ADAPTING MULTIPLICATIVE REASONING TOOLS FOR USE WITH KENYAN STREET CHILDREN

by

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B.S., University of Virginia, 2007

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Master of Science in Education

Mathematics Education

2014

This thesis for the Master of Science in Education degree by

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4/22/2014

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Thesis directed by Professor Ron Tzur

ABSTRACT

This research thesis examined teaching considerations of a teacher who attempted to adapt a unit of instruction on multiplicative reasoning, used in the US, so it capitalizes on the knowledge (formal and informal) of Kenyan street children. The study focused on the teacher's thinking, including what needed to be adapted in the US unit of instruction, her reasoning behind in-class adjustments to lesson plans, and how students' thinking and understanding were assessed and informed subsequent teaching (and adaptations). Specifically, the instructional unit in the US that was adapted employed a platform task, called Please Go Bring Me (PGBM). This game is rooted in a constructivist theory on learning, which focuses on how students can construct new mathematical ideas based on their experiences. The researcher used the grounded theory methodology to guide the collection and analysis of data, including transcribing and coding the lessons and keeping field notes throughout the unit. This analysis reveals how lessons were adapted by considering the students' academic backgrounds as well as topics and manipulatives familiar to them (e.g., using wheels, or beads, in place of cubes/towers). Analysis also focuses on how, during lesson implementation, numerous in-class adjustments were made to handle issues of language, as well as ability to operate abstractly while playing PGBM. Other teaching considerations included shifts from a partner activity to teacher-led activity to support the teacher/researcher's work. Methods used to gauge student understandings included closely monitoring their solution strategies, probing to explain their thinking, looking for non-verbal cues, and cultural issues (particularly those seen as a barrier to student learning). This study contributes to research on mathematics teachers' shift toward a student-adaptive pedagogical approach, so one's teaching addresses mathematical, personal, and cultural (including language) needs of students and thus fosters students' ways of reasoning mathematically. The study specifically contributes to identifying ways in which teaching can be adapted to such needs of children with inadequate schooling experience (e.g., street kids).

The form and content of this abstract are approved. I recommend its publication.

Approved: Ron Tzur

DEDICATION

I dedicate this work to the street kids in Kenya who courageously fight everyday to make tomorrow better than today.

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CHAPTER I

INTRODUCTION

The purpose of this research is to examine ways in which teaching methods and tools developed for multiplicative reasoning for students in the United States can be adapted to teaching street kids[1] in Kenya. Adaptation of these methods and tools is needed because the experiences, needs, and prior knowledge of students in Kenya, especially street kids, differ from those of students in the United States. Therefore, special considerations seem needed in order to take a unit of instruction developed with students, teachers, and school cultures in the United States and teach it to Kenyan street kids. This research will examine the considerations a teacher needs to make in order to adapt a unit of instruction, make in-the-moment adjustments while teaching and gauge student learning all while working with the diverse population of Kenyan street kids.

Depicting the Phenomenon of "Street Kids"

UNICEF (2001) defined a street kid as "any girl or boy who has not reached adulthood, for whom the street (in the broadest sense of the word, including unoccupied dwellings, wasteland, etc.,) has become her or his habitual abode and/or sources of livelihood, and who is inadequately protected, supervised or directed by responsible adults" (p. 89). According to this definition, street kids or street children are children who either work and/or live on the streets. UNICEF (2001) further identified two categories of street children: "on the street" and "of the street." Children "on the street" are those that work on the street by selling goods or

[1] The term "street kids" will be explained later in this chapter; specific information about the street kid population considered for this study will be described in Chapter 3.

begging for money and return to their homes at night. Children "of the street" are homeless and may work or beg for money. They live on the streets in places such as bus stations, city dumps, or in abandoned structures. This study focuses on the latter group, children "of the street," which from here on will simply be referred to as "street kids."

Street kids leave their homes for various reasons; some are kicked out while others choose to leave their homes to escape abuse, find food, in search of adventure, or to make a living. Many have experienced abuse, both at home and while living on the streets. According to a UNICEF study conducted in Zimbabwe (2001a), a majority of street kids are boys. This gender gap is also consistent with my personal experiences working with street kids in Kenya. Possible reasons for this disparity are that many boys are often expected by their families to find their own food, they tend to experience more physical abuse at home, and mothers that re-marry often kick out their sons to satisfy their current husbands' concerns about inheritance. These children are at a particularly high risk for human trafficking, especially girls (Hatloy & Huser, 2005).

The reasons children leave home are often complex, and life on the streets is often much harder than expected. Hatloy & Huser (2005) explained:

> They are children living without a safety net, often seeking new challenges or trying to escape their present circumstances. …The fact that they live in the streets shows that they are courageous. They have made a decision [or been forced] to change their situation, by leaving their place of origin and trying to make a living on their own. (p. 7, 12).

Life on the street is hard for children and formal educational experiences are non-existent. Street kids have a hard time finding food and often resort to stealing or sniffing glue, an addiction that masks feelings of hunger. Interestingly, another UNICEF study (2001b) conducted in Zambia found that street kids' greatest desire and need for assistance was help in receiving an education. There are numerous organizations that work with the street kid population in Kenya providing basic necessities and even rescuing them from life on the streets and returning them to school. Thus, this research thesis is important for developing useful mathematical interventions to improve teaching of this diverse and needy population of children. More specifics about the population of street kids considered in this study will be discussed in the Methodology section. The next section turns to exploring aspects of mathematics education, particularly culture.

Mathematics Education is Culture-Rooted

This research focuses on developing meaningful and lasting mathematical and educational experiences for street kids in Kenya. To this end, it is important to situate mathematics learning and teaching in a view of these processes as culturally rooted, because culture plays a significant role in teaching and learning mathematics (Bishop, 1988; Bishop & Abreu, 1991; Bishop & Pompeu, 1991; Nickson, 1992). Thus, differences between Kenyan culture, particularly Kenyan street culture, and culture in the United States must be considered for a unit of instruction to be adapted.

Bishop viewed culture as a "complex of shared understandings which serves as a medium through which individual human minds interact in communication with one another" (1988, p. 5, as cited by Stenhouse, 1967, p. 16). This definition is particularly insightful in analyzing the role of culture in mathematics education, as it highlights the evolutionary view of culture that is necessary in negotiating social norms and sociomathematical norms in mathematics classrooms (Cobb & Yackel, 1995). Therefore, the culture of these students - on a community, street, school, and classroom level - will need to be considered in the adaptation of a unit of instruction from one culture to another.

In Kenya, much of the instruction in mathematics seems based on a behaviorist view of learning (Gagne, 1965; Guthrie, 1942). In particular, my informal observations of mathematics classes over a two-week period in May 2011, substantiated by research on Kenyan schools, indicated that learning in mathematics class is based largely on memorization and algorithms (Hardman, 2008). The teacher would write a sentence up on the board, have the students copy it three or four times into their notebooks and then repeat in unison the same sentence three to four times (e.g., "The circumference of a circle is the diameter multiplied by Pi."). In this case, they did not talk about what circumference actually is, but rather memorized the formula and completed some practice problems. Much of class time was spent with students on copying verbatim from the board, with little activity or interaction between the teacher and the students, or among students themselves. Many street kids struggle when they return to school, mostly due to lacking academic backgrounds and undiagnosed learning disabilities. They tend to be disengaged, particularly in mathematics class, and this type of instruction seems to do little to engage them.

