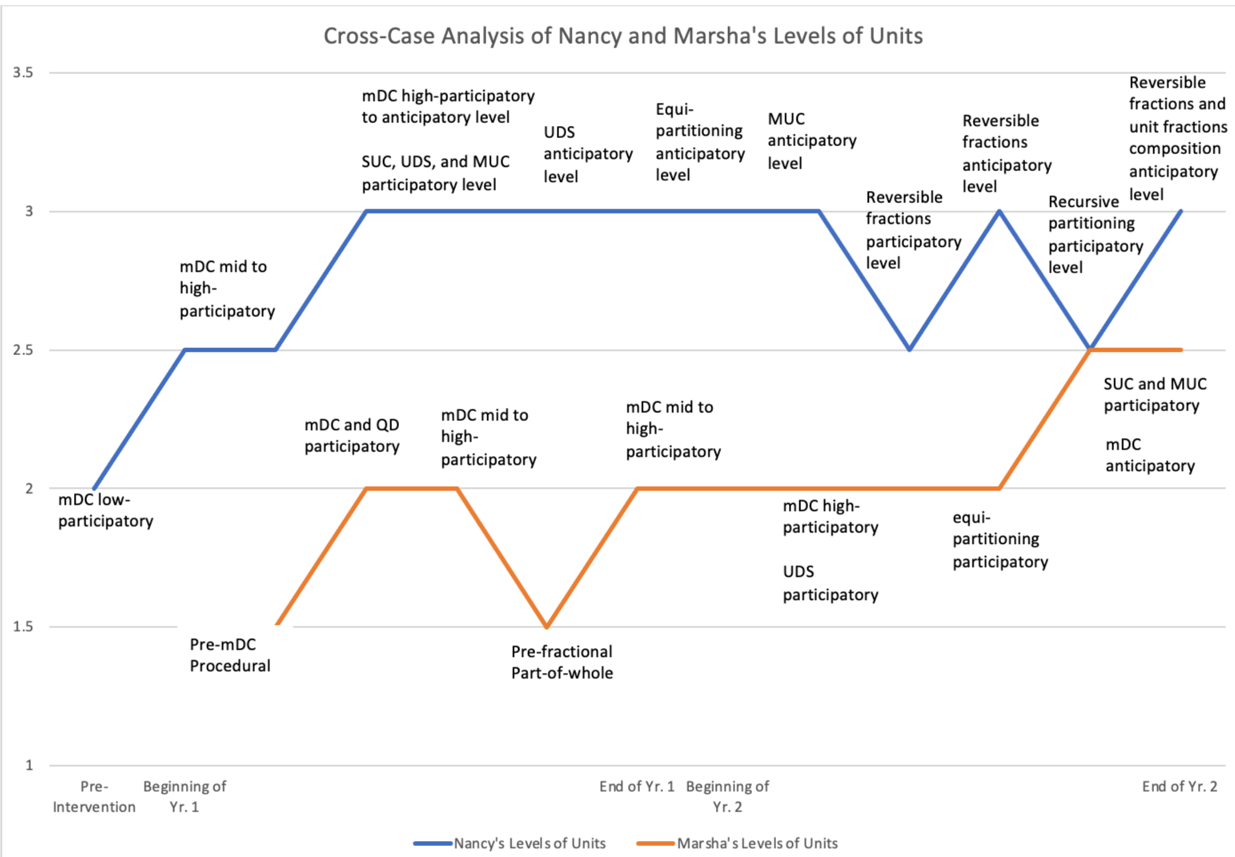

Figure 5.1 - Growth Gradient for Cross-Case Analysis of Nancy and Marsha's Levels of Units Coordination and Multiplicative and Fractional Reasoning (Miles & Huberman, 1994)

A New Methodological Approach to Identifying Teachers' MKT

As I articulated in Chapter II, drawing on Shulman's (1987) notions of content knowledge and pedagogical knowledge, Ball, Hill, and their colleagues (Hill & Ball, 2004; Hill, Rowan & Ball, 2005; Hill, Schilling, & Ball, 2004) have developed a research program that focused on the way teachers may use their mathematical knowledge in practice. In particular, they portrayed two dimensions of teachers' content knowledge that are required for effective mathematics teaching – specialized knowledge of content and common knowledge of content. They termed this twofold dimension of reasoning as *mathematical knowledge for teaching* (MKT), pointing out it greatly impacts one's teaching – especially the ability to consider students' reasoning (Hill et al., 2005). Tzur (2016) further asserted that a teacher's own mathematical knowledge is the limit of what they can teach. Furthermore, lacking the ability to analyze student reasoning can decrease informing teachers' decisions what and how to teach next (Tzur et al., 2016). In other words, a teacher's MKT affords and constrains what they can notice and infer in their students' reasoning and use as a conceptual basis for their teaching. The empirical evidence provided when assessing teacher MKT also provides insight into their own mathematical reasoning. If a teacher is noticing and inferring about specific reasoning in their students, it seems safe to assume the teacher has also constructed at least that same reasoning.

