THE EFFECTS OF TECHTIVITIES ON STUDENTS' REASONING: EXAMINING HOW STUDENTS INTERPRET DYNAMIC SITUATIONS DURING AN ONLINE COVARIATIONAL REASONING ASSESSMENT

by

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The effects of techtivities on students' reasoning: Examining how students interpret dynamic situations during an online covariational reasoning assessment

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ABSTRACT

Covariational reasoning is essential to developing a conceptual understanding of key mathematical concepts, such as function and rate of change. Students lack opportunities to develop covariational reasoning in the classroom, which limits their understanding of these concepts that are fundamental to courses like algebra and calculus. As these classes are necessary for most STEM majors, this is causing a shortage of qualified workers in STEM fields. We examine students' reasoning while working through four online assessment items designed to promote their covariational reasoning. Using the data collected by the *Implementing Techtivities to Promote Students' Covariational Reasoning in College Algebra* (ITSCoRe) research team, we investigate two research questions: (a) *How does the reasoning of students in the control group compare to students in the treatment group?* and (b) *How does the reasoning of students who answered the assessment items correctly compare to students who answered incorrectly?*

The ITSCoRe team worked with college algebra students, who were split into control and treatment groups. The treatment group worked with techtivities (free, online resources that link animations with dynamic graphs) throughout the semester. To collect data, the ITSCoRe team created an online assessment that was designed to assess students' covariational reasoning. Using students' written responses, we coded for evidence of different kinds of reasoning, including covariational, variational, motion, and iconic.

We report two main findings. First, students in the treatment group were more likely than students in the control group to show some type of coded reasoning on the assessment items. Second, students whose responses showed evidence of one of these four types of coded reasoning were more likely to answer the assessment items correctly. Based on these findings, we can conclude that the techtivities impacted students' reasoning, as evidenced by their responses to the assessment items, and that their reasoning impacted their ability to answer the items correctly. This suggests that techtivities can promote students' reasoning. Opportunities to engage in covariational reasoning, such as working with techtivities, may provide students with a better conceptual understanding of dynamic situations, which could help more students find success in the math classroom.

The form and content of this abstract are approved. I recommend its publication.

Approved: Heather L. Johnson

DEDICATION

Lanaghan:

I dedicate this to my wife, Eleanor. Without her love and support over the last few years, I never would have been able to accomplish my goals as an educator and learner. Thank you for always being by my side and understanding the late nights, my stresses, and when I had to choose school over hanging out with you.

Van Wright:

I dedicate this thesis to my husband, Kit. Without his understanding and support (and nights of solo-parenting), I would not have been able to see this through to completion. I also dedicate this to my baby, Colby, who made it especially challenging to choose work over him at times, but who helped me find a healthy balance between being a teacher, being a student, and being a mother. And to my own mother, who taught me everything I know about grammar rules (even though I still forget them at times), and who has always believed in me no matter what. And lastly, to my father, who instilled in me a joy of math from as young of an age as I can remember.

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CHAPTER I

INTRODUCTION

While algebra and calculus are vital to many different science, technology, engineering and mathematics (STEM) fields, students are lacking opportunities to develop conceptual understanding critical to algebra and calculus, which is cutting the supply for STEM education short, and as a result, jobs are going unfilled (Carter, Helliwell, Henrich, Principe, & Sloughter, 2016; Heiny, Heiny, & Raymond, 2017). Students who are considered prime candidates for STEM programs may change majors because they do not have the opportunity to learn the math needed to succeed. Secondary and undergraduate students can struggle with rates of change, interpreting graphs, and understanding how variables relate in a function (Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Ellis, Tasova, & Singleton, 2018; Moore & Thompson, 2015; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) With much of the forecasted job growth being in STEM related fields, teachers need to focus on developing students' mathematical reasoning.

Placement exams can contribute to the problem of placing students in classes for which they do not yet have the conceptual understanding needed to succeed. The current college mathematical placement exams focus on computational procedures rather than students' abilities to reason about functions (Madison, Carlson, Oehrtman, and Tallman, 2015;). This makes placement into a course like college algebra something students can do based on their memorization of different algebraic techniques rather than their conceptual understanding (Madison et al., 2015). Or worse, it may keep students demonstrating conceptual understanding from enrolling in math classes because tests focus too heavily on procedures. One of the reasons that college algebra has such a high attrition rate is because students have not been given the opportunity to develop the conceptual understanding that is essential to finding success in the class (Madison et al., 2015).

Covariational reasoning is a key building block of numerous mathematical ideas, including proportion, rate of change, trigonometry, exponential growth, the Fundamental Theorem of Calculus, and differential equations (Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017; Ferrari-Escola, Martinez-Sierra, & Mendez-Guevara, 2016; Ellis, et al., 2016; Thompson & Carlson, 2017; Carlson et al., 2002). Even though covariational reasoning is essential to mathematics and is accessible to elementary school students, it is not heavily emphasized in most schools, at least not in the United States (Thompson & Carlson, 2017). Covariational reasoning focuses on the *relationship* between two quantities and how they are continuously changing with respect to one another. Carlson and colleagues (2002) define covariational reasoning as "The cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). For example, if a student were asked to describe the way a bowl fills with water, the student might conceive of a relationship between the water level and the volume of the water. The student might conceive that as the water level increases, the volume also must increase. The student conceives of the two attributes as covarying when they coordinate that as the water level increases, the volume of the water increases.[1] Focusing on developing students' conceptual understanding of function and covariational reasoning from a young age can foster their success in secondary and undergraduate math courses (Thompson & Carlson, 2017).

To better understand secondary and undergraduate students' covariational reasoning, researchers often employ tasks that do not use numbers (Saldanha & Thompson, 1998; Johnson,

[1] We use the singular "they" throughout this paper as a gender-neutral pronoun

2012; Johnson, McClintock, & Hornbein, 2017; Thompson et al., 2017; Moore & Thompson, 2015). In designing tasks to measure and better understand a student's ability to engage in covariational reasoning, it is important to consider different types of attributes and their complexities (Johnson et al., 2017). To this end, Johnson and colleagues (2017) adapted a bottle filling task from the Shell Centre (1985). This task showed videos of a bottle being filled with water and asked students to sketch a graph of the relationship between the height of the water and the overall volume of water in the bottle. Johnson and colleagues (2017) found that some types of attributes, like height, promoted opportunities for covariational reasoning more than others, like volume. Giving students opportunities to work on tasks with different types of attributes that do not have numerical values can provide researchers and teachers insight into how students are reasoning.

Student Interpretations of Graphs

In many disciplines of mathematics, graphs are frequently used to represent two varying quantities (Clement, 1989). Although graphs have been a common feature in math classes from a very young age (grade 2 according to the Common Core State Standards), many students have not been given the opportunity to reason about graphs in a way that helps them conceive of how the two quantities covary (Carlson et al., 2002; Thompson & Carlson, 2017). A student may conceive of a graph of a Ferris wheel spinning as showing the literal shape of the Ferris wheel, or as the literal motion of the Ferris wheel as it spins. When students look at a graph representing two varying quantities, they may not conceive of the graph as depicting two varying quantities, or understand how those quantities covary. Students may conceive of the graph as a representation of the motion of an object, or as a representation of the iconic shape of the object or graph (Clement, 1989; Thompson & Carlson, 2017; Kerslake, 1977; Bell & Janvier, 1981;

Moore & Thompson, 2015). If students engage in covariational reasoning, then it may help them to conceive of graphs as representing relationships between attributes (Johnson et al. under review).

Using Techtivities to Promote Students' Reasoning

We report a mixed methods study, in which we analyze data collected by the Implementing Techtivities to Promote Students' Covariational Reasoning in College Algebra (ITSCoRe) research team led by principal investigator Johnson of the University of Colorado Denver (CU Denver). One goal of the ITSCoRe team's research is to help students develop their covariational reasoning through techtivities. These techtivities are free, online resources that link animations with dynamic graphs so that students can see how the attributes change with respect to one another (Johnson, McClintock, Kalir, & Olson, 2018). As part of their research into the effectiveness of these techtivities, the ITSCoRe team utilized the population of college algebra students at CU Denver. The ITSCoRe team developed a fully-online assessment (in place of clinical interviews or a paper and pencil test) to measure how students reasoned about dynamic situations (Johnson, Kalir, Olson, Gardner, Smith, & Wang, 2018). For our study, we analyzed data collected from this covariational reasoning assessment.

The ITSCoRe team's focus on college algebra students allows us to concentrate our research on a class in which covariational reasoning is essential. Many of the students in the college algebra course wish to pursue further STEM studies, and this course is a prerequisite to move on to higher level mathematics courses (such as calculus) at CU Denver. The students at CU Denver come from a variety of backgrounds, with 50% being first generation college students and 44% being minorities ("Quick Facts," 2019). The students in our sample were representative of the larger population.

We were drawn to the ITSCoRe team's research because of their work with college algebra students. As high school teachers, we both work with algebra and calculus students and we see our students struggle on a daily basis to understand graphs, rates of change, and functions. For example, in our algebra classes, interpreting rate of change from a graph is challenging for some of our students. Similar to Clement's (1989) claim, we can see how students could treat a graph as a picture, paying little to no attention to the axes to determine the rate of change. Providing our students opportunities to improve their covariational reasoning may help them better understand graphs, rate of change and functions.

Research Questions

In this study, we examine the ITSCoRe team's data from their covariational reasoning assessment given to college algebra students in the fall of 2018. The Fall 2018 sections of college algebra were split into a treatment and control groups. The treatment group sections were administered the techtivities throughout the semester, and the control group sections experienced business as usual. We used the written responses from four assessment tasks as evidence of student reasoning. These assessment items involve dynamic situations that each relate two covarying quantities. Using students' written responses to these assessment items, we answer the following research questions:

- *How does the reasoning of students in the control group compare to students in the treatment group?*
- *How does the reasoning of students who answered the assessment items correctly compare to students who answered incorrectly?*

By analyzing students' written responses to these four assessment items, we compare how students in the treatment group reasoned about the four assessment items with students in the control group. We also compare the type of reasoning shown by students who answered the assessment item correctly with those who answered incorrectly. By addressing our research questions, we intend to help researchers and educators gain a better understanding of whether applying more emphasis on dynamic situations in the classroom through the use of techtivities helps students develop covariational reasoning.

CHAPTER II

REVIEW OF THE LITERATURE

In this literature review, we look at the different types of reasoning that students might engage in while interacting with dynamic situations. To begin, we briefly address the process of quantification, which is a building block for variational and covariational reasoning. Then, we discuss variational reasoning and covariational reasoning. We finish by discussing how students interpret graphs and how some students conceive of graphs as depicting the motion of the dynamic situation or as an iconic representation of the dynamic situation.

Quantity

When students reason about an attribute of an object, such as the height of a tree, they can reason about it in a number of ways, including wondering how tall the tree is, if it is the tallest tree in the area, or if it is taller than their house. They also might wonder how they could determine the tree's height. When a student reasons about an attribute as something that is possible to measure, they are conceiving of the attribute as a *quantity*(Thompson, 1994). Thompson (1994) explains that *quantities* are conceptual entities that exist in people's conceptions of situations. When a person conceives of an object's attribute (a feature of the object, like the height of a building) in a way that makes the attribute possible to measure, they are conceiving of that attribute as a quantity. Consider, for instance, a student reasoning about a tree. The student may perceive the tree to have a height (an attribute) that they could measure. This gives evidence the student conceiving of the height as a quantity. When students conceive of an object's attribute as measurable, they are thinking about the attribute's quantity, and therefore modeling the attribute with mathematics.

The ability to quantify attributes of objects helps students take a real-world situation and model it with mathematics (Thompson, 2011). For instance, a student might look at a tree and want to determine how tall the tree is so that they can compare it to other trees in the area. Johnson (2015) paraphrases Thompson's (2011) definition of quantification as "conceiving of an attribute of an object, conceiving of a unit of measure for the attribute, and forming a relationship between the attribute's measure and the unit of measure" (p. 65). For example, the height of an object is an attribute that a student can quantify. A student might see a tree and want to know how tall the tree is. Using their own height as a unit of measure, students could determine a relationship between their height and the tree. Students could conceive that the tree is five iterations of themselves tall. In this case, the students created a unit of measure to understand how tall the tree is in relation to their own height. The ability to quantify attributes of objects gives students a way to mathematically model objects they see in the real world.