Additionally, schools in Kenya face many difficulties unknown to more developed countries. For example, textbooks are hard to come by in many schools, teachers frequently have little training, there is a high turnover rate for teachers, and classroom instruction is done in students' second or third language (Glewwe, Kremer, & Moulin, 2009). Kenyan schools have historically performed poorly in comparison to other nations, particularly in mathematics (Ngware, Mutisya, & Oketch, 2012). These circumstances and aspects of culture need to inform the adaptation of a unit of instruction for use in this cultural setting.

Because of these explicit, direct ways of teaching, infusing constructivist-based teaching units from the US will require help for students to adjust from a culture of listening and repeating to a culture of engaging, reasoning, and discussing. For example, students will struggle to shift from watching and listening to interacting and asking questions. The shift in cultural norms and expectations at a classroom level may be difficult to establish because of how, in this culture, much authority is given to teachers and superiors. Students will need to see themselves as holders of knowledge and as people who have meaningful contributions to make in the classroom (Boaler, 2006). In light of this, the teacher of this unit of instruction will need to explicate cultural considerations for further engaging students and assign competence to student contributions (Boaler, 2006). This research will consider strategies for beginning and sustaining the change in classroom culture, particularly changes needed at the student level as lessons unfold to maintain student engagement and participation. The next section discusses the informal knowledge students develop through life on the streets.

Mathematical Understandings Develop through Informal Learning Situations

To guide such cultural considerations, I will build on the research of Carraher, Carraher, and Schliemann (1985). These researchers worked with children who were fruit and candy sellers on the streets in Brazil. They found that such children have developed informal systems for performing and understanding mathematical operations within the context of their business of selling fruit. For example, to sell four coconuts the children would know that 3 coconuts cost 105, since this is a common purchase, and an additional coconut costs 35. They would break apart the 35 into 30 and 5, then add 105 plus 30 to get 135 and another 5 gives 140 (Carraher et al., 1985, p. 26). All of this computational work would be done in the childrens' heads. Yet, when the same children were asked in school to solve 35 times 4, they would attempt to use their formally learned algorithms for multiplication. The "child resolves the item 35 x 4 explaining out loud: 4 times 5 is 20, carry the 2; 2 plus 3 is 5, times 4 is 20. Answer written: 200" (Carraher et al., 1985, p. 26). While being unable to perform such mathematical operations in a school setting, the children's practice of selling indicated relevant understandings of the concepts of composite unit (e.g., 35 as being three units of 10 and one unit of 5 and thus decomposable), place value, and coordinated double counting (Tzur, et al., 2013).

This study seeks to build on Carraher et al.'s (1985) work by recognizing and utilizing the competencies that Kenyan students have developed informally while living on the streets and use this as a foundation for building students' multiplicative reasoning. Street kids in Kenya come with a wide range of schooling experiences. The amount of schooling a street kid has had depends on how long she or he has lived on the streets and if their families were able to pay their school fees prior to them ending up on the streets. Despite a general lack of consistent formal schooling, many street kids have developed their own ways of operating mathematically in their everyday lives (Carraher et al., 1985; Saxe, 1988). It thus behooves the teacher of this unit to determine students' current mathematical conceptions, both correct and incorrect, informal and formal, in order to successfully use these understandings as a sort of "primitive knowledge" (Pirie & Kieren, 1994) from which to build up multiplicative reasoning. For example, many street kids have experience buying and selling goods in markets, especially food. As such, they have experience with money, determining how much to pay, which bills to use and calculating change. They also have experience purchasing typical groupings of items at the market, such as five bananas or three coconuts. While operating mathematically in these various situations, street kids may use informal and formal knowledge of addition, subtraction, and multiplication as a double counting operation[2]. By examining the ways in which they solve such real-life problems a teacher can gain insight into the units they use and how they operate with those units (Steffe, 1985; Steffe & Cobb, 1998), and build on those insights when teaching more formal mathematics.

Utilizing students' experiences and inferences into their informally developed concepts is consistent with Dewey's (1938) assertion that "What [a student] has learned in the way of knowledge and skill in one sitaution becomes an

[2] This way of reasoning multiplicatively is explained further in Chapter 2.

instrument of understanding and dealing effectively with the sitautions which follow" (p. 44). In his view, meaningful experiences should serve to connect previous knowledge (formal and informal) to future learning with a clearly defined end-goal for the growth. However, not all experiences are equal in their usefulness for achieving particular student growth, in this case multiplicative reasoning. Every experience will move a student in a particular direction; the quality of that experience is what determines if the direction is the intended one or not. The direction in which students move can be related to formal mathematics knowledge or in their attitudes towards learning and mathematics, both of which will have an impact on future experiences. The learning that takes place but was unintended, such as learned attitudes towards mathematics, is called collateral learning and needs to also be taken into consideration by teachers when developing future experiences for students (Dewey, 1938). As such, this research will consider ways for determining what prior knowledge students seem to have and how this prior knowledge can best be utilized to adapt a unit of instruction to their benefit.

Mathematics Teaching Should Build on Students' Prior Knowledge

Keeping in line with Carraher et al. (1985) and Dewey (1938), this study is designed to provide a unit plan for multiplicative reasoning that comes from a constructivist stance on learning (Piaget, 1985; Steffe, 1991; von Glasersfeld, 1995). This perspective, with its core notion of assimilation, implies to build instruction on students' current conceptions and experiences (both formal and informal) in order to develop a coherent and lasting understanding (e.g., of multiplication). This unit plan will integrate the Kenyan students' everyday mathematics with school mathematics in a way that can be meaningful for the students. Studies (Masingila, Muthwii, & Kimani, 2011) have shown that home-school connections are powerful ways to develop mathematical understanding, by building on what students already know and understand. "By building on mathematical and scientific knowledge that students bring to school from their everyday experiences, teachers can encourage students (a) to make connections between in-school and out-of-school mathematics and sciences practice in a manner that will help formalize the students' informal knowledge and (b) learn mathematics and science in a more meaningful, relevant way" (p. 10). Additionally, connecting home and school mathematics addresses equity issues in education by going further than just building on students' understanding. "The failure of schools to recognize the mathematical strengths that students bring to school widens the gap between in- and out-of-school mathematical competencies" (Wager, 2012, p. 10).

In order to accomplish adaptation to street kids' experiences/knowledge, this research will focus on how a unit of instruction used in the United States for teaching multiplicative reasoning to elementary students can be adapted to teach street kids in Kenya. Specifically, this research takes the Please Go Bring Me[3] (Tzur et al., 2013) game – a method used to teach multiplicative reasoning to students in the United States – and adjusts it in order to meet Kenyan street kids' educational needs and promote their learning (Steffe, 1990; Tzur et al., 2013). Secondly, students' available knowledge will have to inform the adaptation and implementation of this unit (Dewey, 1938; Wager, 2012; Bishop & Abreu, 1991).

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[3] Please Go Bring Me will be explained in Chapter 2 along with a theoretical background and the method for adapting it to use with Kenyan street kids will be discussed in Chapter 3.