To date, many research studies have been conducted that attempted to analyze teacher MKT. In order to capture the relationship between teacher mathematical knowledge and student learning, the MKT research program has developed assessments of teachers' common knowledge of content and specialized knowledge of content. Through the use of these assessments, Hill et al. (2005) found that MKT can positively predict student success in the elementary mathematics classroom.

However, as I explained in Chapter II, previous methods for assessing teacher MKT were based on teachers' correct or incorrect responses to written items. Thus, those assessments did not seem to tap into the teachers' reasoning. The theoretical framework I have used in my dissertation study and the analysis presented in Chapter IV indicated the importance of such reasoning – particularly the focus on levels of units coordination. My study points out that to assess such reasoning in teachers, a different methodological approach is needed. Specifically, my study revealed how using qualitative data (interviews, observations, coaching, and interventions) to analyze how teachers' levels of units coordination could enrich the understanding not only of what teachers know but also how their MKT shifted as they assimilated more units into their own reasoning.

Key to my study was that the goal was to examine how teachers' levels of units coordination afford (or constrain) their ability to identify and analyze their own students' levels of units coordination as they engage in classroom tasks. Taking this approach to analyze Nancy and Marsha's MKT provided valuable insights into the markers and transitions in their reasoning, which go above and beyond what can be obtained from written assessments of their specialized knowledge of content. My novel way of using qualitative analysis of the case study teachers enabled articulating very specific moments in which these shifts could be pinpointed. For instance, at the beginning of the project, Marsha clearly focused on the accuracy of her students' answers (Excerpt 4.1), but by her fourth Buddy-Pair session, she was analyzing her students' additive counting strategies, having moved beyond simply looking for correct answers. By and during Buddy-Pair #6, Marsha was beginning to analyze her students' assimilation and tracking of multiple units (Excerpts 4.4 and 4.5). These small-yet-noticeable shifts in her MKT continued throughout the two-year project, and by the end of the project, Marsha was paying close attention to her students' ways of operating on assimilated units and making inferences into what that meant about their reasoning (Excerpts 4.24 and 4.25). Overall, the significant leaps from beginning to end of Marsha's participation in the PD intervention were highlighted through this new approach to analyzing specialized knowledge of content in terms of inferences into her growth in reasoning about student assimilation of units. Similarly, the qualitative approach to analyzing MKT highlighted significant shifts in Nancy's MKT. Upon entering into the project, Nancy was already utilizing her specialized knowledge of content to analyze student reasoning (Excerpt 4.27). Over the course of the project, her specialized knowledge of content became stronger, until she was able to use her specialized knowledge of content to make real-time inferences into her students' reasoning and made in-the-moment modifications to her teaching based on those inferences (Excerpt 4.54).

It is important to note that my ability to identify the shifts in Nancy and Marsha's MKT was afforded by the qualitative analysis of data collected over a two-year duration of the project (as opposed to a pre/post quantitative analysis). It was only through the interviews and exchanges with both teachers, and observing their teaching in classrooms, that such significant changes were found from one session to another. A written assessment of correct/incorrect responses alone would not have detected such nuances over time. Sometimes the shifts seemed almost hidden within the data. They could only be teased out through further questioning in the interviews and debriefs of the Buddy-Pair and AOP sessions. Without the ability to further probe into Nancy and Marsha's MKT, some of those shifts might have gone unnoticed. For instance, in many of the debriefs and interviews, the researchers asked Nancy or Marsha why specific inferences they had made about responses of different children were important. Through these types of questions, Nancy and Marsha would expound on those inferences and what it meant about their students' reasoning at the time (Excerpt 4.29 is one example). This is where much of the evidence in the data were found to suggest what their MKT (units, operations, schemes) was at that moment. These explanations changed over time as their MKT increased, and the new methodological approach to their specialized knowledge of the content enabled articulating how it became stronger.