When a student is quantifying and comparing the height of a tree to themself, they are engaging in *quantitative reasoning*. According to Thompson and Carlson (2017), *quantitative reasoning* "is someone conceptualizing a situation in terms of quantities and relationships among quantities" (p. 424). Looking at our previous example, the students reasoned quantitatively because they conceptualized the height of the tree as five iterations of themselves tall, creating a relationship between two quantities: the height of the tree and their own height. Quantitative reasoning is critical for algebraic reasoning, and is interconnected with variational and covariational reasoning. Being able to reason quantitatively can help students in secondary and college-level math courses develop their understanding of algebraic reasoning, covariational reasoning and variational reasoning (Smith & Thompson, 2007; Ferrari-Escola et al., 2016; Johnson, 2015; Thompson, 2011; Moore & Thompson, 2015).

Variation

When students are presented with a dynamic situation, like a Ferris wheel rotating or a bowl filling with water, they can conceive of attributes that are not static, but changing. For example, as a Ferris wheel spins, a student may conceive that the height of one of its carts increases and decreases and the distance the cart travels increases as it spins. In mathematics, changing quantities of attributes are denoted with a *variable.* When the variable for a quantity is something that can change, researchers call this *variation*. Thompson and Carlson (2017) state, "a variable's variation comes from a person's thinking, either concretely or abstractly, that the quantity whose value the letter represents has a value that varies" (p. 424). Using our example in which the Ferris wheel's cart changes height, a student could denote the cart's height with a variable. They could then conceive that as the cart moves, the variable's value changes because the attribute's quantity (the height of the cart) increases and decreases. Even though variables are commonly used to denote changing quantities, thinking of a quantity as varying does not require the use of a variable (Johnson & McClintock, 2018).

Conceiving that quantities are not static but rather something that can vary can help students better understand dynamic situations. Consider a student watching a bottle fill with water. The student might reason that the height of the water is increasing, rather than remaining constant. This kind of reasoning is known as *variational reasoning*. Thompson and Carlson (2017) describe variational reasoning as a cognitive activity where a student conceives of a quantity as changing. For example, a student watching a bottle fill with water could conceive that as more water is poured into the bottle, the height of the water increases. For a teacher to understand how the student is reasoning, the student would need to provide observable evidence to the teacher of how they conceive of the variation of the quantity. For example, the student might provide evidence in the form of the response, "The water level is increasing." Responses are not the only kind of evidence that a student could provide a teacher. Teachers can also make interpretations of a student's reasoning by watching the way they draw a graph, paying attention to the gestures they use, or listening to their explanations and discussions they have with other students.

While it may seem simple for a teacher to describe variation to a student, it is nontrivial for students to develop conceptual operations that allow them to conceive of variation in quantitative situations (Thompson, 2011). Johnson and McClintock (2018) use the term *quantitative variational reasoning* (QVR) to mean "students' reasoning about attributes that they can conceive of as capable of varying and possible to measure" (par. 4). QVR consists of two key conceptions for students: being able to reason about attributes as varying and being able to reason about attributes as being possible to measure (Johnson & McClintock, 2018). For example, a student showing evidence of QVR might describe a tree as having height (something measurable), and that height increasing each year (the varying attribute). QVR expands upon how a student might be reasoning variationally, but does not specifically address the different ways a student might conceive of the variation.

If a teacher determines a student is reasoning variationally based on the evidence the student provides, it may not be clear what kind of variational reasoning the student is conceiving. Thompson and Carlson (2017) created their framework to explain different variational reasoning level a student might be conceiving, ranging from *variable as symbol* to *smooth continuous variation* (see Table 1). These six levels are dependent on how the student is thinking about the varying quantity, which is not always apparent in a student's response.

Table 1.
Major Levels of Variational Reasoning

Level	Description
Smooth continuous variation	The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-size intervals of variation, but not necessarily.
Chunky continuous variation	The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be the same size, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk—like numbers on a ruler—but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values. Chunky continuous variation is not just a person thinking that changes happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.
Gross variation	The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing.
Discrete variation	The person envisions a variable as taking specific values. The person sees the variable's value changing from a to b by taking values $a_1, a_2, \ldots, a_n$ but does not envision the variable taking any value between a_i and a_{i+1}.
No variation	The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.
Variable as symbol	The person understands a variable as being just a symbol that has nothing to do with variation.

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Teachers and researchers can use this framework to help determine the level at which a student is reasoning variationally. All of these levels are based on the students' conceptions, not on what evidence they provide in their reasoning. Using the evidence that the student provides, a teacher can use this framework to try to identify the student's level of reasoning. For example, if a student were asked to describe what happens when a bottle fills with water, they could conceive of the change in a number of different ways. A student reasoning with *smooth continuous variation* might think of the height as smoothly and continuously increasing up to the halfway mark, and then to the brim. A student reasoning with *chunky continuous variation* might think of the height growing continuously from the quarter mark, to halfway, to the three-quarter mark, and then to the brim, in intervals. A student with *gross variation* reasoning might envision the height as measurable and increasing, but not consider the value of the height's measurements. A student reasoning with *discrete variation* might envision specific heights of the water along the way, but not conceive of it attaining all of the heights in between. A student at the *no variation* level could envision the height of the water taking on a fixed value for the entire scenario. A student at the *variable as symbol* level might not think of the height as having a value or being anything besides a symbol. Johnson and McClintock's (2018) QVR encompasses the levels of gross coordination, chunky continuous variation, and smooth continuous variation because all of these levels of variational reasoning involve the student reasoning about the attribute as being measurable and varying. The framework from Thompson and Carlson (2017) and QVR are both ways to identify how students are thinking about variation, beyond just identifying that they are reasoning variationally.

Because variational reasoning levels are cognitive actions, it can be difficult to determine a student's level of variational reasoning. For a teacher to determine a student's reasoning level, they must interpret observable behaviors by the student (Johnson, McClintock, & Gardner, under review). For example, with the bottle being filled with water, a student could conceive of the water level as the attribute that is varying. They could conceive of the water level smoothly and continuously increasing as the bottle fills up to the halfway mark, and then to the brim. If a teacher asks the student to describe what happens as the bottle fills, the student could respond, "The water level is increasing," giving evidence that they are thinking at a gross variation level. Even though a student might be reasoning at a certain level, they may not provide evidence to convey that. Students' variational reasoning is a cognitive action, therefore determining a student's variational reasoning level must be done by inferring their cognitive actions based on observable behaviors.

Covariation

Dynamic situations typically involve multiple changing attributes. In many cases, like a Ferris wheel spinning, there is more than one attribute changing. In this case, the height of the cart and the distance the cart has traveled are two attributes that are varying and dependent on each other. As the cart's total distance traveled increases, the height constantly changes. These two attributes are considered to be *covarying* because they change in relation to each other.

When students encounter dynamic situations, it is useful for them to conceive of how two attributes change in relation to each other. This kind of reasoning is known as *covariational reasoning*. Thompson and Carlson (2017) define covariational reasoning as a cognitive process for a student comparing two different attributes and how they change in relation to each other. Carlson and colleagues (2002) define covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p.354). Both of these definitions complement each other in that covariational reasoning is a cognitive process for the student. To reason covariationally, the student attends to how two attributes change independently and how they change together. When a student engages in covariational reasoning, they can conceive of the two attributes as quantities, attend to how those quantities vary, and then conceive of how the two attributes vary together (Johnson et al., 2017). For example, if a student were asked to describe the way a car drove around a city block, the student could conceive of a relationship between the distance the car had driven and the car's distance from its starting point. The student could first conceive of the distance the car has driven and the car's distance from its starting point as two separate quantities. They could then reason variationally about how the total distance traveled increases, and the distance from the starting point increases and decreases. Lastly, the student could attend to how the two attributes covary; as the total distance increases, the distance from the starting point will increase and decrease as the car drives around the block. Covariational reasoning is a way that students can better understand dynamic situations, and how the changing attributes in those situations relate to each other

Similar to variational reasoning, Thompson and Carlson (2017) identified different levels of covariational reasoning. Thompson and Carlson (2017) created their framework to explain the different covariational reasoning levels at which a student might be conceiving, ranging from *no coordination* of changing attributes to *smooth continuous covariation* (see Table 2).

Table 2.
Major Levels of Covariational Reasoning

Level	Description
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.
Precoordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

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Comparable to variational reasoning, covariational reasoning is a cognitive action for the student, meaning that researchers could infer a student's reasoning level based on the student's observable behaviors. This framework can be used to help determine the level at which a student is reasoning covariationally. All of these levels are based on what the student cognitively understands, not on what evidence they provide in their reasoning. Based on the evidence that the student provides, a teacher or researcher can try to use this framework to identify a student's level of reasoning. To help explain these different levels, refer back to the Ferris wheel spinning and two of its changing attributes: the distance a cart is off the ground and the total distance the cart has traveled. For a student reasoning at a *smooth continuous covariation* level, the student could envision the height of the cart changing simultaneously with the distance the cart has traveled, and this change would be smooth and continuous as the cart traveled around the wheel. At a *chunky continuous covariation* level, the student could envision the height of the cart changing simultaneously with the distance the cart has traveled, but by intervals of change. A student reasoning at a *coordination of values* level could conceive that the height of the cart and the distance the cart has traveled are related values and could form a pair. For a student at a *gross coordination of values* level, the student could note that as the cart's height changes, the cart's total distance also changes. A student reasoning at a *precoordination of values* level could envision that the cart's height above the ground would change, and then the total distance traveled would change, and then the height again, and so on. To the student, these attributes could not be changing at the same time, but in succession. Finally, at a *no coordination,* the student could have no conceptual image of the cart's distance off the ground and total distance traveled varying together. They could only focus on the variation of one attribute or the other.

Because covariational reasoning levels are cognitive actions, it can be difficult to determine a student's exact level of covariational reasoning (Thompson & Carlson, 2017). For a teacher to determine student reasoning levels, they must interpret observable behaviors by the student. Looking back to the problem with the Ferris wheel spinning, a student could conceive of the cart's height off the ground and the distance the cart has traveled as covarying. They could conceive that as the carts distance smoothly and continuously grows, the height of the cart will smoothly and continuously change. If a teacher asks the student to describe what happens as the

Ferris wheel spins, the student could respond, "the cart goes up and down and the cart keeps spinning," giving evidence that they are thinking at a gross coordination of values. Even though a student might be reasoning at another one of the previously described levels, they may not provide evidence to convey that to an outside observer. Students' covariational reasoning is a cognitive action, therefore determining a student's variational reasoning level must be done by inferring their cognitive actions based on observable behaviors.

Students' Interpretations of Graphs

In many disciplines of mathematics, graphs are frequently used to compare two varying quantities (Clement, 1989). When students look at a graph comparing two varying quantities, they may conceive of the graph as depicting many different things. For example, a teacher could present a student with the graph of a car driving around a square track, with one axis representing the distance the car is from the center of the track, and other axis showing the total distance the car has traveled. A student might conceive of the graph as showing the literal shape of the track, and expect the track to look exactly like what the graph is showing (Clement, 1989; Moore & Thompson, 2015). Another student could conceive that the graph shows the motion of the car, and that each turn on the graph is a turn for the car in that direction (Kerslake, 1977; Bell & Janvier, 1981). These students are conceiving of the graph as a representation of either the literal motion of the car, or the literal shape of the track. (Clement, 1989; Thompson & Carlson, 2017; Kerslake, 1977; Bell & Janvier, 1981; Moore & Thompson, 2015). Students conceiving of graphs in these ways are doing something other than attending to the attributes represented on the axes.

Conceiving of Graphs as Representing the Motion of Objects

When teachers present graphs of dynamic situations to students, students might interpret the graph to show the motion of the object (Kerslake, 1977; Bell & Janvier, 1981). Kerslake (1977) presented three graphs to students and asked them to determine which ones represented a journey and what happened in each journey. For all three graphs, the axes represented the same attribute: distance on the y-axis and time on the x-axis. Some of the students who described what happened for their chosen graph described the motion that they saw the graph representing (Kerslake, 1977). The students interpreted the journey as either moving in cardinal directions, going up walls, moving up or down hills or mountains, and so on. (Kerslake, 1977). Even though a graph's axes commonly display attributes, students do not always make a relationship between them; they may interpret the graph as *representing the literal motion* of the journey (Bell & Janvier, 1981; Kerslake, 1977).

When a student interprets a graph to represent the literal motion of the object, they could envision the graph as tracing the physical path of the object (Bell & Janvier, 1981; Kerslake, 1977). Referring back to Kerslake's (1977) example, to the student, each different part of the graph showed the physical motions that took place in the journey. For a graph that had a vertical portion, students in Kerslake's (1977) study envisioned that a wall was being climbed, or that the journey traveled directly north. This interpretation by the student takes into account the shape of the graph and the students conceive that the graph shows the motion that is occurring in the journey (the person travels east, then goes north, and then east again). When students conceive of the graph as representing the literal motion of a dynamic situation, they can conceive of the line of the graph as showing that something moves or as showing the physical path that something takes.