One way to identify and examine these issues can be for the teacher-researcher to examine what changed from the planned lesson to the enacted curriculum and the underlying considerations for these changes. The changes will be evaluated for their worth as a permanent change to the unit/lesson or as something to be included in the teacher "toolbox" for use in special circumstances. Accordingly, the research questions that will be addressed in this research thesis are:

1. What inferences does a teacher-researcher make about students' available conceptions and experiences in order to inform the adaptation of an instructional unit? How do these inferences impact her considerations (rationale) for adapting the lesson planning used in the US?
2. What in-class, real-time adjustments take place during the implementation of the instructional unit and what is the teacher's rationale for making those adjustments?
3. In what ways does the teacher-researcher experience and gauge students' needs and thinking as they learn to reason multiplicatively?

CHAPTER II

REVIEW OF LITERATURE

This chapter discusses the conceptual framework guiding the creation of and adaptation of this unit of instruction. I will first summarize the constructivist theory of learning as it provides the foundational framework for all other ideas. This is followed by an overview of multiplicative reasoning and process of developing multiplicative reasoning. Next, I present the learning theory of how students develop understandings, including the two stages (participatory and anticipatory) of multiplicative reasoning. Lastly, I will discuss the platform game Please Go and Bring for Me (PGBM), which is adapted in order to develop multiplicative reasoning in street children.

Constructivist Theory of Learning

This thesis used a constructivist theory of learning as a guiding framework because this theory focuses on how students can construct their own meaning. A constructivist theory holds that knowledge must be built by each individual knower based on his or her experiences (Dewey, 1938; Piaget, 1985; von Glasersfeld, 1995). This does not mean that students can only construct meaning *by* themselves, but rather that they must do it *for* themselves. Thus, teachers play a crucial role in developing situations in which students can construct meaningful understandings. Additionally, constructivists argue that understanding cannot be transferred to students through other means. Von Glasersfeld (1995) argues that knowledge does not exist independent of an experiencer and the act of knowing is a dynamic process: any knowledge is a result of a subject's action and not externally caused and registered by a passive receiver. He theorized that 'reality' is in continual construction and is not an accumulation of ready-made structures.

Another important aspect of constructivism is that new concepts and knowledge must be connected to and built upon already existing understandings of the student (Piaget, 1985). This is an implication of the core constructs of assimilation and accommodation, whereby building upon knowledge means a student learns through transformation of her extant conceptions. For example, a teacher wishing to help students develop an understanding of multiplicative reasoning need to root these experiences in something the student already understands. Otherwise, students are not likely to create meaningful conceptual understandings of multiplication. Dewey (1938) further emphasized that learning takes place through meaningful experiences, which serve to connect previous knowledge, formal and informal, to future learning with a clearly defined end-goal for the growth. Using this theory of learning then, teachers need to be aware of how students construct meaning, their existing internal structures, and how to create situations in which students become active learners by constructing meaning for themselves. The next section takes a deeper look at constructivism and the three-part notion of scheme used to explain construction of knowledge.

Construction of Schemes

A scheme is a three-part mental structure, which the mind uses to organize and connect one's experiences (Piaget, 1985). The three parts of a scheme include: a situation perceived by the learner, an activity involving an object, and a result of the activity (von Glasersfeld, 1995). Students can perceive the situation entirely through things they notice on their own or a teacher or other learners may influence the perception of the situation. All of these factors can contribute to how a student perceives their present reality as it relates to the given situation. The activity follows the perceived situation and involves an action on some object, be it tangible and/or mental. For example, when presented with a multiplication problem of 6 times 5, a student may be in a situation where they are skip counting by fives using 6 towers of 5 cubes to know when to stop skip counting or they can skip count and mentally keep track of when to stop by coordinated double counting (i.e., 5 is 1, 10 is 2, 15 is 3, 20 is 4, 25 is 5, 30 is 6). Lastly, the result is expected to follow the activity and is what the learner notices about the outcome of the activity and how that fits into their existing schemes.

This three-part framework helps to explain how learning takes place, including assimilation and accommodation in the learning process. Schemes constitute both what a learner knows and what he or she is learning. They are the foundation on which new knowledge is formed. For example, when learning to multiply a double-digit number by a single-digit number, students have an existing scheme for composite unit, which allows them to think of groups of individual ones as a unit, and one that allows them to see multiplication as a faster way to find out the number of units by counting them as groups. This existing understanding of problems such as 5 x 8 gives them a way to think about a new problem such as 25 x 8 (e.g., 5 x 40). This scheme was built through many experiences, and without it, 25 x 8 is likely difficult or impossible for students to reason about in any meaningful way. Because this problem introduces a two-digit number, students may be unsure. Piaget (1985) asserted that these new problems can create perturbations in the learner, and result in accommodation of the available schemes.

Assimilation into available schemes and accommodation of those schemes allow a student to learn by transforming their existing schemes into new ones upon perturbation. Students assimilate new information into an existing scheme when the new information is closely related to an existing scheme. For example, if a student knows how to solve 5 x 6 and 5 x 7, they may assimilate the problem 5 x 8 into their existing scheme for multiplying by 5. Students accommodate new information by creating a new scheme or changing an existing scheme to allow for the new information. For example, when encountering the problem 5 x 18 without a scheme for multiplying by double-digits, this perturbation must be resolved by accommodating the scheme to include the new problem (e.g., 5 x 18 = 5 x 8 + 5 x 8 + 5 x 2).

Assimilation and accommodation are important for teachers to understand because these processes require utilizing students' existing schemes. Teachers need to determine students' existing schemes and create appropriate perturbations for students to accommodate new information within those schemes. Because learning is this continual process of assimilation and accommodation brought about by perturbation, resulting in the elaboration or modification or existing schemes, teachers need to also have a firm understanding of schemes required prior to developing multiplicative reasoning, which are discussed in the next section.

Building Blocks for Multiplicative Reasoning

Prior to learning multiplicative reasoning, students must have developed schemes for operating numerically, such as counting-on and additive double counting (Clark & Kamii, 1996). A student understanding of counting-on serves as an indication the student understands a number as an abstract, symbolized composite unit, which is used to represent the action of counting, without actually having to go through the action of counting (Steffe & von Glasersfeld, 1985). For example, the number 6 represents a composite unit of 6 ones. The number can be arrived at by counting objects, "1, 2, 3, 4, 5, 6," but once the number of objects has been established, it is not necessary to re-count from one while adding it to another number (e.g., 5). "Six" now refers to the six units of one being considered as a composite unit called "six." For example, students who can count-on, as opposed to count-all, are able to add 5 objects to 6 objects by starting at the composite unit 6, and counting "7, 8, 9, 10, 11" while keeping track of the items added so they know when to stop. By using counting-on, it shows that a student understands 6 as an abstract composite unit because he or she does not need to re-count the first 6 objects. This is an important scheme for a student to have prior to learning to reason multiplicatively, because the latter requires a student to repeatedly add composite units. For example, when a student is multiplying 7 x 4, it means that they are taking 4 composite units of 7, each made up of 7 ones, and determining the product, or total number of ones.

Another important understanding demonstrated by a student's ability to count-on is that of additive double counting. In the previous example of 6 + 5, the student counted-on from 6 by counting "7, 8, 9, 10, 11." However, we must consider how the student knew to stop counting at 11. This is inferred to occur through additive double counting. Because five is being added, the student uses a method for keeping track of how many ones have been added as she counts to make sure only five ones are added to six. To keep track of the adding process, students may use fingers, manipulatives, or operate mentally. A student who is double counting verbally might sound like, "7 is 1, 8 is 2, 9 is 3, 10 is 4, 11 is 5." The student knew to stop counting at eleven because they were keeping track and knew when five ones had been added to six. Double counting is an important scheme for students to have prior to learning multiplication because multiplicative double counting, a more advanced form of additive double counting, is one of the schemes students develop when developing multiplicative reasoning. These will be discussed in the next sections.