I believe that using qualitative data such as interviews and observations can benefit researchers and provide evidence into teacher MKT that written assessments alone do not. If those qualitatively-rich opportunities are used to dig into teachers' MKT, stronger evidence can be attained to suggest their MKT stages, and even more importantly – how they transition from one stage to another. Knowing when and how these transitions occur over time can provide valuable insight into how teachers' MKT changes over time and how we may provide interventions or professional development opportunities to improve their MKT. The two first contributions noted so far were both rooted in the constructivist perspective I have been using. Next, I continue this line of discussion by pointing out how researchers can further expand such a perspective by linking MKT to Simon's (1995) core construct of *hypothetical learning trajectory* (HLT).

A Constructivist Lens of MKT – Linking MKT to HLT

Previous research into MKT was mostly rooted in frameworks other than a constructivist approach to teacher knowledge. While MKT research program has yielded significant connections between teacher mathematical knowledge and student success in mathematics, I emphasize a missing tie to a constructivist perspective on learning and teaching. Silverman & Thompson (2008) argued for the inclusion of a constructivist framework within MKT, claiming that "it is not until the teacher transforms [their] knowledge into knowledge that is pedagogically powerful that the teacher has developed MKT" (p. 509). While my study builds on their theoretical claim, it provides empirical evidence to illustrate how teachers may transform their knowledge into "pedagogically powerful" knowledge.

There are two significant gaps in the MKT research that can be filled by bringing in a constructivist lens: (a) How teachers understand their students' assimilatory schemes and (b) How teachers understand the learning trajectories in advancing along those assimilatory schemes. In this study, I attempted to fill that gap by bringing in that missing constructivist lens, with ties to Simon's (1995) HLT construct. One reason for me to do so was that his construct of HLT, while implicitly alluding to the researcher's mathematical knowledge, does not explicitly address MKT as part of the HLT notion. My dissertation study helps examine possible links between MKT and HLT in two ways. First, it shows how a researcher can utilize an HLT to help teachers build upon their MKT. Second, it highlights how teachers' mathematical knowledge can expand on the HLT construct.

Simon's (1995) HLT construct refers to teachers' use of their own mathematical knowledge to articulate potential paths in student conceptual growth. As I explained in Chapter II, this construct consists of three components: 1) a learning goal for students, 2) learning tasks and activities expected to lead students to that learning goal, and most importantly 3) a researcher's (or teacher's) hypothesis of a potential learning path through which students may transition as they engage in the learning tasks from Step 2. As my study of the cases of Marsha and Nancy showed, a teacher's own mathematical knowledge will afford or constrain each of these components. For example, if a teacher's own assimilatory scheme includes an understanding of fractions as a multiplicative relationship between the whole and its fractional pieces, then that teacher's HLT for fractions can include goals for student learning that is tied to the teacher's same understanding (e.g., coordinating the unit of 1, or whole, with unit fractions through the mental activity of unit iteration).

Step 3 in the HLT construct differentiates the stages of the learning path as conceptual markers that students pass through. My study attempted to take this differentiation further by not only analyzing conceptual markers Nancy and Marsha passed through but also how they transitioned between the markers. This expansion on the HLT construct is informed by Tzur's (2019) contention of the importance to conduct transition studies and foregrounding the conceptual transitions occurring as learners move from one marker to the next. By utilizing HLT and the transitions that occur as learners shift from one conceptual marker to the next, I am explicating a constructivist-informed MKT in linkage with the HLT construct of a constructivist-informed pedagogy.

In my analysis I found, for example, that Nancy and Marsha experienced many transitions in their mathematical knowledge and MKT, which led to hints that they may have been heading toward an incorporation of HLT into their teaching practices. As they transitioned from one conceptual marker to the next (e.g., from two to two and a half levels of units coordination), they were also able to assimilate the same conceptual markers into their MKT and the analyses they were able to perform on their students' reasoning. For instance, as Nancy transitioned through the fractional reasoning schemes, she began to use her own mathematical knowledge to identify and analyze her students' assimilatory schemes within those same conceptual markers. She thus could identify particular goals for different students' learning. A specific example of this occurred during her eighth Buddy-Pair session, in which she worked through a task involving the reversible fractions scheme. The researcher provided a task that she assumed would help Nancy construct the reversible fraction scheme. Nancy required some prompting as she worked through the task and began to construct the scheme at the participatory stage. Not long after that Buddy-Pair session, Nancy participated in her third AOP session. In that session, she fully explained the reversible fraction scheme, indicating that she had constructed the scheme at the anticipatory stage. As a result of the newly constructed scheme, Nancy was able to set the goal of teaching the same conceptual marker (the Reversible Fraction Scheme) to her students for whom this would be a sensible goal. Not only was she able to teach her students the same scheme but she was also able to articulate what it meant for her students to "have" that knowledge. Specifically, she explicated the importance of her students' ability to track units as they worked through Reversible Fraction Scheme tasks. Furthermore, she was able to analyze her students' ways of operating as they worked through the task, utilizing her MKT to make inferences into their assimilatory scheme as a basis for what they may learn next (which could eventually develop into a full HLT).