Reasoning that the graph could represent the literal motion of the object can cause issues when students encounter digital tasks that link an animation with a dynamic graph (Johnson et al., under review). Because the animations are moving, students may think that the graph is representing the object's movement in the animation (Johnson et al., under review). For example, suppose a teacher presented a student with an animation of a car driving around a square track and a dynamic graph of the car's distance from the center of the track compared to its total distance traveled. The student might expect the graph to show the movement of the car at each moment it drives around the track because they saw the graph being made as the car moved around the track.

Conceiving of Graphs as Representing the Iconic Object

Even though graphs are often used to represent dynamic situations, students can struggle to understand what information is represented by the graph. One common error Clement (1989) reports is that students can treat graph as a literal picture. When students do this, they appear to believe that the physical shape of the object and the graph should look the same, regardless of what attributes the axes are showing (Clement, 1989). When a student reasons about a graph this way, they conceive of the graph as representing the *iconic shape* of an object. Basically, the student conceives of the graph as a literal representation of the shape and physical features of the object (Clement, 1989; Leinhardt, Zaslavsky, & Stein, 1990). The student envisions the graph to represent the look of the physical object, and could conceive of the graph as resembling it. For example, given a situation involving a Ferris wheel spinning, the student could expect the graph to have the literal shape of a Ferris wheel. With iconic interpretations of graphs students are less focused on the attributes on the axes and more focused on the shape they see, whether that be the shape of the object or the shape of the graph.

When a student reasons about a graph, they may conceive of the graph as representing something other than two covarying quantities. The student might conceive of the graph as a *static* shape. Moore and Thompson (2015) use the construct called *shape thinking* to characterize a way that students think about graphs. Using their two forms of shape thinking -*static* shape and *emergent* shape- they explain how students' conceptual operations underlie their interpretations of graphs (Moore & Thompson, 2015). *Static shape* thinking is when the student conceives of the graph as a physical shape (e.g., thinking of a graph of a parabola as a "U") (Moore & Thompson, 2015). Moore and Thompson (2015) describe this as "essentially treating the graph as a wire" (p. 784). For example, consider a bowl being filled with water and a graph that compares the height of the water to the diameter of its surface area. A student thinking with static shape thinking could just see the graph as a physical line, and that line as something that they can manipulate. Static shape thinking is a form of iconic interpretation because it extends the iconic interpretation to include familiar shapes, not just physical objects.

Conceiving of Graphs as Representing two Covarying Quantities

Students may see a graph as representing two covarying quantities, and this is called *emergent shape thinking* (Moore & Thompson, 2015). *Emergent shape* thinking is when the student conceives that the trace of the graph is made of two covarying quantities (Moore & Thompson, 2015). A student engaging in emergent shape thinking could conceive of the two covarying quantities creating the line of the graph in real time (Moore & Thompson, 2015). For example, when graphing the situation of a bowl being filled with water, a student with emergent shape thinking could envision the height of the water and the diameter of the water's surface area as creating the line of the graph as they change together in real time. Ideally, teachers would want to promote emergent shape thinking in their students because this leads the student to thinking about graphs using a covariation perspective (Johnson & McClintock, 2018; Moore & Thompson, 2015; Moore, Stevens, Paoletti, Hobson & Liang, in press). One major difference between *static* shape and *emergent* shape is that a student engaging in static shape thinking would be less focused on the attributes on the axes and more focused on the graph as an object, while a student engaged in emergent shape thinking could conceive of the quantities on the axes as covarying to create the image of the trace in their mind (Moore & Thompson, 2015).

Conclusion

Because covariational and variational reasoning are cognitive actions, it can be difficult to determine the presence of these types of reasoning when a student reasons about a dynamic situation (Thompson & Carlson, 2017). When students are presented with dynamic situations, teachers and researchers must rely on the evidence the student provides to determine how they might be reasoning. Using Thompson & Carlson's (2017) frameworks (see Tables 1 and 2), teachers and researchers can determine levels at which a student is reasoning covariationally or variationally based on the evidence the student provides.

Students can conceive of graphs in different ways. Ideally, students would interpret a graph as an emergent shape, where they can envision the smooth continuous covariation of the two attributes (Moore & Thompson, 2015; Johnson & McClintock, 2018). Not all students envision a graph in this way. Some students might conceive of a graph as showing the motion of an object, while others may view the graph as an iconic representation of the physical features of the object (Bell & Janvier, 1981; Kerslake, 1977; Clement, 1989; Leinhardt et al., 1990). Similar to determining a student's covariational and variational reasoning levels, determining if a student is viewing a graph as an iconic representation, as representing the motion of the object, or as an emergent shape is based on the evidence provided by the student (Bell & Janvier, 1981; Kerslake, 1977; Clement, 1989; Leinhardt et al., 1990).

CHAPTER III

METHODS

We drew on data obtained from one part of the ITSCoRe research team's project: the online covariational reasoning assessment (Johnson, Kalir et al., 2018). We analyzed the type of reasoning evidenced by students' written responses to the four items in this assessment.

Research Questions

We aim to address the following research questions: (a) *How does the reasoning of students in the control group compare to students in the treatment group?* and (b) *How does the reasoning of students who answered the assessment items correctly compare to students who answered incorrectly?* By addressing our research questions, we can assess whether students in the treatment group were more likely to demonstrate covariational reasoning and whether students whose responses conveyed covariational reasoning were more likely to answer the assessment item correctly.

Sample

The data for this study were gathered from an in-class covariational reasoning assessment, given to students in the college algebra course at CU Denver. The participants included the 250 students who were enrolled in 13 sections of this course in Fall 2018 and consented to having their data used for this study. Although specific demographics of these students are not known to us, CU Denver's undergraduate enrollment for Fall 2018 consisted of 56% White, 22% Hispanic, 12% Asian American, 7% African American, 2% Native American, and 1% Pacific Islander students ("Quick Facts," 2019). Approximately half of the students at CU Denver are first-generation college students ("Quick Facts," 2019).

The ITSCoRe team has been working with students enrolled in the college algebra course at CU Denver to find ways to measure and improve their covariational reasoning and promote their success as mathematicians. The college algebra course has provided the ITSCoRe team a group of students that it can directly benefit with its study, and a group of students for whom covariational understanding is essential to their success. Many of these students are taking college algebra as a way to proceed down a STEM route at CU Denver, as this class is necessary for STEM majors.

Students in the college algebra course attended lecture twice per week. Each lecture had a smaller group recitation meeting with a different instructor. Before the semester, recitation instructors were given the option to participate in the ITSCoRe team's professional development to learn how to implement covariational reasoning activities, known as techtivities, during their recitation meetings. Of the 13 recitation sections, ten sections were taught by instructors who implemented techtivities and three were taught by instructors who did not implement techtivities. The ten sections that implemented techtivities formed our treatment group and the three that did not were our control group. Both groups received equal instruction time over the course of the semester. The control group's instruction was no different than it had been in previous years, with no extra emphasis on covariational reasoning.

The techtivities were created in Desmos (www.desmos.com), in collaboration with Meyer, the Chief Academic Officer, to help develop students' covariational reasoning through dynamic situations (see Table 3). For example, one techtivity involved a man getting fired out of a cannon, and students compared the man's distance traveled with his height (Johnson, McClintock et al., 2018). In total, there were 7 different techtivities that the students in the treatment group completed, which are available on www.desmos.com ("How Graphs Work," 2018).

Table 3.
A Blueprint for a Techtivity

A Blueprint for a Techtivity
1. View animation of a situation involving changing attributes. Identify the changing attributes on which to focus in this situation. 2. Move a dynamic segment to show how one attribute is changing. 3. Move a second dynamic segment to show how the other attribute is changing. 4. View both dynamic segments changing together, appearing in conjunction with an animation. (In 2-4, dynamic segments are located on horizontal or vertical axes on a Cartesian Plane.) 5. Sketch a Cartesian graph representing how both attributes are changing together. 6. View a computer-generated Cartesian graph, appearing in conjunction with an animation. 7. Reflect on an aspect of the Cartesian graph. For example, is the graph what you expected? Is there anything about the graph that surprises you? Why might it make sense for a graph to look that way? Is it possible for two different looking graphs to represent the same situation? 8. Repeat 2-7 for a new Cartesian graph representing the same situation, with attributes on different axes.

Note: Reprinted from Johnson, H. L., McClintock, E., Kalir, R., & Olson, G. (2018), p. 1229

Each techtivity provides a dynamic video of the situation and shows students the attributes on which to focus. Students get a chance to play around with the different attributes and see how they are changing before creating their own graph relating the two attributes. Next, students see the correct graph and they can compare it to their own. While working on the techtivities, students create and interpret multiple graphical representations of the same situation, but with the changing attributes on different axes.

Near the end of the Fall 2018 semester, all 13 sections of college algebra were given an online covariational reasoning assessment. Students had the option of using a computer, tablet, or mobile phone to complete these assessment items. Students' responses were not assessed by their teacher, and they had no impact on students' grades for the course.

Covariational Reasoning Assessment Description

The ITSCoRe team distributed a fully online, validated assessment of students' covariational reasoning. To validate the assessment, Johnson conducted individual interviews with 30 college algebra students. The team found a statistically significant degree of correlation between students' correct answers and their engagement in covariational reasoning, $p < 0.01$. Furthermore, the variation in the total number of students' correct answers could be explained by students' engagement in covariational reasoning, $p < 0.001$ (Johnson & Wang, unpublished document).

In the assessment, there are four items involving four different situations: a cart moving around a Ferris wheel, a person (Nat) walking to and from a tree, a fish bowl filling with water, and a toy car going around a square track (Johnson, Kalir et al., 2018). For each of the items, students view a dynamic video of the situation and are told the attributes on which to focus. Each video highlights these attributes with segments or arcs representing each attribute's measure, and the video shows how the attributes change throughout the situation (see Figure 1). The order of the four assessment items are randomized for each student. One assessment item asks students to compare the relationship between a Ferris wheel cart's height from the ground and total distance traveled as it moves around the Ferris wheel. Another item involves Nat walking towards a tree on a straight path, and then away from the tree again. Students can compare Nat's distance from the tree with Nat's total distance traveled. In the Fish Bowl item, students compare the height of the water to the diameter of the water surface as the spherical bowl fills with water. In the Toy Car item, students compare the car's total distance traveled around the square track and its distance from the center of the track.

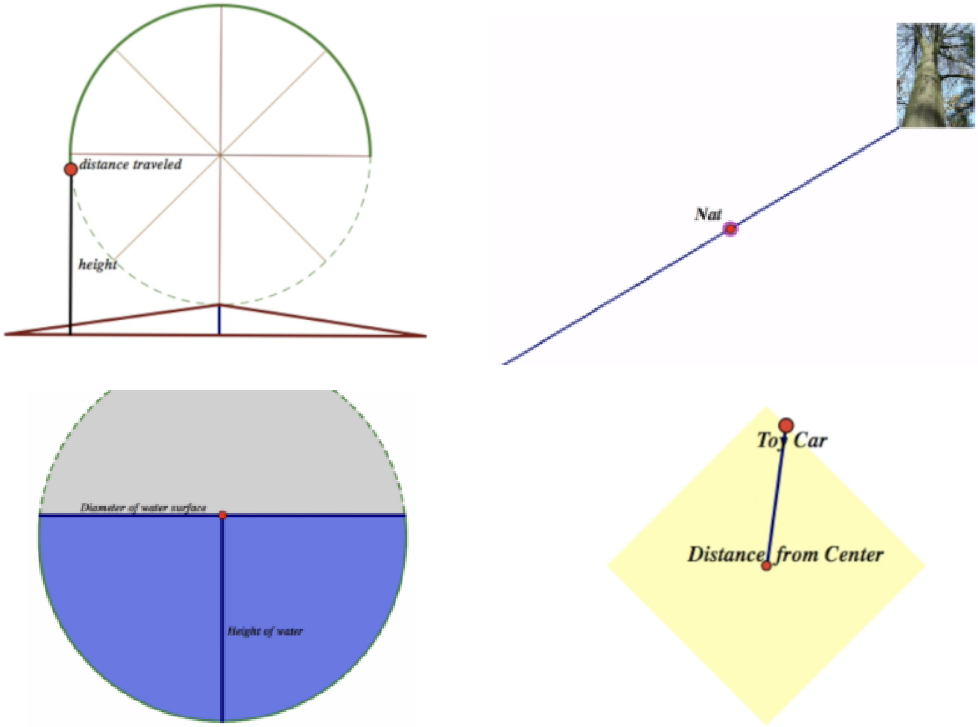

Figure 1. Screenshots from the Ferris Wheel, Nat+Tree, Fish Bowl, and Toy Car dynamic videos. Adapted from *Networking theories to design a fully online assessment of students' covariational reasoning,* by Johnson H.L., Kalir, R., Olson, G., Gardner, A., Smith, A., & Wang, X. (2018), p. 1345.