Multiplicative Reasoning

The transition from addition to multiplication is a big hurdle for many students in primary school, yet multiplication is often inadequately taught as repeated addition (Steffe, 1988; Clark & Kamii, 1996). One of the most important differences between additive and multiplicative reasoning is that in multiplicative reasoning double counting is done with composite units rather than ones as in additive reasoning (Steffe, 1992; Tzur, et al., 2010). As explained above, in additive double counting students simultaneously keep track of the ones being added as well as the total sum. However, in multiplicative double counting, students keep track of the number of composite units being added as well as the total sum. For example, in determining the product of 7 x 4, a student who is using multiplicative double counting would count “7 is 1, 14 is 2, 21 is 3, 28 is 4.” This is more complex than additive double counting because a student is operating on composite units instead of just units of one.

Another important difference between additive and multiplicative reasoning is the coordination of units in multiplication (Steffe, 1992; Tzur, et al., 2010). In multiplicative reasoning, the “items of one composite unit are distributed over the items of a second composite unit to produce a third composite unit” (Tzur, et al., 2010). For example, to determine the total number of wheels on six cars, the units involved in this operation are 6 *cars* X 4 *wheels per car* = 24 *wheels*. When reasoning multiplicatively, students must keep track of the units for each number with which they operate and distribute the four wheels across each of the six cars to find the product. This operation, which involved three different units, is different from repeated addition because repeated addition just considers one unit (4 wheels + 4 wheels + 4 wheels + 4 wheels + 4 wheels + 4 wheels = 24 wheels).

A student with an understanding of multiplication as repeated addition has an incomplete understanding of multiplicative reasoning and may experience frustration later in his or her mathematical education. Therefore, this distinction is important for teachers to understand and to help their students develop this understanding.

Schemes for Reasoning Multiplicatively

Tzur et al. (2012) developed a framework of six schemes of multiplicative reasoning, the first four of which will be considered here. The first scheme is that of multiplicative double counting, when students keep track of two quantities, one of which is being distributed over the other, as discussed in a prior section. For example, a student with this scheme could determine 5 x 4 = 20 by counting 5 is 1, 10 is 2, 15 is 3, 20 is 4. The second scheme is same-unit coordination. Students who have this scheme are able to operate additively on groups of composite units. For example, a student with this scheme could answer, "how many towers of 5 do both of us have altogether if you have 3 towers and I have 5 towers?" The student is able to view the composite unit as a "thing" in and of itself, as well as being composed of 1s—and operate on the composite unit. Unit differentiation and selection is the next scheme. This scheme allows a student to be able to distinguish between composite units and 1s and find similarities and differences between them. For example, if student A has 3 towers of 8 cubes and student B has 7 towers of 8 cubes, a student with this scheme would be able to operate multiplicatively to determine the difference in 1s between the two sets. The student might determine the difference between the towers (4 towers) and multiply by the number in each tower (8) to determine that student B has 32 more cubes. The fourth, and last scheme considered here, is called mixed-unit coordination. In this scheme, students can find totals of 1s and/or composite units when presented with both unit types. For example if given 4 towers of 5 cubes and 15 individual cubes, this student would be able to determine the total number of cubes by placing the individual cubes into towers of 5 by recognizing 5 as the unit rate. This student determines the unit rate by understanding that 5 ones comprised each tower unit.

The detailed nuances of each of the four schemes discussed are important for designing lessons that build on students' existing schemes and promote the development of subsequent schemes. Teachers without an understanding of the progression of these schemes might ask a student to answer a question for which they cannot determine an answer because the student has yet to develop the necessary schemes. Additionally, the schemes highlight the need to pay close attention to what units a student is operating on (Tzur et al., 2012), as this will help to shed light on what schemes are available to the student.

Learning through Reflection on Activity Effect Relationship

Reflection on activity effect relationship (ref-AER), presented by Tzur and Simon (2004), describes the mechanism for learning within a constructivist framework. Ref-AER happens when a student uses a current scheme to accurately predict what will happen when a certain activity is carried out by reflecting on that scheme. When working on a problem, such as 5 x 4, students would be asked to reflect on how their prediction compares with the result. By continually exploring problems in this way, students can reconfigure their intuition and develop an analytic way to think about such problems. Students accommodate their schemes of multiplication to include this new idea, by reflecting on what they've seen and done with the activity, until the activity is no longer necessary, and the object is mental. By using Ref-AER to understand how this reasoning may occur, we can better understand how students can learn to reason more effectively, and how teachers can promote this process.

In this model, a problem is solved through the following steps. An attempt to assimilate the problem is made, goal-directed activity sequences are engaged and one is selected as preferable. The activity is carried out while its effects are monitored to determined compatibility between the anticipated goal and actual effect. Depending upon the results of these comparisons, other activity sequences may also run with careful monitoring and adjustments, with the ultimate goal being correct systematic analytical reasoning as the new norm for the student. This process of reasoning takes place with every problem a student is asked to solve and can happen at two levels, which will be discussed in the next section.

Participatory and Anticipatory Stages

The two levels of reasoning involved in reflection on activity effect relationship are participatory and anticipatory (Tzur & Simon, 2004). At the participatory level, a student needs to have the activity on which the conception is based to be able to predict, At the anticipatory level, a student can make the predictions in the absence if the activity. For example, if a student can find 5 x 4 by using cubes arranged into 4 towers of 5 cubes each, students might be able to see and articulate that the product of the two numbers can be determined via skip counting by fives. However, when the towers of cubes are not available this knowledge is no longer clear to a student at the participatory stage, who may thus revert to the original thinking (e.g., adding 1s) or not be able to operate at all. Participatory students may be able to reactivate the knowledge if another person (e.g., the teacher) somehow prompts for the activity, but they cannot reason about it independently of the activities until they are at an anticipatory level. When anticipatory, students can predict the outcome of 5 x 4 without prompting or carrying out any activity.

The prompt needed for students to be able to access schemes at the participatory level may come from a teacher, student, or from oneself. A self-prompt is often in the form of an "oops" moment when a student begins to carry out an activity but part-way through realizes they have made a mistake and self-correct. Self-correcting students are considered to be at a high participatory level because the prompt came from within the student's own mental system of schemes (Tzur & Lambert, 2011). Students may then advance to the anticipatory level in which schemes can be accessed without any prompts and with some time passing between uses of the given scheme. Many teachers have experienced the frustration of students "forgetting" how to solve problems the day after they seemed to have mastered them. This may happen because students were at the participatory level the prior day, with the learning activity freshly prompted in their minds. They were able to prompt themselves by recalling the learning activity. However, it is only once students no longer need to recall the learning activity that they are at the anticipatory level.

Teachers should have an understanding of this phenomenon as it can help them to move at a pace appropriate for the students' levels of understanding. As a teacher it is easy to assume that students are ready to move on if they are able to self-correct, but in reality, these students are still prompt-dependent and may fall behind if pushed ahead without the proper underlying understandings. The fine line between participatory and anticipatory levels can be hard to distinguish, especially for students at a high participatory level who are self-prompting. If students fall behind, a teacher should consider what underlying schemes a student may not yet have at an anticipatory level and seek to move them towards this level. The game presented in the following section describes ways in which teachers can help students to develop an anticipatory understanding of the various stages involved in developing multiplicative reasoning.