Linking between MKT and the HLT construct is important for the field, because it sheds light on how mathematics educators and researchers can utilize the HLT construct to increase teachers' MKT, which in turn may result in teachers being able to utilize the HLT construct in their own classrooms (Figure 5.1). However, my study suggests that before teachers can utilize the HLT construct they may need to first understand the learning trajectories required for hypothesizing on the potential learning their students may engage in. Therefore, the HLT construct could be expanded to include the teacher's constructivist-informed MKT and how transitions in their own knowledge afford their ability to utilize the HLT construct. For example, before Nancy can incorporate the HLT construct into her teaching practice, she first needs to construct the mathematical knowledge herself and then also begin to understand the learning trajectories that could lead to the incorporation of the HLT construct.

In Figure 5.1, I capture this relationship between the MKT and HLT of teachers and mathematics educators (researchers). When teacher educators utilize the HLT construct (outer circle in Figure 5.1), they can put interventions in place that may lead to transitions in teacher reasoning, such as the ones Nancy and Marsha experienced. By analyzing the effectiveness of those interventions, we can then hypothesize future teacher learning. Then, as teachers transition between conceptual markers, the teachers can begin to utilize their MKT to create and use HLT to improve instruction in their classrooms (inner circle of Figure 5.1). In those HLT, teachers MKT will enable them to identify student reasoning at least at the level of their own reasoning (i.e., the assimilation of levels of units), analyze that reasoning in order to make future goals for student learning, choose appropriate tasks for student learning, hypothesize the types of reasoning students may bring to those tasks, and modify those tasks when needed. To this end, the work of mathematics educators begins with an in-depth assessment of a teacher's current assimilatory schemes. From there, the suggested path in Figure 5.1 may begin, eventually leading to higher levels of student reasoning. If researchers know what a teacher's assimilatory schemes are, they can utilize the HLT construct, which eventually may lead to teachers implementing the HLT construct in their own classroom (requiring the teacher to assess their students' assimilatory schemes) and finally to transitions in student reasoning (the ultimate goal of all educators).