The ITSCoRe team carefully considered the types of attributes and situations they used when designing the covariational reasoning assessment (Johnson, Kalir et al., 2018). The assessment items purposely avoided using time as one of the graphed attributes. When students conceive of time as "experiential time," they experience time as simply progressing, and may or may not conceive of time as something they need to measure in the situation (Thompson & Carlson, 2017). Inspired in part by Moore, Silverman, Paoletti, and LaForest (2014), the ITSCoRe team also varied the types of attributes that were on the horizontal and vertical axes between items so that increasing and decreasing attributes could be found on either axis. Additionally, the team incorporated both linear and nonlinear graphs for each situation, as well as different backgrounds with which to measure the changing attributes, such as a round Ferris wheel and a square track.

Coding

We coded students' responses to each item based on the type of reasoning that their response conveyed. Johnson and the ITSCoRe team developed a coding rubric with seven categories for coding (Johnson, McClintock et al., 2018). The seven categories are: covariation, variation, motion, iconic, best answer, guess/unsure, and off topic (see Table 4). We used a priori (predetermined) codes, which enabled us to directly answer our research questions and compare the types of reasoning students used for each item (Saldaña, 2013).

We interpreted this table to mean that if students mentioned both changing attributes, their response would receive a COV code. If they mentioned one changing attribute, their response would be coded as VAR. Similar to Johnson and colleagues (under review), we did not code for levels of covariational and variational reasoning, as laid out in Thompson and Carlson's (2017) framework (see Tables 1 and 2). Rather, we coded for the presence of either type of reasoning. Students' written responses tended to be brief, and we did not feel as though we had enough information to accurately distinguish gradations in responses that conveyed COV and VAR reasoning. Responses would be coded as MO if they discussed the motion of the object in the video, without explicitly mentioning either of the attributes in question. Responses would receive an IC code if they talked about the shape of the object or the shape of the object's motion. Responses would be coded as ANS if they justified their graph choice simply by saying it was the right one. If students indicated they were unsure or guessed, their response would receive an IDK code. And if they gave an answer that was unrelated to the problem, the response would be coded as OT.

Table 4.
Covariation measure qualitative coding key

Types of Reasoning	Description of Reasoning
Covariational Reasoning (COV)	Students coordinate variation in unidirectional change in a quantity with another varying quantity. (E.g., as the height increases, the diameter increases gradually, then begins to decrease). AND/OR Students coordinate an amount of change in one quantity with a value of (or amount of change in) another quantity. (E.g., when the Ferris wheel turns about ¼ of its distance, the height is at a maximum). AND/OR Students coordinate direction(s) of change in one quantity with direction(s) of change in another quantity. (E.g., x will increase and decrease, while y will continue to increase).
Variational Reasoning (VAR)	Students represent variation in unidirectional change in a single quantity. (E.g., the distance increases gradually at first, then starts to increase faster.) AND/OR Students provide different values of one quantity (E.g., the Ferris wheel is at its max here, then its min here, then its max here) AND/OR Students represent variation in the direction of change in a single quantity. (E.g., the distance increases, then decreases.)
Representing Motion of Objects (MO)	Students represent motion of objects in an animation (E.g., Nat walks out, then back, so I need a graph that goes out then back.). Students conceive of objects (e.g., a tree) as occupying a literal location on a graph.
Representing Iconic/ Familiar Objects (IC)	Students represent the shape of objects (E.g., the fishbowl has a curved edge, so I'll pick the graph with a curved edge.)
Find Correct Answer (ANS)	Students state that they chose the best, most logical, or correct answer.
Guess/Not sure (IDK)	Students state that they guessed, or don't know why
Off Task (OT)	Students provide an off-task response. (E.g., "I like bananas.")

Note: Adapted from Johnson, H.L., (2018). Coding for covariation. Unpublished manuscript, School of Education and Human Development, University of Colorado Denver. These codes are adapted from Johnson, H. L., McClintock, E., & Gardner, A. (2019). Designing digital task sequences to promote students' conceptions of graphs as relationships between quantities. Manuscript under review.

When coding, we did not know which sections were treatment and which were control. We also did not look at whether a student's response matched to a correct or incorrect answer. This helped us to guard against bias in our coding. After all of our respond codes had been agreed-upon and finalized, we added binary codes for which sections were treatment and control and which answers were correct and incorrect.

Validity and Reliability

Coding began with a small calibration sample. We coded the written responses for each of the assessment items independently using the seven codes in the above coding key. After we each completed our coding, we identified the codes on which we differed, and we engaged in disagreement calibration. We spoke with one other and shared our reasoning for why we used a certain code. We referred to the coding key, and tried to come to an agreed-upon decision. When we agreed, we coded the response as such. If we did not agree, we spoke with Johnson to determine what the code should be. We then recorded the final codes in our spreadsheet.

Once the initial calibration sample was complete, we repeated the process with a larger calibration sample using responses from one section of college algebra from the Spring 2018 semester. This second calibration helped us establish intercoder reliability, as we agreed on 92 out of the 98 total responses (94%) for this data set when coding in isolation (Campbell, Quincy, Osserman, & Pedersen, 2013). Bernard (2000) and Campbell and colleagues (2013) do not recommend that intercoder reliability be measured as a strict percentage of agreed-upon answers as it does not take into account chance agreement. Although our initial percentage agreement was strong, through our disagreement calibration, we came to an agreed-upon code for every response. Johnson reviewed our codes for this sample and agreed that our coding was sufficient to continue with the Fall 2018 data.

After completing coding and disagreement calibration for this larger sample and receiving Johnson's approval, we began coding the full set of data from the Fall 2018 semester. We each coded every response for each of the four assessment items independently for all 13 sections of students. As previously stated, we were blind to which responses came from students in the treatment group and which responses came from students in the control group, as well as which responses came from students who answered the item correctly and which responses came from students who answered incorrectly. Throughout the coding process, we talked about the coded responses on which we disagreed and either came to a decision or sent the responses to Johnson for her input. We did not measure our intercoder reliability, but rather focused on our intercoder agreement, meaning that we were able to come to an agreement about each coding discrepancy through discussion (Campbell et al., 2013).

When coding responses, we looked at whether students' explanations provided evidence of certain types of reasoning, and we inferred how students were engaging in various types of reasoning based on their responses. It is important to note that the codes we gave are only based on students' written explanations. It is possible that students engaged in other types of reasoning, but did not show evidence of that type of reasoning in their responses.

Table 5 provides examples of students' responses, how they were coded, and our reasoning why. The sample responses in Table 5 were representative of the kinds of responses that we saw frequently, and they were relatively straightforward in terms of how they fit into our coding rubric (as explained in the right column).

Table 5.
Student responses and codes

Response	Code	Reasoning
The distance kept increasing, while the height decreased and increased.	COV	This response references both attributes and it coordinates variation in the total distance traveled with variation in the height of the Ferris wheel cart.
Because the height increased then decreased.	VAR	This response references the changing height of the Ferris wheel, but it does not coordinate it with the changing distance.
Because it gradually covers the surface.	MO	This response discusses the movement of the water as it fills the fish bowl.
This is the one that fits the best and it has sharp edges because the shape.	IC	This response coordinates the shape of the square track with the shape of the graph, stating that the graph the student chose is correct because its sharp edges mimic those of the square shape.
It best shows the situation.	ANS	This response simply states that the graph the student chose is the best one without giving any mathematical reason why.
I honestly just guessed between D and C.	IDK	This response shows that the student was not sure which graph was correct and guessed.
I am really tired.	OT	This response does not attempt to answer the question.

As we coded, we noticed that many responses showed evidence of multiple types of reasoning. To be consistent and to ensure that each response had only one code, we coded in a systematic manner. Furthermore, we coded the response with the most sophisticated type of reasoning that it conveyed. For example, if a response showed any signs of covariational reasoning, we gave it a "COV" code, even if it also showed evidence of another type of reasoning. We grouped some categories of reasoning together when we interpreted them to involve the same level of sophistication. We grouped the codes in the following way: (1) COV, (2) VAR, (3) MO/IC, and (4) ANS/IDK/OT. For responses that satisfied multiple codes that were considered the same level of sophistication (i.e., MO and IC or ANS, IDK, and OT), we worked together to determine which response code was most fitting based on which type of reasoning or justification was the most prominent.

We faced some challenges when coding students' responses. We encountered three main issues: unclear pronoun references (e.g., "it increases and decreases"), spelling and grammatical errors (e.g., "I can it's b cuz of the way it is"), and an overall lack of clarity in what the student was trying to say (e.g., "the bumps and places that it's farther away"). As we coded, we worked hard not to assume any unstated information in a student's response, and only to use the information they provided to determine their reasoning. For example, one student justified their graph choice for the Toy Car problem by stating, "Decreases from center then bounces and increases." Although the response may appear to give evidence of variational reasoning, as the student is making reference to something decreasing and increasing, it is unclear which attribute they are referring to, or if they are even referring to one of the attributes in question (the distance from the center of the track or the total distance traveled). Because of this, we coded this response as "motion," as they are describing what they see the toy car doing in the animation. We disregarded spelling and grammatical errors when it seemed clear what the student was trying to say. For example, one student responded, "As height increases do does distance." We felt confident that the student meant "so" instead of "do", and coded it as if they had written "so." In general, while coding, we tried to presume students' competencies rather than deficits, without inferring unstated reasoning.

Data Analysis

After we coded all of the data and agreed upon the codes, we created a final compiled spreadsheet with all 13 sections of responses. We added a column into our spreadsheet to indicate whether the responses came from a student in the treatment or control group, and we added a column to show whether the student answered the question correctly or incorrectly. We determined counts for each of the different codes for each of the four assessment items, as well as counts for each code in the treatment group and control group, and counts for each code among students who answered the problem correctly and incorrectly.

Chi Square Analysis

To answer both of our research questions, we used a chi-square test for association. We chose to use a chi-square test as it allowed us to determine if there was an association between the different types of students' reasoning and (a) the section they were in (control or treatment) or (b) whether they answered the problem correctly or incorrectly. Lerner and Lerner (2014) explain that chi-square tests deal with categorical observations, which was appropriate for our coding structure that separated students' responses into seven categories: covariation, variation, motion, iconic, answer, I don't know, and off task.

We performed a number of different chi-square tests between student response reasoning codes (COV, VAR, MO, IC, ANS, IDK, OT) and section type (control or treatment) and between student response reasoning codes and answer type (correct or incorrect). We used the chi-square test on our observed counts for each of these reasoning codes (within treatment/control or correct/incorrect groups) to calculate expected frequencies for each type of reasoning within each group. We used this to calculate the average squared difference between our observed counts and the expected counts to see if there was a statistically significant association between students' reasoning and whether they were in the control or treatment group or between students' reasoning and whether they answered the problem correctly or incorrectly.

For chi-square tests, Vidacovich (2015) recommends that no more than 20% of the cells have expected frequencies smaller than 5. To accomplish this, we combined the "off task" (OT) and "I don't know" (IDK) responses into one group. This seemed appropriate, as both codes indicated that the student's response did not include a relevant justification for the graph they chose.

Research Question 1:

To address our first research question (*How does the reasoning of students in the control group compare to that of students in the treatment group?)*, we performed a chi-square test comparing the six response code groups (COV, VAR, MO, IC, ANS, and IDK/OT) with the section type (control or treatment). The null hypothesis was that there was no association between the type of reasoning conveyed by students' responses and if they were in the control or treatment group. The alternate hypothesis was that there was an association between the type of reasoning conveyed by students' responses and if they were in the control or treatment group.

We performed two other chi-square tests as well, to better inform our understanding of how the reasoning evidenced by students' responses may be associated with their section. These tests included the following breakdowns of response codes: (a) COV/VAR, MO/IC, and ANS/IDK/OT; and (b) COV and non-COV. These tests gave us many options for how to interpret our data, which we discuss in the next chapter.

We found our first breakdown (COV/VAR, MO/IC, and ANS/IDK/OT) to be appropriate because the responses in each combined group showed different overall approaches to how the students may have been reasoning. Codes of covariational and variational reasoning indicated that students' responses gave evidence of attending to at least one of the attributes and how it changed. The motion and iconic codes indicated that the response talked more about the motion or shape of the object, as opposed to the specific attributes in question. The final group (ANS/IDK/OT) included all of the responses that indicated a lack of evidence of a specific type of reasoning (i.e., COV/VAR/MO/IC).