Please Go Bring Me Platform Game

Please Go and Bring for Me (PGBM) is a platform game developed to teach students in the US multiplicative reasoning (Tzur et al., 2012). A detailed explanation of this unit will be given in Chapter 3, however, in order to discuss the conceptual underpinnings of PGBM, a brief description is included here. When playing PGBM, students work in pairs, with one student retrieving towers of a given size and the other student asking the follow-up questions. The retrieving student will get a certain number of towers (e.g., 5 towers), each with the same number of cubes (e.g., 3 cubes in each tower). Once the desired number of towers has been retrieved, the other student ("sender") asks four follow up questions designed to focus the "bringer's" attention to the coordination of units involved in multiplication. The questions are: (1) How many towers do you have? (2) How many cubes are in each tower? (3) How many cubes do you have total? and (4) How did you figure that out? The first two questions focus students' attention to the composite units involved, number of towers and cubes per tower. The third and fourth questions "foster coordination counting [of composite units] (e.g, raising one finger per tower) while accruing the total of cubes (e.g., 3-6-9-12-15) based on the size of the distributed [composite unit] (e.g., 3 cubes per tower)" (Tzur et al., 2012, p. 157). The fourth question also prompts a student to reflect on the methods used to find the product by having him or her articulate the method used, that is, on the relationship between her activity of producing each composite unit from 1s and accruing a few such composite units into a collection.

This game allows students with schemes at various levels of additive and multiplicative reasoning to participate. This is an important aspect of the game as it keeps in line with Piaget (1985) and Dewey's (1938) assertion that learning must build on existing schemes of students and take place through meaningful experiences. Students with an understanding of number as a composite unit (indicated by, say, counting-on) can participate while using the manipulatives (tangible objects). For example, with three towers of five cubes, a student who can count on can find the total number of cubes by counting-on from five and stopping at fifteen. Students with more complex schemes might be able to skip count by fives (e.g., 1-is-5, 2-is-10, then 11-12-13-14-15). In more complex problems, students can use the towers to aid them in multiplicative double counting. For example, with five towers of six cubes, a student can use the towers to keep track of how many composite units of six have been added. A student may count "6 (move one tower to the side), 12 (move another tower to the side), 18 (move a third tower to the side)," and so forth until all 5 towers have been moved to the side and the student knows to stop adding composite units of 6.

Another benefit of using PGBM to foster multiplicative reasoning in students is that it allows students to progress in their construction of knowledge at their own pace. Teacher interaction plays a crucial role in prompting students to explore other ways of solving the problems. For example, a teacher can ask a student to use a different object to represent the towers, say 6 pens instead of 6 towers. This prevents the student from being able to count the cubes in each tower and forces the student to think of other ways of determining the total number of cubes, like using skip counting. If the student is not at the appropriate level to complete this task the teacher can ask the student to retrieve one tower or even all the towers if the student is still at the participatory level with skip counting. The teacher can also ask students to retrieve the towers, but to cover some or all of them prior to determining the total number of cubes.

In order to prompt students to reason more abstractly, the teacher can also ask the student to draw sketches of the towers in gradually more abstract ways. For example, the teacher may ask the student to draw all the towers with all the cubes, and then move to drawing the towers without indicating individual cubes and instead the number of cubes indicated above the tower. This process can continue to push students to think in gradually more abstract ways in order to foster "a shift from attending to 1s that constitute a [composite unit] to the numerical value resulting from how each [composite unit] was produced" (Tzur et al., 2012).

Lastly, when students first play PGBM, they are limited in the number of cubes allowed in each tower as well as the total number of towers. Because students can more easily skip count by 2s or 5s, towers of these two sizes are ideal for introducing the game. Additionally, small numbers of towers (less than 7 towers) are also used to begin playing the game. Once students are at an anticipatory level with multiplicative double counting in these situations, different numbers can be introduced, such as towers of size 3 and 4 and later towers of size 6, 7, 8, and 9. The flexible nature of this activity allows each student to operate at a level appropriate for their current understandings, while providing the teacher with a platform from which he or she can prompt students to expand their thinking and methods. The next chapter describes the ways in which this game has been adapted for use with Kenyan street kids, as well as the methods used for collecting and analyzing the data collected in this research.

CHAPTER III

METHODS

This chapter discusses the methods used to gather and analyze data for studying how teaching multiplicative reasoning can be adapted to street kids' experiences and understandings. However, students' background is provided because it influences teacher considerations of such adaptation. Thus, the section below presents information about the participant pool.

Participant Pool

Teaching of mathematics in this study was adapted to street kids (boys) at a Christian ministry in Kisumu, Kenya, named Agape. This ministry rescues boys from the street by providing a safe place to live while seeking to rehabilitate and reintegrate them with their families. There are 8 teachers at the Agape school, all of which have a high school education and have completed training at a Kenyan teacher college. The children live on a campus, which provides for all of their basic needs (food, bed, hygiene, etc.) and counseling. The children are also expected to complete daily chores and to attend a school that focuses on remedial schooling and small group tutoring.

The Agape campus has a maximum capacity of approximately 100 boys, with actual numbers that fluctuate on a daily basis. The ages of the children range from 7 to 16 years, with a majority of the students in the 10 -14 range. The population at Agape is ever changing since the goal is to rehabilitate the kids, as well as their families, and eventually return the children to their homes. Most of the children are reintegrated with their families within four to six months of arriving at Agape. Once the kids have been reintegrated back at home, they re-enroll in the local Kenyan school.

To further support the children's reintegration, the Agape campus follows the Kenyan curriculum for instruction in mathematics. As is required in Kenyan schools, beginning in grade three all instruction is done in English, which for many of the students is their second or third language. New students to the campus spend four weeks exclusively attending a course designed to help them transition from life on the street to a more structured environment. After completion of this course, they are evaluated academically, incorporated into Agape's general school population, and begin attending regular courses with the rest of the students.

Five Agape students were chosen to participate in this study. The teachers at the school recommended them to participate in this unit of instruction because these students: (1) demonstrated a strong command of the English language as used in academic settings, (2) have been found by the researcher to be at an anticipatory level with counting-on and double-counting (with addition) and (3) were thus deemed ready to begin learning multiplication while having struggled to learn multiplication previously. The schools' mathematics assessment test, along with teacher insights, placed the students at this level for learning multiplication.

Unit of Instruction

In order to promote students' multiplicative reasoning, I used a modified version of the Please Go Bring Me (PGBM) platform game, which has been used to teach multiplicative reasoning to students in the United States (Tzur et al., 2012). Students play this game in pairs. In the original version of PGBM, one student

(student A) asks her or his partner (student B) to go to a box of single Unifix cubes and come back with a connected tower of a specified number of cubes. For example, student B may be asked to go to the box, produce a tower made of 5 cubes, and bring it back to student A. Next, student A again asks student B to get another tower of the same size. This process continues until student B has retrieved the number of towers desired by student A (e.g., 4 towers with 5 cubes each). Subsequently, student A asks student B four questions designed to focus both students' attention on the composite units and singletons involved: (1) How many towers did you bring? (2) How many cubes are in each tower? (3) How many cubes do you have in all? (4) How did you get that? Student B is expected to answer each question in full sentences. For example, when answering the first question, "There are 5 cubes in each tower" is an acceptable response; whereas a response like, "5 cubes" is not acceptable.