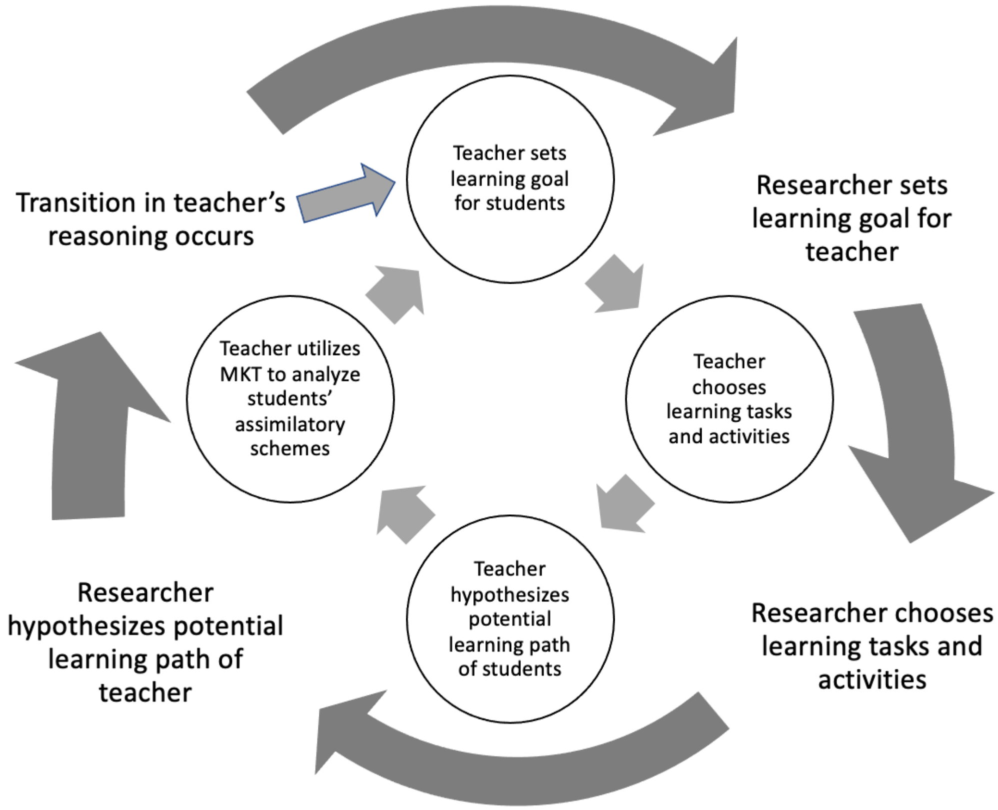

Figure 5.2 - Researcher's Use of HLT Triggers Teachers' Use of HLT

Limitations of the Study

One obvious limitation of this dissertation study is the small sample – only two case studies. This limitation was directly affected by another, central limitation of this study, namely, my choice to focus on mathematical knowledge for teaching (MKT) in terms of levels of units coordination and related schemes for multiplicative and fractional reasoning. I acknowledge that, aside from a teacher's level of units coordination, there are various other factors that may have contributed to the similarities and differences in Nancy and Marsha's learning trajectories. Clearly, such factors would also be relevant for studying a larger number of teachers. Those other factors may include differences in their entering points, differences in their professional (including mathematical) identities, or differences in how they positioned themselves within the intervention and or within their work environments. For example, quite early on Nancy seemed to have taken a leadership role of her grade-level team and the entire school team. Marsha, on the other hand, seemed to position herself as a participant (possibly following Nancy's leadership).

Furthermore, Nancy and Marsha entered into the intervention at different times. Nancy entered the intervention at the very beginning of the project and was able to participate in the very first Summer Institute provided by the researchers. Marsha did not enter the intervention until after the first Summer Institute, and therefore missed key foundational understandings that were needed to construct harder mathematical understandings. Accordingly, Marsha repeatedly indicated she was bothered by this missing content. She would often reference the fact that she was further behind her peers in the intervention, because they had all been at the first Summer Institute. That is, not only did she miss this important content but also seemed very self-conscious about her lack of content knowledge when compared to her peers.

In addition, it is natural that Nancy and Marsha began the intervention with different mathematical identities. From the very beginning, Nancy viewed herself as a learner and enjoyed teaching mathematics. Marsha, on the other hand, often referenced her lifelong difficulties with mathematics and did not view herself as a strong mathematical learner. These differing perceptions likely have served as additional factors in how each of them engaged with the new approach to what constitutes mathematics knowing, learning, and hence teaching. While Nancy's learner self-perception led to seeing the challenges as something to work through, Marsha often seemed to feel defeated while engaging in the mathematical tasks.

These are all possible factors which may have contributed to their different learning trajectories that my study did not address. Yet, I have chosen to focus on their levels of units coordination, because it is my belief that their levels of units coordination was a key factor in their ability or inability to engage with the mathematics explored and promoted during the PD intervention. Future research could focus, for example, on how teachers like Marsha struggle with a negative mathematical identity due to early assimilations of only one or one and a half levels of units, which in turn made multiplicative and fractional reasoning rather difficult. Such research might also shed light on how her explicated feeling as incompetent within the PD may be rooted in levels of units coordination. Such a lack of confidence in one's own reasoning and mathematics may lead to not positioning oneself as a leader.

REFERENCES

Aguirre, J., Herbel-Eisenmann, B. A., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements D. (2017) Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education*, *48*(2), 124-147.

Arbaugh, R., Herbel-Eisenmann, B., Ramirez, N., Knuth, E., Kranendonk, H., Reed Quander, J. (2010). *Linking research and practice: The NCTM Research Agenda Conference report.* Reston, VA: National Council of Teachers of Mathematics.

Armstrong, B. E. & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 85-119). Albany: State University of New York Press.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90(4)*, 449-446.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching, 4th ed.* (pp. 433-456). New York: Mackmillan.

Bishop, J. (2012). "She's Always Been the Smart One. I've Always Been the Dumb One": Identities in the Mathematics Classroom. *Journal for Research in Mathematics Education*, *43*(1), 34–74.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers give up too easily? *Journal for Research in Mathematics Education, 23(3),* 194-222.

Boyce, S. (2014). *Modeling students' units coordinating activity*. (Publication No. 14933) [Doctoral dissertation, Virginia Polytechnic Institute and State University]. Retrieved from http://vtechworks.lib.vt.edu.

Boyce, S., & Norton, A. (2019). Maddie's units coordinating across contexts. *Journal of Mathematical Behavior*, 55, 1-13.

Creswell, J. W. (2013). *Qualitative inquiry & research design* (3rd ed.). Los Angeles, CA: Sage Publications.

Davies, B. & Harré, R. (1999). Positioning and personhood. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (p. 32-52). Oxford, UK: Blackwell Publishers.

Easterby-Smith, M., Thorpe, R., & Jackson, P. R. (2008). *Management research* (3rd ed.). London: Sage Publications.