We ran the test comparing COV and non-COV (all of the other response codes grouped together) with section type to see if there was an association between students' responses that conveyed covariational reasoning and whether they were in the control or treatment group. One of our goals (and a goal of the ITSCoRe team's research) is to see whether opportunities to interact with techtivities improves students' covariational reasoning (Johnson, Kalir et al., 2018 and Johnson, McClintock et al., 2018). Therefore, it made sense to focus on the COV response code compared to all of the others.

Research Question 2:

To answer our second research question (*How do the responses of students who answered the problem correctly compare to students who answered incorrectly?)*, we performed a chi-square test comparing the six response code groups (COV, VAR, MO, IC, ANS, and IDK/OT) and whether students answered the question correctly or incorrectly. In this case, we did not distinguish between the treatment and control groups, as we wanted to determine whether there was an overall association between the type of reasoning evidenced by students' responses and whether they answered the problem correctly, regardless of their class section. The null hypothesis was that there was no association between the type of reasoning conveyed by students' responses and whether or not they answered the problem correctly. The alternate hypothesis is that there was an association between the reasoning conveyed by students' responses and their correct/incorrect answer.

Similar to the first research question, we also performed two more chi-square tests: one comparing COV/VAR, MO/IC, and ANS/IDK/OT with correct/incorrect answers and one comparing just two response code groups (COV and non-COV) with correct/incorrect answer. We ran the first test to determine if there was an association between responses that conveyed similar types of reasoning and whether students selected the correct answer. We ran the second test to determine if there was an association between students engaging in covariational reasoning and whether they answered the problem correctly or incorrectly. As the assessment items were designed to assess students' covariational reasoning, we found it appropriate to focus our attention on COV compared to all of the other response codes.

CHAPTER IV

RESULTS

In this chapter, we present our findings and discuss how we analyzed our data to answer our research questions. First, we discuss the types of reasoning conveyed by students' responses across the four assessment items. Next, we compare the types of reasoning evidenced by students' responses in the treatment group and in the control group (RQ1). Then, we compare how this reasoning differs for students who answered the problem correctly versus incorrectly (RQ2). For both research questions, we address and interpret the results we obtained from the chi-square tests for association that we ran.

Students' Reasoning Across Assessment Items

First, we give an overview of the types of reasoning that students provided in their written responses to each of the four assessment items. Table 6 shows the percentages, by assessment item, for the type of reasoning evidenced in students' responses. This table demonstrates how students' responses differed across the assessment items. We provide some observations about this data before discussing the results of the chi-square tests that we performed to address our two research questions.

Table 6.
Type of Response Reasoning for each Assessment Item (by percentage)

Type Of Reasoning	**Ferris Wheel**	**Nat+Tree**	**Fish Bowl**	**Toy Car**
COV	30.4	18.8	50.8	10.4
VAR	25.6	28.4	20.8	31.6
MO	20.8	30.8	11.6	32.8
IC	6.8	4.0	3.6	3.2
ANS	10.8	12.4	8.8	14.4
IDK	5.2	4.0	2.8	6.0
OT	0.4	1.6	1.6	1.6

It is interesting to note the variation in percentages of responses giving evidence of covariational reasoning across the four assessment items. For example, the Toy Car item had the smallest percentage of responses showing covariational reasoning, with only 10.4%, while over half of students' responses to the Fish Bowl item showed that they were attending to both attributes. Overall, the Fish Bowl item had much higher rates of responses coded as COV compared to the other three assessment items, suggesting that there was something specific to this problem that made it more likely for students to attend to both attributes in their written responses. The Fish Bowl item was the only assessment item that involved two attributes that had the potential to both increase and decrease (height and diameter), whereas the other three items had one attribute that could only increase (total distance traveled).

We also noticed that the Ferris Wheel, Nat+Tree, and Toy Car items had higher percentages of responses coded as MO when compared to the Fish Bowl item. These three items all involved a physical object moving (a cart, person, or car), whereas the Fish Bowl item involved water rising and changing shape. It seems that students were more likely to discuss motion in their responses when the dynamic situation involved a tangible object that they could track, rather than something extending across an object, such as the height or surface diameter of water.

The last observation we discuss compares motion and iconic reasoning. For every assessment item for both groups, we coded MO more frequently than IC. This may be because of the medium in which the items were presented. These items were presented with an animation of moving attributes, so students may have perceived that the graph should show the motion of the dynamic situation (Johnson et al., under review). It may have been less likely for students to conceive of the graph iconically when they were presented with an animation of the motion in the dynamic situation.

Treatment and Control Groups

Next, we address the data relevant to our first research question: *How does the reasoning of students in the control group compare to that of students in the treatment group?* First, we compiled the data into the frequencies of the responses that occurred, split into treatment (n = 206 students) and control (n = 44 students) for each of the four assessment items, and then broke these down into response percentages for each item (see Table 7).

Table 7.
Types of Response Reasoning for each Assessment Item for Treatment and Control Groups

Type of Reasoning	Ferris Wheel		Nat+Tree		Fish Bowl		Toy Car	
	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %
COV	32.5	20.5	18.0	22.7	54.4	34.1	12.1	2.3
VAR	25.7	25.0	29.1	25.0	18.9	29.5	30.6	36.4
MO	20.9	20.5	33.0	20.5	11.7	11.4	34.0	27.3
IC	6.8	6.8	3.9	4.5	3.4	4.5	2.9	4.5
ANS	9.7	15.9	12.1	13.6	7.8	13.6	13.1	20.5
IDK	3.9	11.4	2.9	9.1	2.9	2.3	5.8	6.8
OT	0.5	0.0	1.0	4.5	1.0	4.5	1.5	2.3

Chi-Square Results

We used a chi-square test for association to determine if these differences were statistically significant (see Table 8). We found with over 95% confidence that we can reject our null hypothesis for the Nat+Tree item. This shows that there was a statistically significant association between the type of reasoning conveyed by students' responses and whether they were in the control or the treatment group. For the Nat+Tree item, more students in the control group gave responses that did not show evidence of a specific type of reasoning (IDK/OT) than the expected value predicted. This seems to suggest that students in the control group were more likely to guess or be unsure about how to approach the Nat+Tree item than we would have expected.

Table 8.
Chi-Square Results for Response Code vs. Treatment and Control

	Chi-Square value	p-value
Ferris Wheel	6.21	0.09
Nat+Tree	8.78	0.04
Fish Bowl	7.16	0.07
Toy Car	6.09	0.10

As discussed in our methods, we combined the reasoning codes into different groups to further interpret our findings. With so many different options for response codes, it seemed appropriate to compare different combinations of response codes that conveyed different approaches to how students may have been thinking. Also, with our focus on covariational reasoning, combining response code groups to compare against COV could lend us different insights.

To start, we looked at COV/VAR, MO/IC, and ANS/IDK/OT (see Table 9). Using these chi-square results, we can no longer reject our null hypothesis for the Nat+Tree item with over 95% confidence, but we can now reject our null hypothesis for the Ferris Wheel item. Grouping ANS with IDK and OT lowered the chi-square value for the Nat+Tree item, as there was less of a discrepancy between the control group's expected values and actual values for students whose responses received one of these three codes. Similarly, this grouping increased the chi-square value for the Ferris Wheel item, as the control group's actual value of responses receiving one of these three codes was greater than the expected value. We grouped these three response codes together as all three of them indicated that the student's response did not convey a specific type of reasoning. Yet, the ANS response code implied that the student felt they knew the correct answer, but did not provide an explanation for why. It is possible that students whose responses were coded as ANS were thinking about the problem quite differently from those whose responses were coded as IDK or OT. This may explain why our findings for Nat+Tree were no longer statistically significant.

Table 9.
Chi-Square Results for COV/VAR, MO/IC, ANS/IDK/OT vs. Treatment and Control

	Ferris Wheel		**Nat+Tree**		**Fish Bowl**		**Toy Car**	
	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %
COV/VAR	58.3	45.5	47.1	47.7	73.3	63.6	42.7	38.6
MO/IC	27.7	27.3	36.9	25.0	15.0	15.9	36.9	31.8
ANS/IDK/OT	14.1	27.3	16.0	27.3	11.7	20.5	20.4	29.5
	χ^2 (2) = 4.91 p = 0.04		χ^2 (2) = 4.02 p = 0.07		χ^2 (2) = 2.62 p = 0.13		χ^2 (2) = 1.79 p = 0.21	

Next, we looked at COV compared to every other group (see Table 10). This allowed us to see if there was an association between responses with evidence of covariational reasoning and whether a student was in the treatment or control group. With these results, we can reject our null hypothesis for the Fish Bowl and the Toy Car items. The statistical significance for the Fish Bowl item was especially strong, with p = 0.008. For both of these assessment items, fewer students in the control group gave responses that showed evidence of covariational reasoning than the expected value predicted. This means that students in the control group were less likely to attend to both attributes in their responses to these items than expected. This seems to suggest that the control group (as evidenced by students' responses) was significantly less likely to engage in covariational reasoning for these two items compared to the treatment group.

Table 10.
Chi-Square Results for COV, Non-COV vs. Treatment and Control

	Ferris Wheel		Nat+Tree		Fish Bowl		Toy Car	
	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %	Treatment %	Control %
COV	32.5	20.5	18.0	22.7	54.4	34.1	12.1	2.3
Non-COV	67.5	79.5	82.0	77.3	45.6	65.9	87.9	97.7
	χ^2 (1) = 2.50 p = 0.07		χ^2 (1) =0 .54 p = 0.41		χ^2 (1) = 5.97 p = 0.008		χ^2 (1) = 3.79 p = 0.03	

Chi-Square Test Summary

Our chi-square tests provided a number of different interpretations depending on how we grouped the response codes. Across the different tests, we found different kinds of statistical significance. It seems that the overarching trend among our chi-square results was that the treatment group was more likely to provide a response that conveyed some level of reasoning (COV, VAR, MO, or IC) compared to the control group. Additionally, for most of the assessment items, the control group's responses were less likely to convey covariational reasoning and more likely to convey a lack of reasoning than our expected values predicted.

Correct and Incorrect Response Comparison

Next, we address our second research question: *How does the reasoning of students who answered the assessment items correctly compare to that of students who answered incorrectly?* Table 11 shows the percentage of students who answered the problem correctly and incorrectly for each of the different response codes. The first column for each assessment item shows the percentage of students who were correct, given the type of reasoning evidenced by their response. For example, of the students whose responses gave evidence of covariational reasoning in the Ferris Wheel problem, 61.8% selected the correct graph to represent the situation. The second column for each assessment item breaks down all of the students who answered the problem correctly with the type of reasoning they showed in their responses. In this case, of the students who answered the Ferris Wheel item correctly, 30.7% of their responses showed covariational reasoning and 30.1% showed variational reasoning.

Table 11.
Types of Response Reasoning for each Assessment Item for Correct and Incorrect Responses

	Ferris Wheel		**Nat+Tree**		**Fish Bowl**		**Toy Car**	
Type Of Reasoning	With given reasoning, % correct	If correct, % with reasoning	With given reasoning, % correct	If correct, % with reasoning	With given reasoning, % correct	If correct, % with reasoning	With given reasoning, % correct	If correct, % with reasoning
COV	61.8	30.7	51.1	17.5	51.2	58.6	19.2	10.0
VAR	71.9	30.1	64.8	33.6	34.6	16.2	24.1	38.0
MO	67.3	22.9	57.1	32.1	44.8	11.7	20.7	34.0
IC	52.9	5.9	60	4.4	55.6	4.5	25.0	4.0
ANS	44.4	7.8	41.9	9.5	31.8	6.3	8.3	6.0
IDK	30.8	2.6	30.0	2.2	42.9	2.7	6.7	2.0
OT	0	0	25.0	0.8	0	0	75.0	6.0

We start by noting a few observations about these percentages, and then we discuss our statistical findings from the chi-square analyses. The percentages in Table 11 reveal whether students whose responses conveyed more sophisticated types of reasoning (i.e., COV or VAR) were more likely to answer the problem correctly and whether students who answered correctly were more likely to show higher types of reasoning in their written responses. It is intriguing to see that student responses that conveyed covariational reasoning were not the most likely to answer the problem correctly for any of the four assessment items. Students whose responses showed evidence of variational reasoning were the most likely to correctly answer the Ferris Wheel and Nat+Tree items. For the Fish Bowl and Toy Car items, students whose responses conveyed iconic reasoning were the most likely to select the correct answer. It is interesting that for both the Nat+Tree and Toy Car items, students whose responses showed evidence of variational, motion, or iconic reasoning answered the problem correctly at higher rates than students whose responses showed evidence of COV reasoning. We provide possible interpretations for these observations in our next chapter.