Building on Dewey (1938) and Carrahar et al.'s (1985) emphasis on children's learning via genuine experiences, my teaching plan modifies PGBM in order to better incorporate students' informal and out-of-class experiences. Instead of playing PGBM with just Unifix cubes, I introduced the game using items familiar to the children in their life that come in sets. For example, the students enjoy making bracelets with small beads. Each row of the bracelet contain anywhere from five to eight beads. The number of tires on bicycles and motorcycles (2 tires), tuk tuks (3 tires) and cars (4 tires) is another good example. The boys enjoy making cars out of pieces of wire and bottle caps, so manipulatives for counting tires on cars is a great connection to the boys' life outside of school. The boys at Agape love anything with wheels and enjoy making the bracelets, which is why these manipluatives were chosen as an adaptation to PGBM. The wording of the questions from the original PGBM was altered slightly to reflect the various manipulatives. To save time and to propel the children's use of figural items, I often asked them to make a schematic drawing of items (1s) to be included in larger (composite) units.

The numbers selected for use in this activity play an important role in ensuring that students did not experience the learning paradox (Pascual-Leone, 1976, cited in Tzur, 2008) and instead are able to make use of existing understandings and skills to complete the task (Dewey, 1938). For example, starting with easier numbers that comprise sets of tires (2) is beneficial for the students because they can more easily count pairs of 1s or multiply a number by 2, with the option of counting-all always available. For example, if a student has tires of five bicycles, they can more easily count by twos, "2, 4, 6, 8, 10," than they could if they needed to count by, say, sevens.

Once students became familiar with the PGBM game, cubes as in the original PGBM game were introduced to incorporate numbers not available in sets of numbers used naturally by the children in real life. Additionally, using the cubes emphasized how the game is designed to foster students' differentiation and coordination between making composite units from 1s, and making a set of composite units from a few of them, so counting of both 1s and composite units simultaneously is supported. Towers of size five were good to move on to, since students typically have an easier time counting units of five than larger numbers (6, 7, 8, 9).

Lastly, perturbations (Dewey, 1938; Piaget, 1985) in the students' experience were created in order to cause students to examine their processes and ways of understanding, set their own learning purpose/goal, and assimilate new experiences into their existing and evolving understandings. This was done by asking students to cover up all or some of the towers (or manipulatives, or drawings) before they determine their answer. Eventually, tasks shifted to abstract operations by asking students to imagine they retrieve manipulatives and solve the problem without actually performing the action. For example, a student may be asked, "Imagine that you went and got 6 towers of 4 cubes per tower. How many cubes would you have in all?" If a student is at an appropriate level of understanding to complete this task, they may operate abstractly by counting-all, or counting composite units, or possibly multiplying (retrieving the fact) to determine the answer. This would foster students to move from a concrete understanding to an abstract understanding. If students were not yet ready to operate on covered items, they could still peek under their hand or use another method to determine the answer. If they could not perform this operation mentally, they could be asked to draw a picture depicting one tower, all of the towers, or even acting out the scenario by physically retrieving towers. This type of imagining can cause students to engage in Type 1 Reflection on Activity-Effect Relationship (Tzur & Lambert, 2011), particularly if they are prompted to guess prior to needing to use drawings or manipulatives to determine the answer.

As preparation for teaching this unit, I engaged in thought experiments in order to think through how I would handle particular situations I thought were likely to arise during this unit of instruction. For example, if a student could skip count by 2s when working with 6 towers of 2 cubes but reverted to counting all when working with 5 towers of 4, how would I handle this? I then thought through a plan of action or line of questioning that would be helpful in assisting the student. Continuing with this example, if this situation, or a similar one came up, I planned to ask the student if they could think of a way to figure out the total number of cubes without counting all of them. I could also ask the student what they would do if I broke the 5 towers of 4 into 10 towers of 2. These thought experiments were useful for reducing the number of off the cuff adjustments and adaptations I made while teaching.

Data Collection

This research takes on an autobiographical tone (Smith, 1994), because I act as both the teacher and researcher. I focused on my personal experiences as the teacher, including my ways of experiencing the students through interacting with them, rather than on specific students' experiences and learning outcomes. I share the struggles and triumphs of teaching in a particular classroom with a small group of students, by taking a successful unit of instruction used in the United States and implementing it with a small group of rescued street kids in Kenya. In this sense the study can be considered as Action Research, because it boils down to a "teacher studying their own teaching" (Smith, 1994, p. 301). As such, I was "proposing, planning, implementing, observing, recording (through diaries and journals), [and] reflecting" (p. 301) as a means to analyze my practice and improve a unit of instruction based on the particular needs, abilities, and background knowledge of my students.

As a participant observer (Atkinson & Hammersley, 1994), I videotaped each of the six multiplicative reasoning lessons taught as a cohesive unit of learning to collect data for this study. These lessons were taught on consecutive weekdays and lasted approximately 45 minutes each. After each lesson, I videotaped my reflections (Clandinin & Connelly, 1994) on the overall lesson, including the quality of the lesson plan in terms of goals set for student learning, tasks used to promote accomplishing those goals and the rationale for these tasks (e.g., bridging), and inferences I made while teaching the lesson. The video was used solely as a recall instrument, and not for using any specific information about the students. My reflective notes also included how and why I adapt the lesson as circumstances indicted, reasoned changes I would make for the next time this lesson is taught, and my assessment of the extent to which students learned the intended concepts. These reflections were based on both the classroom culture and mathematical needs of the students.

Data Analysis

I use the work of Glaser and Strauss (1967) in *The Discovery of Grounded Theory* to inform my analysis of data to ensure that this is rigorous qualitative research. In doing this, I follow Jin (2011), who summarized Creswell's (2005) perspective on grounded theory as "a cohesive system of collecting data, identifying and coding themes and categories within the data, explicitly linking these themes/categories, and formulating a theory about a substantive human phenomenon in terms of social-cultural processes that explain it" (Jin, 2011, p. 72). To generate a grounded theory, the research process begins with a broad and open lens and becomes gradually more defined as data are alternatingly collected and analyzed. The data collection and analysis happens in a zigzag fashion, with ongoing data analysis throughout the data collection process, as is consistent with grounded theory. This process happens naturally for the teacher/researcher, as the teacher reflects on their practice and uses those reflections to further refine their practice. Grounded theory emphasizes the use of interviews and field notes. However, in this research, these data are the same as roles of the researcher and teacher are intertwined.

To analyze the data, all the lessons were first transcribed and then important events were coded (Glaser & Strauss, 1967; Tzur, 2007; Powell, Francisco, and Maher, 2003). Important events included: (1) changes in student understanding (though not specifically discussed, these are coded to see if I picked up on them in my personal reflections), (2) classroom culture successes and failures, and (3) changes in the lesson from the plan to implementation. The codes were then compared across cases to delineate themes and categories developed throughout the process of analyzing data because "a grounded theory emerges from the bottom up rather than from the top down" (Bogdan and Biklen, 1992, as cited in Jin, 2011, p. 72). Secondly, I triangulated the coding with my personal reflections to strengthen accuracy and reliability of the analysis. I noted areas of discrepancy as events overlooked/missed by me as the teacher during the lessons. The results of this data analysis are presented in the next chapter.