Erickson, F. (1992). The interface between ethnography and microanalysis. In M. D. LeCompte, W. L. Millroy, & J. Preissle (Eds.), *The handbook of qualitative research in education* (pp. 201-225). San Diego, CA: Academic Press.

Fischbein, E., Deri, M., Nello, M. S., Marino, S. M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16(1),* 3-17.

Glaser, B. G. & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 276-295). New York: Simon & Schuster Macmillan.

Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction*, 28(4), 383-432.

Hackenberg, A. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, 32(2013), 538-563.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior 28*, 1–18.

Harré, R., & van Langenhove, L. (1999). Positioning theory: Moral contexts of intentional action. Oxford: Blackwell.

Herbel-Eisenmann, B. A., Wagner, D., Johnson, K. R., Suh, H., & Figueras, H. (2015). Positioning in mathematics education: revelations on an imported theory. *Educational Studies in Mathematics*, *89(2)*, 185–204.

Heritage, J. (1984). *Garfinkel and ethnomethodology*. New York: Polity Press.

Hill, H. and Ball, D. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education, 35*, 330-351.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects on teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42(2)*, 371-406.

Hill, H.C., Schilling, S., Ball, D.L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *The Elementary School Journal, 105(1)*, 11-30.

Hodkowski, N. M. (2018). *Manifestations of elementary mathematics teachers' shift towards second-order models.* (Order No. AAI10842329). [Doctoral dissertation, University of Colorado Denver] Retrieved from https://search-proquest-com.aurarialibrary.idm.oclc.org/dissertations-theses/manifestations-elementary-mathematics-teachers/docview/2120611232/se-2?accountid=14506.

Hunt, J. H., Tzur, R., & Westenskow, A. (2016). Evolution of unit fraction conceptions in two fifth-graders with a learning disability: An exploratory study. *Mathematical Thinking and Learning, 18(3)*, 182-208.

Izsák, A., Jacobsen, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4).

Kiczek, R. D. (2000). *Tracing the development of probabilistic thinking: Profiles from a longitudinal study.* Unpublished doctoral dissertation, Rutgers University, New Brunswick.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). Columbus, OH: ERIC/SMEAC.

Lamon, S. J. (2012). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). New York: Routledge.

Lemke, J. L. (2000). Across the scales of time: Artifacts, activities, and meanings in ecosocial systems. *Mind, Culture, and Activity*, *7(4)*, 273-290.

Lovin, L. H., Stevens, A. L., Siegfried, J., Wilkins, J. L. M., Norton, A. (2018). Pre-K-8 prospective teachers' understanding of fractions: An extension of fractions schemes and operations research. *Journal of Mathematics Teacher Education, 21(3)*, 207-235.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understandings of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Maher, C. A. (2002). How students structure their own investigations and educate us: What we've learned from a fourteen year study. In A. D. Cockburn & E. Nardi (Eds.), *Proceedings of the Twenty-Sixth Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-46). Norwich, UK: University of East Anglia.

Maher, C. A. & Davis, R. B. (1996). Children's explorations leading to proof. In *Proceedings of the Mathematical Sciences Institute of Education Conference on Proof*. University of London, UK.

Maher, C. A. & Martino, A. M. (1996a) The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27(2)*, 194-214.

Maher, C. A. & Martino, A. M. (1996b). Young children invent methods of proof: The gang of four. In P. Nesher, L. P. Steffe, P. Cobb, B. Greer, & J. Golden (Eds.), *Theories of mathematical learning* (pp. 431-447). Mahwah, NJ: Lawrence Erlbaum Associates.

Maher, C. A. & Martino, A. M. (2000). From patterns to theories: Conditions for conceptual change. *Journal of Mathematical Behavior, 19(2)*, 247-271.

Maher, C. A., Pantozzi, R. S., Martino, A. M., Steencken, E. P., & Deming, L. S. (1996). *Analyzing students' personal histories: Foundations of mathematical ideas*. Paper presented at the American Educational Research Association, New York.

Maher, C. A. & Speiser, R. (1997). How far can you go with block tower? Stephanie's intellectual development. *Journal for Research in Mathematics Education, 16(2)*, 125-132.

Miles, M. B. & Huberman, A. M. (1994) *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Moghaddam, F. M., Harré, R., & Lee, N. (2008). Positioning and conflict: An introduction. In F. M. Moghaddam, R. Harré, & N. Lee (Eds.), *Global conflict resolution through positioning analysis* (p. 3-20). New York: Springer.