Chi-Square Results

In order to determine whether there was a statistically significant association between the type of reasoning students conveyed in their written responses and their correct/incorrect answer to the problem, we performed a chi-square test comparing each of the response codes (with IDK/OT combined) to whether students answered the problem correctly or incorrectly (see Table 12). After computing our chi-square values, we rejected the null hypothesis for the Ferris Wheel and Nat+Tree items with over 95% confidence.

Table 12.
Chi-Square Results for Response Code vs. Correct and Incorrect Response

	Chi-Square value	p-value
Ferris Wheel	13.86	0.007
Nat+Tree	9.36	0.04
Fish Bowl	7.56	0.06
Toy Car	4.05	0.14

For the Ferris Wheel item, we found that there was an especially strong statistically significant association between the type of reasoning conveyed by students' responses and their correct/incorrect answer, $p = 0.007$. Using the expected values that we calculated during the test, we found that significantly fewer students answered the problem incorrectly with VAR responses than expected, more students answered the problem incorrectly with ANS or OT/IDK responses than expected, and fewer students answered the problem correctly with OT/IDK responses than expected. This seems to suggest that students were more able to answer the Ferris Wheel item correctly with responses conveying variational reasoning and less able to answer the problem correctly with responses that did not show evidence of reasoning (ANS or OT/IDK) than we would have expected.

For the Nat+Tree item, we found that there was a statistically significant association between the type of reasoning evidenced by students' responses and their answer, $p = 0.04$. Although not as strong as our Ferris Wheel findings, we can still reject the null hypothesis with over 95% confidence. Using our expected values, we found that significantly more students answered the problem incorrectly than expected with OT/IDK responses. Similar to the Ferris Wheel item, this suggests that students were less able to answer the problem correctly with responses that were off-task or implied they were unsure.

Although there was not a statistically significant association between response codes and correct/incorrect answers for the Fish Bowl or Toy Car items, our calculations showed that fewer students answered the Toy Car item correctly than the expected value predicted with an ANS response. This means that, for these assessment items, students who justified their answer by saying they knew it was the correct graph were less likely to select the correct graph than we would have expected. This seems to suggest that students did not know the correct graph for the Fish Bowl and Toy Car items, even though they thought they did.

Similar to our analysis with the treatment and control groups, we combined the response codes into three groups based on similar types of reasoning: COV/VAR, MO/IC, and ANS/IDK/OT (see Table 13). Running the chi-square test with these three groupings provided somewhat similar results to our previous chi-square test where we only grouped IDK and OT. For this test, the statistical significance of the association between response type and answer became stronger for the Ferris Wheel ($p = 0.003$) and Nat+Tree ($p = 0.02$) items; however, the association became even weaker for the Fish Bowl item. For the Ferris Wheel and Nat+Tree items, there was an even bigger discrepancy between expected and actual values of student responses coded as ANS/IDK/OT for both correct and incorrect answers. Many more students answered the problem incorrectly with an ANS/IDK/OT response than expected and far fewer students answered the problem correctly with that type of response. These discrepancies had previously existed for both ANS and the IDK/OT groups for these assessment items, so combining them into one group made the chi-square value even greater. For the Fish Bowl item, the discrepancies between the expected and actual values for COV and VAR negated each other when grouped together, which is why the overall chi-square value decreased.

Table 13.
Chi-Square Results for COV/VAR, MO/IC, ANS/IDK/OT vs. Correct and Incorrect Responses

	Ferris Wheel % Correct	**Nat+Tree** % Correct	**Fish Bowl** % Correct	**Toy Car** % Correct
COV/VAR	66.4	59.4	46.4	22.9
MO/IC	63.8	57.5	47.4	21.1
ANS/IDK/OT	39.0	37.8	30.3	12.7
	χ^2 (2) = 10.29 p = 0.003	χ^2 (2) = 6.48 p = 0.02	χ^2 (2) = 3.07 p = 0.11	χ^2 (2) = 2.42 p = 0.15

Next, we performed a comparison of COV and non-COV response codes and answer type to see whether there was a significant association between responses conveying covariational reasoning and answering the problem correctly. This test provided quite different results (see Table 14). We can reject the null hypothesis for the Fish Bowl item; however, none of the individual expected vs. actual counts was significant.

Table 14.
Chi-Square Results for COV and non-COV vs. Correct and Incorrect Responses

	Ferris Wheel % Correct	**Nat+Tree** % Correct	**Fish Bowl** % Correct	**Toy Car** % Correct
COV	61.8	51.1	51.2	19.2
Non-COV	60.9	55.7	37.4	20.1
	χ^2 (1) = 0.02 p = 0.89	χ^2 (1) = 0.33 p = 0.57	χ^2 (1) = 4.81 p = 0.03	χ^2 (1) = 0.01 p = 0.92

For the Ferris Wheel, Nat+Tree, and Toy Car items, the percentages correct were similar for responses giving evidence of covariational reasoning and responses giving evidence of some other form or reasoning. The Fish Bowl was the only item where there was a statistically significant difference between how students whose responses conveyed COV performed compared to those whose responses did not convey COV. These chi-square results indicate that students whose responses showed evidence of covariational reasoning on the Fish Bowl problem were more likely to correctly answer the problem and students whose responses did not show evidence of covariational reasoning were less likely to correctly answer the problem than our expected values predicted. These findings add to our observation from Table 6 in which we noted that students' responses were most likely to convey covariational reasoning for the Fish Bowl item compared to the other three assessment items. Now, we also know that students' responses that conveyed covariational reasoning were more likely to answer the assessment item correctly than expected.

Chi-Square Test Summary

The statistical significance of our chi-square results varied depending on how we grouped our response codes. The most common observations that we made from these results were that students were (a) more likely (than the expected value predicted) to choose the correct graph with a response coded as VAR and (b) less likely to answer correctly with a response code of ANS, IDK, or OT.

Our first observation may suggest that students could determine the correct graph by only attending to one attribute. Since each assessment item only had one attribute that both increased and decreased in the given dynamic situation, it is possible that students could select the correct graph by only considering that attribute's behavior. Many student responses on single items seemed to suggest this as a possibility. For example, one student justified their graph selection for the Ferris Wheel item by saying, "The height from the ground increases at first and then decreases after the peak distance from the ground, it is not a linear relationship." Some students were able to identify the correct graph by determining which one showed height increasing first and by recognizing that the graph needed to be curved and not linear. This may help explain why our chi-square results comparing COV to non-COV were especially weak for the Ferris Wheel, Nat+Tree, and Toy Car items, as many of the students chose the correct graph for those items without showing evidence of covariational reasoning in their responses.

Our second observation implies that students who stated that they knew a graph was correct without justifying why (ANS) and students who said they were unsure or provided a response that was not related to the problem (IDK/OT) had more trouble determining the correct graph than we would have expected. This seems to suggest that students who were unable to justify their answers were less likely than expected to answer correctly, even if they thought they had.

Conclusion

Across both research questions, our chi-square results suggest that students in the treatment group were more likely to provide a response that conveyed some level of reasoning (COV, VAR, MO, or IC), and students whose responses conveyed some level of reasoning answered the assessment items correctly more often. We discuss the implications of this, and our other findings, in the next chapter.

CHAPTER V

DISCUSSION

In this chapter, we interpret our results and discuss how they connect to the literature we reviewed about covariational, variational, motion, and iconic reasoning. Next, we connect our research to the overarching ITSCoRe project. We discuss the limitations that we faced and the implications they had on our study. Then, we suggest how our study can guide future research on students' reasoning. We provide a reflection on our experience working together on this thesis. Lastly, we offer closing remarks to summarize our findings.

Interpreting our Results and Connecting to Literature

In this section, we interpret our findings for each of our research questions and discuss how they fit into the broader literature that we reviewed in Chapter II. A summary of our statistically significant results is provided in Table 15 below.

Table 15.
A Summary of Statistically Significant Results

	Statistically Significant Items	*Statistically Significant Details*
Treatment vs. Control		
COV VAR MO IC ANS IDK/OT	Nat+Tree (p = 0.04)	Control group had higher rates of IDK/OT than expected*
COV/VAR MO/IC ANS/IDK/OT	Ferris Wheel (p = 0.04)	Control group had higher rates of ANS/IDK/OT than expected
COV Non-COV	Fish Bowl (p = 0.008) Toy Car (p = 0.03)	Control group had lower rates of COV than expected (both items)
*Based on expected values from chi-square calculations		

Table 15 continued.

	Statistically Significant Items	*Statistically Significant Details*
Correct vs. Incorrect		
COV VAR MO IC ANS IDK/OT	Ferris Wheel (p = 0.007) Nat+Tree (p = 0.04)	More students answered the problem incorrectly with IDK/OT (both items) and ANS (FW) than expected* Fewer students answered the problem incorrectly with VAR than expected (FW)
COV/VAR MO/IC ANS/IDK/OT	Ferris Wheel (p = 0.003) Nat+Tree (p = 0.02)	More students answered the problem incorrectly with ANS/IDK/OT than expected (both items)
COV Non-COV	Fish Bowl (p = 0.03)	No statistically significant details
*Based on expected values from chi-square calculations		

Treatment and Control Responses

Our first research question (*how does the reasoning of students in the control group compare to students in the treatment group?*) focuses solely on students' reasoning. Similar to Johnson and colleagues' (under review) study, for this research question, we were not interested in whether students chose the correct graph. Rather, we focused on the reasoning they gave for making their selection. We compared students' reasoning in the treatment and control groups to see whether an emphasis on techtivities could help promote students' covariational reasoning. From our results, we made a few observations that may be relevant to the use of techtivities.

First, we found that the treatment group was more likely to respond in a way that conveyed one of the four types of reasoning (i.e., COV/VAR/MO/IC). When we compared the two groups (see Table 9), the control had a higher percentage of responses coded as ANS, IDK, or OT for the four assessment items. While the control group did have active learning experiences in class, they did not do the techtivities. The treatment group worked through the techtivities, which involved watching an animation, interacting with dynamic graphs, drawing a sketch of what the graph should look like, and then reflecting on how their graph compared to the actual graph (See Table 3). The reflections in the techtivities regularly asked students to explain their reasoning about the dynamic situation. Having students explain their reasoning for their graph selection in the covariational reasoning assessment was something with which they were familiar.

One reason the treatment group had a lower percentage of ANS/IDK/OT responses may have been that they were used to being asked to explain their reasoning with the dynamic situations. Since the control group did not have experience with the techtivities, they may have decided to not say why they chose the graph rather than to try to explain why, giving a response that did not convey their reasoning. Another reason this might have happened was because the treatment group was used to thinking about dynamic situations in a digital setting. They had seen animations of dynamic situations during the techtivities, whereas the control group may have been unfamiliar with these types of animations. This unfamiliarity might explain the control group's lack of reasoning in some responses. Overall, the treatment group's experience with the techtivities, their experience with the dynamic situations in those techtivities, and being asked to regularly explain their reasoning during the techtivities may explain why they were more likely to explain their reasoning during the assessment items, as compared to the control group.

Looking across our four assessment items and our tests for significance (see Tables 8-10), there were a few observations that stood out. Each time we compared different groupings of responses, we had different assessment items show significance. Because the overall lens of this study involved looking at how techtivities might affect students' covariational reasoning, we focus on our significance tests for COV and non-COV (see Table 10). The Fish Bowl had the highest percentage of COV for the treatment and control across all four items, showing that something was different that encouraged students to address both changing attributes. This assessment item compares the height and the diameter of the surface area of the water. One reason the responses showed COV more often may have been that this is the only assessment item that did not have total distance as one of the attributes. We found total distance to be the most common thing for students to omit in responses that were coded as VAR. Even though the height of the water is constantly increasing in the fish bowl, it is a different measurement than total distance traveled because the height can only increase to the top of the bowl, while the total distance could continue to grow indefinitely. We think the difference between the two attributes might explain why it was addressed more often by both groups. Another reason might be that the animation and the graph choices for the Fish Bowl looked different from the motion of the volume increasing in the animation. For the other three tasks, the graphs had aspects that looked similar to the shape or movement in the animation. These similarities may have made it difficult for students to disconnect the animation from how the graph should look, causing students to focus on the motion, rather than the attributes on the axes (Kerslake, 1977; Johnson, et al., under review). Since none of the graphs resembled the motion of the water's volume changing (the motion in the Fish Bowl animation), the students may have been less likely to see the graph as representing the literal motion of the graph.