CHAPTER IV

RESULTS

This chapter presents the results of the study, including inferences made by the teacher (myself) in lesson design, in-class adjustments and the teacher's rationale for those adjustments, and the teacher's experiences in assessing students' needs and thinking.

Inferences Made by Teacher in Lesson Design

Most of my planning for this unit focused on big picture ideas. One of those ideas was my decision to have the students work with wheels on a car instead of just cubes in a tower. Another idea I planned in advance was how and when I planned to transition from using cars to towers. At the outset, I also thought through my goals for the unit as a whole. From a mathematical perspective, my goals for the unit were to get the students to a point where they could anticipate and apply the multiplicative double-counting scheme to smaller, easier numbers, such as three and four, and have the knowledge and tools available to also do so with larger, more difficult numbers like seven and eight. From a classroom culture perspective, my goal was to get the students comfortable with explaining their mathematical thinking.

I also planned in detail my first lesson, focusing on which numbers and manipulatives to use. Subsequent lessons were only planned generally, and I waited until I could reflect on the previous day's lesson to make concrete plans. For example, I knew that in the second lesson I wanted to focus exclusively on working with smaller numbers and skip counting. However, I didn't set goals for the students' level of abstraction until after seeing how they performed in the previous day's lesson working with smaller numbers, like 2, 3, 4, and 5. Even once my lessons were planned I was constantly adjusting in the moment as I taught each lesson. The next section describes considerations of student interests that informed my planning.

Students' Interests

The biggest factors that influenced the creation and adaptation of this unit were the personal interests and background knowledge of the students. First, the boys were extremely competitive, and turned PGBM into a competitive experience that helped them develop multiplicative reasoning. It should be noted that PGBM in its original form is not a game that has a winner and a loser. However, the boys would make a competition out of who could go and get their cars or towers the fastest. This helped to keep them engaged in the lesson and focused on the task at hand. There were times when they literally could not wait to jump up out of their seat to grab the manipulatives and would race back and forth until the desired number had been retrieved.

An important interest of the students that went into the adaptation of this unit was their interest in cars, tuk tuks, and bikes. During free time at Agape, many of the boys enjoyed building cars out of wire and bottle caps, as well as ride bikes around the campus. These two interests went into the adaptation of this unit for use with this particular group of students. The game was introduced initially with the use of matchbox cars instead of towers. These boys had never seen matchbox cars, so they were excited to be able to use them to learn math. While the cars were initially a distraction to the boys, once they got past the novelty of the cars they became effective and concrete tools for the boys to use.

I did have concerns about using the matchbox cars because they don't lend themselves to a view of the 4 wheels per car as a composite unit. This is because the boys were not actively putting the wheels on the cars. Rather, they were passive in the recognition that there are four wheels on a car. Though the number of wheels on a car is common knowledge for the boys, from a constructivist perspective about the importance of personal, hands-on experience, the students were not actively creating composite units of four because the wheels were already on the cars. This is unlike the original game of PGBM when the students take an active role in creating towers of size 4 from single cubes.

To make up for the potential conceptual gap created by using matchbox cars, I quickly shifted to using bottle caps to represent wheels. This adjustment was made to reflect my noticing of how the boys frequently used bottle caps for the wheels on cars they made out of pieces of metal, so it was a natural jump for them to use bottle caps. Using the bottle caps required the students to take an active role in creating composite units of four bottlecaps to represent the number of "wheels" on each "car" they were counting. Using bottle caps to represent wheels also allowed the teacher to incorporate composite units of sizes other than 4 (for car wheels), such as 2 (for bike wheels) and 3 (for tuk tuk wheels).

Along with using the cars-and-wheels context, the use of towers-and-cubes context was also engaging for the boys because they enjoyed making towers using the colors from flags of countries with which they were familiar. For example, the boys would often use the colors of the Kenyan, Tanzanian, and American flags to make their towers. As the teacher, this was a surprising, but welcome interest of the students as it helped to keep them engaged in the activity even when they were not using manipulatives related to wheels on a vehicle. Generally, the students did not get tired of repeatedly making the towers, which was a concern I had going into the unit. Though students were engaged with building and using the towers, there were some difficulties that I recognized in the use of these particular manipulatives. These difficulties will be discussed in a later section.

Another big shift for me was that I decided not to use beads on a bracelet as a manipulative for larger numbers. The boys loved to make beaded bracelets with anywhere from five to eights beads in each row of the bracelet. I thought using beads would be a great way to play PGBM with a familiar object while working with bigger numbers. However, there were issues on campus just before the start of the unit where boys stole beads from one another and even got in fistfights over them. Since many of our boys have a history of theft, I did not want to put the temptation in front of them to steal beads from the ones I planned to use in class. Therefore, I made the decision between Lesson 2 and Lesson 3 not to use the beads and instead have the boys work only with cubes once we began working with numbers larger than 4. This is another example of how I linked my choice for manipulatives to my students' real life experiences. Not only did I consider which manipulatives were familiar to the boys and what would get them engaged, I also considered what manipulatives would distract from the goal of learning multiplicative reasoning. The use of cubes was not my first choice manipulative, because of the students' lack of familiarity with these objects, but I felt the use of beads would have been a bigger distraction from the boys' learning. Thus, I decided to invest the extra time and effort needed to learn how to play the PGBM game with cubes. The next section describes how students' academic backgrounds informed my planning.

Students' Academic Background

The academic backgrounds of my students also played an important role in how I structured the unit and each specific lesson. Each of the 5 students in this study had been assessed in mathematics using Agape's school mathematics assessment. This assessment gave me a sense of the mathematics level of each of the students. Specifically, all five students have learned multiplication previously, but needed to develop a conceptual understanding of multiplication as discussed in Chapter 2. As a reminder, multiplicative double counting is a scheme a child constructs and use to solve problems in which the total number of units and the number of composite units are counted simultaneously. For example, a student who can multiplicatively count 4 towers of 5 would count, "1-is-5, 2-is-10, 3-is-15, 4-is-20." Needing to develop a conceptual understanding of multiplication seemed to be a major reason why they fell behind as their class moved on to learning the mechanics of how to multiply a two-digit number by a one-digit number. In informal observations, the students were slowly able to skip count to recite a multiplication table for a given number. However, they were only able to do so because they were taught the "trick" of reciting a times table and not because there is an understanding of this process or underlying multiplicative reasoning. For example, students could recite multiplication facts for the number 5, from 5 times 1 up to 5 times 12, because they know to start at 5 and add 5 each time to get the next product. However, if asked what is 5 times 7, the students would not be able to answer the question unless they recited the whole times table from 5 times 1 up to 5 times 7.

Knowing that the students were at this level of multiplicative reasoning – somewhat proficient in the mechanics, but needing to develop conceptual understanding – greatly impacted the development of this unit. I began the first lesson by having the students work with cars (composite units of 4) to get a better feel for their ability to multiplicatively double count and reason mathematically. I knew that working with composite units of 4 would likely be difficult and planned to move quickly to working with composite units (unit rate) of 2 and then on to 3 and 5. Interestingly, some of the students had an easier time multiplicatively double counting by 3s than they did by 2s. Because of this finding, I allowed the students to choose between working with composite units of 2 and composite units of 3. I wanted the students to start building towards multiplicative double counting with whatever composite units they found easiest to work with, rather than just focusing on working with composite units of 2 as I had originally planned.