Murtadha-Watts, K. & D'Ambrosio, B. S. (1997). A convergence of transformative multicultural and mathematics instruction? Dilemmas of group deliberations for curriculum change. *Journal for Research in Mathematics Education, 28(6),* 767-782.

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal, 45(4)*, 1080-1110.

Norton, A., & Boyce, S. (2013). A cognitive core for common state standards. *Journal for Mathematical Behavior*, 32, 266–279.

Norton, A., & Boyce, S. (2015). Provoking the construction of a structure for coordinating $n + 1$ levels of units. *Journal of Mathematical Behavior*, 40, 211-232.

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J.L.M. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education*, 10(2), 111-136.

Norton, A., Boyce, S., Ulrich, C., & Phillips, N. (2015). Students' units coordination activity: A cross-sectional analysis. *The Journal of Mathematical Behavior*, *39(2015)*, 51-66.

Norton, A., & Hackenberg, A. J. (2010). Continuing research on children's fraction schemes. In L. P. Steffe & J. Olive (Eds.), *Children's Fractional Knowledge* (p. 341-352). New York: Springer.

Norton, A. & Wilkins, J. L. M. (2012). The splitting group. *Journal for Research in Mathematics Education, 43(5),* 557-583.

Norton, A., Wilkins, J. L. M., & Xu, C. Z. (2016). A progression of fractions schemes that crosses Chinese and US classrooms. *Journal for Research in Mathematics Education, 49(2)*, 210-226.

Onwuegbuzie, A. J., & Collins, K. M. T. (2007). A typology of mixed methods sampling designs in social science research. *The Qualitative Report*, *12*(2), 281-316.

Piaget, J. (1976). Piaget's theory. In B. Inhelder, H. H. Chapman, & C. Zwingmann (Eds.), *Piaget and His School* (p. 11-23). New York: Springer.

Piaget, J. (1980). *Adaptation and intelligence*. Chicago: The University of Chicago Press.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development.* Chicago: The University of Chicago Press.

Pirie, S. & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, *26*, 165-190.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22(2003)*, 405-435.

Shulman, L.S. (1987). Knowledge and teaching: Foundations of the New Reform. *Harvard Educational Review, 57(1)*, 1-22.

Siegler, R. S. (1995). How does change occur: A microgenetic study of number conservation. *Cognitive Psychology, 28*, 225-273.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26(2),* 114-145.

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking & Learning, 8(4),* 359-371.

Simon, M. A. (2009). Amidst multiple theories of learning in mathematics education. *Journal for Research in Mathematics Education*, *40*(5), 477-490.

Simon, M. A. (2018). An emerging methodology for studying mathematics concept learning and instructional design. *Journal of Mathematical Behavior, 52(2018),* 113-121.

Simon, M. A., Kara, M., Placa, N., & Avitzur, A. (2018). Towards an integrated theory of mathematics conceptual learning and instructional design: The learning through activity theoretical framework. *Journal of Mathematical Behavior, 52,* 95-112.

Simon, M. A. & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30(3)*, 252-264.

Simon, M.A. & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6(2),* 91-104.

Simon, M.A., Tzur, R., Heinz, K., Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35(3),* 305-329.

Silverman, J. & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11,* 499-511.

Steencken, E. P. (2001). *Tracing the growth of understanding of fraction ideas: A fourth grade case study.* Unpublished doctoral dissertation. Dissertation Abstracts International 3009381, Rutgers University.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4(3)*, 259–309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3–39). Albany: State University of New York Press.

Steffe, L. P. (2007). *Problems in mathematics education.* Paper presented for the Senior Scholar Award of the Special Interest Group for Research in Mathematics Education (SIG-RME) at the annual conference of the American Educational Research Association in Chicago, Illinois.

Steffe, L. P. (2011). A theory bite on learning through mathematical activity. *Cognition and Instruction, 29*(2), 256-263. doi:10.1080/07370008.2011.557601

Steffe, L. P. & Cobb, P. (1988). *Construction of arithemetical meanings and strategies.* New York, NY: Springer.

Steffe, L. P., Liss, D. R. I., & Lee, H. Y. (2013). On the operations that generate intensive quantity. In L. P. Steffe, K. C. Moore, & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 49-79). University of Wyoming: Wyoming Institute for the Study and Development of Mathematical Education (WISDOMe).

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York, NY: Praeger Scientific.

Stevens, A. L., Wilkins, J. L. M., Lovin, L. H. Siegfried, J., Norton, A., & Busi, R. (2018). Promoting sophisticated fraction constructs through instructional changes in a mathematics course for PreK-8 prospective teachers. *Journal of Mathematics Teacher Education, (2018)*, https://doi.org/10.1007/s10857-018-9415-5.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Boston: Kluwer.