The Fish Bowl assessment item had a 20.3% difference between the treatment and the control group's COV response rates that was statistically significant. We believe that this may have been attributed to the techtivities that the treatment group completed. The treatment group did multiple techtivities that compared the height of something to its width. Diameter can be seen as another way to describe width, as it is the width of a circle through the center. These techtivity animations showed similar attributes as the Fish Bowl's animation, changing in a similar way because the width would get larger and smaller as the height increased. Familiarity with the techtivities, and that this item did not have total distance as an attribute, may have been the reason why students in the treatment group addressed both the diameter and the height more often in their responses.

Across the four assessment items, the Toy Car had the lowest rates of responses coded as COV for both the treatment and control groups. This might be explained by looking at the attributes highlighted in the animation. Since the Toy Car animation only shows the distance from the center changing, students may have felt that they only needed to explain this attribute in their justification for why they chose their graph. This may explain why the Toy Car had such a low coding rate for COV and such a high coding rate for VAR compared to the other assessment items (see Table 6). The Toy Car item also had a high coding rate for MO, possibly showing that some students relied on the motion in the animation to determine which graph they chose (Kerslake, 1977). Again, these findings seem to suggest that the animations may have caused students to pay more attention to the motion in the animation and how the graph might show that motion (Johnson et al., under review).

For the Toy Car item, there was a difference of 9.8% between the COV rates for the treatment and the control group that was statistically significant. Even though the treatment group only had 12.1% for COV, which is very low compared to the other tasks, we think their experience with the techtivities may be why they did better than the control group. Being familiar with the techtivities may have helped them address the overall distance the car traveled along with the distance from the center in their explanations because they were used to looking for two changing attributes in the animations.

From these observations, there are a few things that we can say about the use of techtivities to improve students' covariational reasoning. The treatment group's familiarity with the techtivities seemed to have helped them explain their reasoning more frequently than the control group. We base this on the evidence that with all four assessment items, the treatment group had a higher percentage of answers that coded for some type of reasoning (COV, VAR, MO, or IC). We believe the techtivities show promise in helping engage students in covariational reasoning. Looking at our comparison of COV to non-COV for the treatment and control groups, we had two of the four assessment items show significance, which is notable given our small sample size. These items also showed that students in the treatment group were more likely to convey evidence of covariational reasoning in their responses. While it would have been stronger to see significance across all four assessment items, we still believe that it is likely that the techtivities helped engage students in covariational reasoning. From these observations, we believe the techtivities helped students to explain their reasoning about dynamic situations, and that the techtivities show promise in helping students develop their covariational reasoning.

Correct and Incorrect Responses

We interpret our findings for our second research question (*how does the reasoning of students who answered the assessment items correctly compare to that of students who answered incorrectly?*) and discuss how they relate to the broader literature. We made a couple of interesting observations from the percentage table analyzing how responses of students who answered the problem correctly compared to those of students who answered incorrectly (see Table 11). For the Nat+Tree and Toy Car items, a much larger percentage of students who answered the problem correctly had responses conveying VAR or MO reasoning compared to COV reasoning. This may mean that students were able to figure out the correct graph without considering both changing attributes. Another possibility is that students considered both attributes, but did not explicitly mention them in their responses. As we coded the data, we noticed many students referred to the attribute that increased and decreased (i.e., distance from tree or distance from center) in their justifications, but did not mention the attribute that continually increased (i.e., total distance). Akin to Thompson and Carlson's (2017) notion of students viewing time as simply progressing ("experiential time"), it is possible that students viewed total distance in a similar manner, as it continually increased throughout the situation. Another possibility is that students may have considered both attributes, but neglected to mention the ever-increasing attribute for the Ferris Wheel, Nat+Tree, and Toy Car items. This would have prevented their responses from being coded as COV, resulting in a "false negative." The percentage results in Table 11 gave us some insight into how students' response codes may have been related to their answers; however, there were numerous interpretations for what these percentages meant.

Next, we focus on covariational reasoning across the assessment items. With most of the items, students who showed covariational reasoning in their written responses were more likely than not to choose the correct graph, yet this was not the case with the Toy Car. This seems to suggest that students could think about both attributes and how they were changing, without understanding what the graphical representation would be. In this case, they may have had trouble connecting the changing relationship between the attributes to the shape of the graph, possibly treating the graph more as a picture of the situation, similar to what Clement (1989) described in his research about graphing misconceptions. For the Toy Car item, there were two graph options that showed total distance continually increasing while distance from the center increased and decreased. Comparable to Kerslake's (1977) findings relating graph shapes to journeys, it is possible that students chose the linear graph instead of the curved graph for the Toy Car because the square track that the car travels around appears to be linear. Although our results show that evidence of covariational reasoning increases students' likelihood of choosing the correct graph, attending to both attributes can still lead to students selecting an incorrect graph.

The other three assessment items' graph options were set up similarly (with a mix of linear and curved options); however, the Toy Car was the only item whose background shape (a square track made up of straight lines) and motion (along a straight path) did not appear to correspond with the correct graph shape (a curved graph with round bumps). It is possible that some students viewed each of these graphs as a picture of the motion they saw in the dynamic video (a Ferris wheel cart moving in a circle, water filling a round fish bowl, or Nat traveling along a straight path), and still selected the correct answer because the graphical representation fit that shape, similar to what Bell and Janvier (1981) and Kerslake (1977) found. If this were the case, we may have had responses that were coded as COV in which students may have combined both covariational and iconic reasoning in selecting their graph. Our results drew from the more sophisticated type of reasoning; however, our findings may have been different if we coded for multiple types of reasoning, which we discuss more in our limitations below.

Extending Beyond our Study

Our research findings connect to the results of other covariational reasoning studies. In addition to the covariational reasoning assessment we used, Johnson's teams conducted a number of other studies to better understand how students reason when working on tasks involving multiple changing attributes. The coding rubric we used was an adapted version of Johnson and colleagues' (under review) coding process, in which they used four codes (COV, VAR, MO, and IC) to assess the type of reasoning students used while working through a progression of covariational reasoning tasks. Their study focused on individual students' conceptions of graphs and how they changed while working through a sequence of tasks. Their goal was to help students shift to COV conceptions of graphs. Johnson and colleagues (2017) had a similar goal of examining how students transfer covariational reasoning across tasks with different backgrounds and attributes. The authors' findings in both of these studies could provide a helpful next step to our study. While we assessed the types of reasoning evidenced by students' responses, Johnson's teams explored ways to guide students from one type of reasoning to another and to transfer that type of reasoning across tasks.

Johnson and colleagues (under review) explicitly mentioned their study's focus on students' competencies, rather than deficits. Instead of presenting their findings as reasons why students were unable to shift from one way of conceiving of graphs to another, they focused on whether their effort to create learning opportunities to promote these shifts was successful. We tried to have a similar mindset in our study. Rather than assuming students lacked the ability to reason in a certain way, we examined the aspects of our study that may have prevented them from showing evidence of such reasoning in their written responses. As educators, we work to build on what our students know and can do, rather than identify what they do not know and cannot do.

Throughout our study, we examined the relationship between students attending to both changing attributes and students selecting the correct graphical representation. Even though the items in the ITSCoRe team's covariational reasoning assessment were specifically designed to promote covariational reasoning, many students' responses conveyed that they only focused on one changing attribute or that they focused more on the motion and shapes involved in the dynamic situation. Ellis and colleagues (2018) made a similar observation in their case study. They concluded that "simply relying on the use of quantitatively-rich contexts is not sufficient; it does not guarantee that students will attend to both quantities or develop images of coordinated change" (p. 198). Ellis and colleagues (2018) recommended that teachers explicitly encourage students to attend to both attributes represented in graphs. This is consistent with what the ITSCoRe team found during the interviews they conducted to validate the covariational reasoning assessment (Johnson & Wang, unpublished document). We agree with this recommendation, and we also encourage teachers to promote students' explanations of how they are reasoning about these attributes in their written responses.

Limitations

In this section, we discuss the limitations we encountered during this study. We address the limitations with the coding process as well as the limitations with our chi-square analysis.

Limitations with the Coding Process

We used students' responses as a proxy for their reasoning, yet it is important to recognize that students may have engaged in certain types of reasoning even if their responses did not provide such evidence. While using online tasks allowed us to collect a lot of data very quickly, the types of written responses students gave created challenges with the coding process. We describe three main limitations with student responses and the coding process: (a) responses with ambiguous pronouns or unclear descriptions of the varying attributes did not receive COV codes, even if it seemed that they were attending to both attributes; (b) some responses lacked detail, which impacted our coding; and (c) responses met the criteria for multiple codes in the rubric. We discuss these limitations and possible ways to mitigate them.

As we discussed in the methods section, students' written responses sometimes used ambiguous pronoun references or imprecise language when discussing how attributes were changing. Often times, it seemed that students were trying to describe two covarying quantities, but the phrasing they used did not warrant a COV rating. For example, one student's response to the Nat+Tree assessment item was, "Her distance is always increasing. She gets closer to the tree and then immediately goes straight back away from the tree." Even though it seems that the student addressed both total distance traveled and distance to and from the tree, we coded this as VAR instead of COV because they did not explicitly mention both attributes. Coding this way allowed us to be more consistent with our coding choices; however, re-coding with a more expansive view of COV and VAR could impact our findings.

As we coded student responses, we often found ourselves wanting to ask follow up questions to better understand what students were trying to convey. Another way to improve upon the data we analyzed could be to change the way answers were collected to give us more insight into students' reasoning and avoid ambiguities. This could be done by providing more structure that encouraged students to include a more detailed explanation of their reasoning, for example by raising the minimum word limit on written responses. Alternatively, researchers may interview students to better understand their thinking, similar to the interviews conducted by Johnson during the validation of the ITSCoRe covariation assessment (Johnson & Wang, unpublished document). Giving students the opportunity to elaborate on their reasoning might have provided us with more detailed responses, which could have impacted our results.

Many students' responses showed evidence of multiple types of reasoning. A student's response could start by discussing how one attribute was changing and then talk about the motion or shape of the object. For example, one student's response to the Toy Car item was "The distance traveled is continuous, however the distance from the center of the track changes. It starts high, but then gets low, and so on. The lines are straight because of the shape of the track." This student mentioned both changing attributes, which is why their response was coded as COV. Yet, they also talk about the car's movement and the shape of the track, which could suggest they were engaging in motion and iconic reasoning as well. As discussed, we used a systematic approach to determine one code for each response; however, this meant that we did not code for other types of reasoning in which the student may have been engaging. Coding for multiple types of reasoning may have allowed us to more fully assess all of the ways in which students approached these assessment items, which also could have led to different results.

Limitations with Chi-Square Results

Chi-square tests are sensitive to sample size. Vidacovich (2015) explains that large sample sizes are more likely to produce statistical significance even if the association is small, and small sample sizes are unlikely to achieve statistical significance even if there is an association between the variables. Our sample size of 250 students for this study is relatively small. Despite the small sample size, our statistically significant findings (see Table 15) are notable and can help advise future research in this area.

Future Research

Our study was inspired by the ITSCoRe research team's work with techtivities to develop covariational reasoning. In our study, we found some promising results about the validity of using techtivities to improve students' covariational reasoning. As more data are collected from future sections of college algebra, we anticipate that findings will continue to reveal significant associations.

We believe additional research should be conducted to see what type of activity can best develop students' covariational reasoning. For example, although virtual techtivities are easily accessible, it may be more impactful for students to see these dynamic situations presented to them in real life or to physically act them out themselves. Johnson et al. (under review) suggests that students who demonstrate a motion-based conception of graphs could benefit from "embodied tasks" - tasks that allow students to model the motion themselves (Duijzer, Van den Heuvel-Panhuizen, Veldhuis, Doorman, & Leseman, 2019). It is not unreasonable to have real life opportunities to work with dynamic situations in a classroom (such as filling a bowl with water), and it may help students to experience the varying attributes firsthand. Giving students the opportunity to walk back and forth in class or drive a real toy car around a track might help them understand how the attributes are changing in relation to one another. Providing students embodied types of experiences in conjunction with the techtivities may have impacted our results on the covariation assessment. We concur with Johnson and colleagues (under review) that it could be productive to implement embodied tasks in the future. We also feel that online techtivities are beneficial and could be used in conjunction with in-person simulations to allow students to explore the different attributes independently and at their own pace.

Reflecting on Moore et al.'s (2014) study, it would be interesting to see whether having the increasing and decreasing attribute on different axes impacted how students approached the assessment items. The current covariational reasoning assessment included two items for which the graph choices had the increasing and decreasing attribute on the horizontal axis and two items for which it on the vertical axis. For example, with the Ferris Wheel item, it could be beneficial to vary the graph choices for different students so that the correct graph had the increasing and decreasing attribute on the horizontal axis for one student and on the vertical axis for another. This would allow a direct comparison of how students responded when the attribute was on different axes, which would facilitate future data analysis to see whether students were more likely to choose the correct graph when the increasing and decreasing attribute was on the vertical axis (allowing the relationship to pass the well-known "vertical line test"). Additional analysis could be done to see whether students' responses were more likely to convey covariational reasoning when the graph they selected had the increasing and decreasing attribute on the horizontal axis, as this might draw students' attention to something changing on that axis (instead of the horizontal axis representing something that keeps increasing, such as time or total distance).

We also suggest performing data analysis to compare a given student's responses across each of the assessment items. This could allow us to determine if a student engaged in similar types of reasoning for each of the assessment items or if their reasoning changed based on the situation and the types of attributes. Johnson et al. (2017) believe that students are more likely to engage in covariational reasoning with tasks involving attributes that are easy to conceive of as being measurable, such as height and distance. In the covariational reasoning assessment, all four of the items' attributes are some variation on height and/or distance. The Fish Bowl item measures diameter of its surface area as its "distance" attribute, whereas the other three items all have distance traveled as one of their attributes. It would be interesting to see if the data from the covariational reasoning assessment provided insight into the types of attributes that were most likely to promote students' covariational reasoning. This analysis may also shed light on which assessment items were most effective at developing or invoking students' covariational reasoning, which could inform future techtivity design.

Reflection

In this section, we reflect on our experience working together on this study, what we learned from the experience, and how we can apply what we learned to our role as educators. As high school teachers, we were both eager to work on a study that would have implications on our classroom instruction. Johnson suggested that we join her ITSCoRe team as graduate research assistants. We appreciated the opportunity to work on part of an ongoing project, with an experienced and established team of researchers. We began working independently with separate research questions that drew from the same set of data (the online covariational reasoning assessment). It soon became apparent that our studies had significant overlap, and it seemed that we could learn more, dig deeper, and make more meaningful connections by working together.

We combined efforts and integrated our ideas and findings to focus both on students' reasoning between the treatment and control groups and students' reasoning based on whether they answered correctly or incorrectly. This gave us a wider lens with which to interpret our results, considering both the impact techtivities had on students' reasoning and the correlation between students' reasoning and their graph selection. We met weekly (and texted far more frequently) to discuss ideas, offer writing suggestions, keep each other on track, and commiserate when we found out that we were not on track after all. We wrote this thesis using Google docs, so that we could work concurrently and provide real-time feedback. We did, at times, try to edit the same thing at the same time, causing unintended confusion, but overall, the online collaboration worked well. This thesis is the result of four (combined) semesters of work, which involved many hours of coding and engaging in disagreement calibration, numerous drafts of each chapter, dozens of pages of scrapped and re-written work, and hundreds of late night and early morning texts to each other. Although it was a challenging endeavor, we feel fortunate to have had the opportunity to work together and learn from one another throughout the experience.

After coding over a thousand student responses, we developed an increased appreciation for students who could clearly and concisely justify their reasoning. Fortunately, our data also suggested that students who could explain their reasoning were more likely to answer the problem correctly. This reinforces the importance of asking students to explain their thinking from a young age, and it is a practice that we encourage in our own classrooms.

Throughout the coding process, we became more familiar with the different ways in which students might convey their reasoning. We observed trends in the ways students responded, including the attributes on which they focused and the words they used to describe how these attributes changed. We often wondered about how students were thinking, beyond what they said in their written responses. The coding process helped hone our “noticing and wondering” skills, as popularized by Fetter (2015), in ways that we could use with our own students when asking them to justify their reasoning. We found ourselves using the observations and categorizations we made while coding to better understand how our own students were reasoning in our classes.

With our own students, we can ask follow-up questions and infer meaning from gestures and motions that they make. It is challenging to use students’ written responses as the sole proxy for their reasoning. As teachers, we are fortunate to be able to observe and assess students’ reasoning in many different ways, including orally, in writing, and graphically. Our experience with coding students’ responses shows us that it is important not to put too much weight on any one form of assessment.

Throughout this project, we learned a lot about data collection and analysis. Using data from the ITSCoRe team's project gave us the opportunity to work with a much larger pool of student responses than we would have had in our own classrooms. We discovered different ways to examine and interpret data, and we learned how to determine if our findings were statistically significant. We performed calculations for over a dozen chi-square tests for association (many of which did not make the final cut for this thesis), and we learned how to convey and describe our results in a meaningful way for other researchers.

From our work with this thesis, we feel as though we have grown as educators and are better-equipped to address our students' needs. Our new knowledge about how students develop covariational reasoning has opened up opportunities for differentiation in our classrooms and has positively impacted our curriculum. Working predominantly with algebra and calculus students in our respective high schools, this research has helped us understand the struggles students face when reasoning about dynamic situations. It is common for teachers, parents, and students to believe that some students are bad at math. With the insight we have gained from this study, we can steer the conversation away from being bad at math and towards how we can develop students' reasoning and understanding. Using techtivities as a model for dynamic situations has helped our students engage in covariational reasoning in our classrooms. Hopefully, this will help our students become better problem solvers when exploring dynamic situations with multiple changing attributes.

Closing Remarks

This experience has improved our understanding of the types of reasoning that students may engage in when working on tasks involving dynamic situations. While conducting background research, we learned about many important connections between covariational reasoning and functions (Thompson et al., 2017; Ferrari-Escola et al., 2016; Ellis et al., 2016; Thompson & Carlson, 2017; Moore et al., 2019; Johnson et al., 2017). Functions are a foundational part of our high school math curriculum, thus our understanding of how students develop covariational reasoning can help us better understand how students reason about functions in our own classrooms. Our findings also reaffirmed the importance of having students explain their reasoning to justify their answers, following the old Latin proverb, *docendo discimus* (by teaching, we learn). Asking students to explain their approach and justify their reasoning helps them better understand what they are doing and why.

Our motivation to complete this study came from our desire to help students feel more engaged and successful in math. As reflective educators, we both try to foster student-centered learning environments in our classrooms. We want our students to be active "doers" of math, rather than passive receivers of our knowledge. Activities such as the ITSCoRe research team's techtivities give students the opportunity to explore mathematical relationships at their own pace. We believe that this type of independent investigation allows students to take ownership of their learning and feel more invested in the learning process. The frameworks and coding rubric that we used provide us a way to better understand and assess how our students may be reasoning through these problems. When students explain what they are doing and why, they can apply that type of reasoning to other problems, and in turn become better overall problem-solvers.

REFERENCES

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34-42. Retrieved from http://www.jstor.org/stable/40240746

Bernard, H. R. (2000). *Social research methods: Qualitative and quantitative approaches*. Thousand Oaks, Calif: Sage Publications.

Campbell, J. L., Quincy, C., Osserman, J., & Pedersen, O. K. (2013). Coding in-depth semistructured interviews: Problems of unitization and intercoder reliability and agreement. *Sociological Methods & Research, 42*(3), 294-320.

Carlson, M., Jacobs, S. Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: a framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carter, J., Helliwell, D., Henrich, A., Principe, M., & Sloughter, J. (2016). Improving Student Success in Calculus at Seattle University. *Problems, Resources, and Issues in Mathematics Undergraduate Studies, 26*(2), 105-124.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 11*(1-2), 77-87.

Duijzer, C., Van den Heuvel-Panhuizen, M., Veldhuis, M., Doorman, M., & Leseman, P. (2019). Embodied learning environments for graphing motion: A systematic literature review. *Educational Psychology Review*.

Ellis, A., Ozgur, Z., Kulow, T., Dogan, M.F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning, 18*(3), 151-181.

Ellis, A., Tasova, H.I., & Singleton, B. (2018). How quantitative reasoning can support graph understanding in algebra. In Hodges, T.E., Roy, G.J., & Tyminski, A.M. (Eds.) *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 195-202. Greenville, SC: University of South Carolina & Clemson University.

Ferrari-Escola, M., Martinez-Sierra, G., & Mendez-Guevara, M.E.M. (2016). "Multiplying by adding": Development of logarithmic-exponential covariational reasoning in high school students. *The Journal of Mathematical Behavior, 42*, 92-108.

Fetter, A. [The Math Forum at NCTM] (2015, June 8). *Ever wonder what they'd notice?: Annie Fetter.* [Video File]. Retrieved from https://www.youtube.com/watch?v=a-Fth6sOaRA

Heiny, R.L., Heiny, E.L., & Raymond, K. (2017). Placement Model for First-Time Freshmen in Calculus I (Math 131): University of Northern Colorado. J*ournal of College Student Retention: Research, Theory & Practice*, 19(3), 270-283.

How Graphs Work. (2018). Retrieved from https://teacher.desmos.com/curriculum/view/5b60949dd5e0c30f58408405

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal for Mathematical Behavior*, *31*, 313-330.

Johnson, H.L. (2015). Secondary students' quantification of ratio and rate: a framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17,* 64-90.

Johnson, H.L., (2018). Coding for covariation. Unpublished manuscript, School of Education and Human Development, University of Colorado Denver.

Johnson, H.L., Kalir, R., Olson, G., Gardner, A., Smith, A., & Wang, X. (2018). Networking theories to design a fully online assessment of students' covariational reasoning. In Hodges, T.E., Roy, G.J., & Tyminski, A.M. (Eds.) *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 1343-1346. Greenville, SC: University of South Carolina & Clemson University.

Johnson, H. L., & McClintock, E. (2018). A link between students' discernment of variation in unidirectional change and their use of quantitative variational reasoning. *Educational Studies in Mathematics*, 1–18.

Johnson, H.L., McClintock, E., & Hornbein, P. (2017). Ferris wheels and filling bottles: a case of students' transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM Mathematics Education, 49*, 851-864.

Johnson, H. L., McClintock, E., & Gardner, A. (Under Review). Designing digital task sequences to promote students' conceptions of graphs as relationships between quantities. Manuscript under review.

Johnson, H.L., McClintock, E., Kalir, R., & Olson, G. (2018). Networking theories to design dynamic covariation techtivities for college algebra students. In Weinberg, A., Rasmussen, C., Rabin, J., Wawro M., & Brown, S. (Eds.) *Proceedings of the 21st annual conference on research in undergraduate mathematics education*, 1224-1233. San Diego, CA: RUME.

Johnson, H. L., & Wang, X. (unpublished document). Covariation assessment validation study: Working paper.

Kerslake, D. (1977). The Understanding of Graphs. *Mathematics in School, 6*(2), 22-25. Retrieved from https://www.jstor.org/stable/30212405.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research, 60*(1), 1-6.

Lerner, K.L., & Lerner, B.W. (2014). Chi-square test. In *The Gale Encyclopedia of Science*. (5th ed., pp. 897-898). Farmington Hills, MI: Gale.

Madison, B. L., Carlson, M., Oehrtman, M., & Tallman, M. (2015). Conceptual Precalculus: Strengthening Students Quantitative and Covariational Reasoning. *The Mathematics Teacher, 109*(1), 54.

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Education, 2*(2), 141–157.

Moore, K.C., Stevens, I.E., Paoletti, T., Hobson, N.L.F, & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *Journal of Mathematical Behavior* (article in press).

Moore, K.C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp. 782-789). Pittsburgh, PA.

Quick Facts. (2019). Retrieved June 9, 2019, from http://www.ucdenver.edu/about-us/facts/Pages/default.aspx

Saldaña, J. (2013). *The coding manual for qualitative researchers*. Los Angeles: SAGE Publications.

Saldanha, L., & Thompson, P.W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S.B. Berensah & W.N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education – North America.* Raleigh, NC: North Carolina State University.

Shell Centre for Mathematical Education (University of Nottingham). (1985). The language of functions and graphs: An examination module for secondary schools. Nottingham, UK: JMB/Shell Centre for Mathematical Education

Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), Algebra in the early grades (pp. 95-132). New York: Erlbaum.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P.W. (2011). Quantitative reasoning and mathematical modeling. In S.A. Chamberlain & L.L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: WISDOMe Monographs, 1*, 33-57. Laramie, WY: University of Wyoming.

Thompson, P.W. & Carlson, M.P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston, VA: The National Council of Teachers of Mathematics, 421-456.

Thompson, P.W., Hatfield, N.J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95-111.

Vidacovich, C. [Courtney Vidacovich]. (2015, October 27). *Chi-square #1 test of association* [Video File]. Retrieved from https://www.youtube.com/watch?v=gxkMqQhKido