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31(1)*, 5-25.

Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1998). Prospective elementary teachers' conceptions of rational number. The United States-Israel Binational Science Foundation. Retrieved from http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html.

Toluk-Ucar, Z. (2009) Developing pre-service teachers understanding of fractions through problem posing. *Teaching and Teacher Education, 25(1)*, 166-175.

Tzur, R. (1996). *Interaction and children's fraction learning*. Ann Arbor, MI: UMI Dissertation Services (Bell & Howell).

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30(4)*, 390-416.

Tzur, R. (2011). Can dual processing theories of thinking inform conceptual learning in mathematics? *The Mathematics Enthusiast, 8(3),* 597-636.

Tzur, R. (2013). Reorganization of anticipation: A hard core principle in Steffe's research program on children's progression from numerical to algebraic reasoning. *Wyoming Institute for the Study and Development of Mathematical Education (WISDOMe), 4*, 175-197.

Tzur, R. (2019a). Elementary conceptual progressions: Reality check + implications. In J. Novotná & H. Moraová (Eds.), *Proceedings of the international Symposium on Elementary Mathematics Teaching* (Vol. 1, pp. 29-40). Prague, Czech Republic: Charles University.

Tzur, R. (2019b). Developing fractions as multiplicative relations: A model of cognitive reorganization. In A. Norton & M. W. Alibali (Eds.), *Constructing number: Merging perspectives from psychology and mathematics education* (pp. 163-191). Cham, Switzerland: Springer Nature.

Tzur, R. (2019c). HLT: A Lens on Conceptual Transition between Mathematical "Markers." In D. Siemon, T. Barkatsas, and R. Seah (Eds.). *Researching and learning progressions (trajectories) in mathematics education* (pp. 56-74). Rotterdam, The Netherlands: Sense.

Tzur, R., (2019d). Two-stage changes in anticipation: Cognitive sources of Aha!Moments. In B. Czarnocha & W. Baker (Eds.), *Creativity of Aha!Moment and mathematics education.* Leiden, The Netherlands: Brill.

Tzur, R., & Depue, B. E. (2014a). *Brain processing of whole-number vs. fraction comparisons: Impact of constructivist-based task design on reaction time and distance effect*. Paper presented at the Annual meeting of the American Educational Research Association, Philadelphia, PA.

Tzur, R., & Depue, B. E. (2014b). Conceptual and brain processing of unit fraction comparisons: A CogNeuro-MathEd study. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 5, pp. 297-304). Vancouver, Canada: PME.

Tzur, R., Hodkowski, N. M., & Uribe, M. (2016). A grade-4 teacher's mathematics: The case of Annie's understanding of decimal fractions. In *Proceedings of the 14th Annual Hawaii International Conference on Education*. Honolulu, HI: Author.

Tzur, R., & Hunt, J.H. (2015). Iteration: Unit fraction knowledge and the French fry tasks. *Teaching Children Mathematics, 22(3),* 149-157.

Tzur, R., Johnson, H. L., Hodkowski, N. M., Nathenson-Mejia, S., Davis, A., & Gardner, A. (2020). Beyond getting answers: Promoting conceptual understanding of multiplication. *Australian Primary Mathematics Classroom, 25(4)*, 35-40.

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., Woordward, J., Hord, C., Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA*, 7(3), 85-101.

Tzur, R., Johnson, H., Nathenson-Mejia, S., Davis, A., Uribe, M., Ferrara, M., Want, X. (2015). Adaptive pedagogy for elementary teachers: Promoting multiplicative and fractional reasoning to improve students' preparedness for middle school mathematics. *National Science Foundation Grant*.

Tzur, R., & Simon, M. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2004(2),* 287-304.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively Part 1). *For the Learning of Mathematics, 35(3)*, 2-7.

Ulrich, C. (2016) Stages in constructing and coordinating units additively and multiplicatively (Part 2). *For the Learning of Mathematics, 36(1)*, 34-39.

von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In Benseler, F., Hejl, P. M., Kock, W. K. (Eds.) *Autopoiesis, communication, and society.* Frankfurt: Campus, 75-85.

von Glasersfeld, E. (1981). An attentional model for the conceptual construction of units and number. *Journal for Research in Mathematics Education, 12(2)*, 83-94.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning.* London: The Falmer Press.

Wilkins, J. L. M. & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education, 42(4)*, 386-416.

Yin, R. K. (2017). *Case study research and applications: Design and methods* (6th ed.). Thousand Oaks, CA: Sage Publications.

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