The students' level of academic English was also a consideration in my planning of the unit. All five students were conversational in English. However, their reading levels were lower than their level of verbal language comprehension. Therefore, I limited the amount of reading required of students while playing PGBM. In the original PGBM game, the four questions the students ask each other are typically written on the board for all students to see and read. Additionally, response sentences with blanks where students are meant to fill in the appropriate number are written on the board. For example, a response on the board might read, "I brought _________ towers." I decided not to write the questions or responses on the board, to avoid difficulties in learning the mathematics due to reading issues. Rather, I had the students memorize the questions, as they appeared to be very good at memorizing sentences and stories and have done so on a regular basis. However, because the responses, and therefore units, were not explicitly written on the board some confusion arose with the units the students were operating on. This confusion and its possible implications will be discussed in more detail in a later section.

Other adaptations to the PGBM platform game were also made. Many of these adaptations occurred during the implementation of the unit. Thus, these in-class adjustments are discussed in the next section.

Teacher Rationale for In-Class Adjustments

No matter how well planned a unit or lesson may be, there are inevitably changes and adaptations that a teacher needs to make. This is particularly needed when a teacher adheres to a student-adaptive pedagogy, because the fluid development of thinking in one lesson must be accounted for as the lesson unfolds. The driving force behind these changes should be to improve the learning experience for the students by adapting goals and activities for their learning based on what they bring to the situation (Tzur et al., 2013). In this section I discuss some of the more important in-class adjustments made and my reasons for doing them, including manipulatives used, level of abstraction, and student-student and teacher-student interactions.

Manipulatives Used

In my layout of this unit, I planned to use matchbox cars followed solely by using bottle caps to represent wheels for numbers 2, 3, and 4. I also planned to go on to using beads in a bracelet and towers of cubes for numbers larger than 4. However, as I began the lessons, I ended up using the matchbox cars in more than just the first lesson as I had originally planned. The major reason was that the boys begged me to let them use the cars again. So, I used the cars as a motivational manipulative to reward for good behavior, ending each lesson, with the exception of one, by playing a round of PGBM with the matchbox cars. Even after the unit ended I had students coming up to me and asking when they would be put in my class so they could learn with the cars. In my field notes after lesson one, I noted: "They loved picking out the cars. They'd race over to try and get the ambulance first. I'm glad I decided to start [the unit] with the cars to get them excited and engaged." The boys' motivation also allowed me to begin using the cars by putting restrictions on the number of cars the boys could get at a time, up to 6 cars during the first lesson. But, the boys liked using the cars so much that I had to put additional restrictions on how many they got. As I mention in my field notes:

> Rather than letting them pick the number of cars, I chose the number of cars they got each time. They kept wanting to get 6 cars or whatever the most number of cars I would let them get was and then they kept getting the same answer.

The use of matchbox cars was successful because it was engaging for the students, but it also was conducive to their learning. The students were very motivated to not only use and play with the cars, but to learn mathematics with them. Their motivation reinforced my decision to use manipulatives that were of interest to them, but also aided in developing multiplicative reasoning.

Another minor shift in how manipulatives were used in the implementation of this unit was that the students could choose to use either cubes or bottle caps when working with numbers 2, 3, or 4. For example, if a student was working with composite units of 3, the student could choose to make towers of size three or work with sets of three bottle caps to represent the three wheels on a tuk tuk. In my field notes from Lesson 3, I said:

> When I noticed [how frequently the boys were mixing up the language of cubes and towers] I decided to let them use bottle caps instead of the cubes when we worked with composite units of 2, 3, and 4. I did not want the language to become a barrier to their learning. Whenever they were using the bottle caps, there were no issues with confusing the units they were operating on.

I had initially planned to transition to using just cubes for both big and small numbers once the cubes were introduced. However, because of a language issue with the boys expressing relationships in the context of cubes and towers, I allowed the boys to use bottle caps instead. I will discuss the language issue surrounding the use of towers and cubes later in this chapter section.

<u>Level of Abstraction</u>

Each student was able to operate at a different level of abstraction, which led to many other in-class adjustments. The level of abstraction was dictated by the individual students' needs and, therefore, adjustments were made on an individual basis. For example, I set a goal in the second lesson for the students to be able to multiplicatively double count by fives without all towers visible, that is, to promote the students' advance to operating on figural items. I hid one or two of the towers for each of the boys so they could not see them. Yet, for some students, counting hidden towers involved operating on 1s (singletons) rather than, say, 2 composite units of 5, so I had to adapt. For example, counting with hidden towers proceeded with fives all the way up to 20, and then the two hidden towers were wrongly counted as 21, 22. Such an activity indicated a student needed to develop an understanding of number as a composite unit to be iterated at a level of application to towers, and required adjusting to operations on visible composite units.

To adjust to a student like this, I had to pull back on the level of abstraction at which I expected him to operate. For example, I removed one of the towers from underneath the paper, which led to counting, "20, 25, and 26" for the one remaining tower. I then removed the last hidden tower from underneath the paper, which enabled counting to 30. These teaching moves proved useful as they allowed me to assess students' operations on 1s but not yet on 5 as a composite unit, and create a task they could solve while noticing and reflecting on the "tower" as a composite unit.

Of course, this in-class intervention and adjustment also required I determine my next steps. While some students were performing well at this level of abstraction, I had to adjust to individuals' needs of those who were struggling to operate with invisible composite units. In my teacher notes after the last lesson in this unit, I reflected:

> I was surprised by [S's] confusion with composite units of 5. I'm curious if he would have treated a hidden car in the same way as he treated the hidden towers. Was it a lack of familiarity with the object that caused his confusion or a true problem with working with hidden composite units?

The data excerpt above indicates how my noticing of potentially conceptual-rooted difficulties during class led to my reflection on teaching and to recognizing an alternative way for working with a particular student. Such noticing and adjustment had since become a major aspect of what I look for during class.

<u>Shift from a Partner Game to Teacher-Led</u>

The last major shift I made during Lesson 3 came about as a result of my needs as a teacher/researcher. As I began to transcribe the first lesson, I had great difficulty doing so because of the multiple conversations among students that were going on at one time. There were 4 or 5 students in each class, resulting in more than one group working and talking at a time. Additionally, I found that most of my teacher-student interactions focused on the lowest students in the class and I was not able to properly push the higher performing students along.

For these two reasons, partway through Lesson 3 I decided to slightly change how the PGBM game was played. Instead of students playing PGBM with a peer, I decided to serve as a partner ("Sender") for each of the students simultaneously, while they played the "Bringer." This adjustment allowed me to determine a proper nature of composite units, be it towers or vehicles, that each student was to get, and how many such units they would retrieve. It also allowed me to then ask the four questions to each of the students. I would ask the first question and go around and have every student respond. Generally, each student was using a different number of units (e.g., cubes and different number of towers). Then, I would ask the second question of each of the students and so forth until each student had answered each of the questions. This adjustment reflected a variation to the game that does not compromise student learning, because it actively engaged each of them in creating and accounting for composite units and 1s. Concurrently, it allowed the teacher to monitor creating of learning experiences suitable for each student (which was not supported by their playing the game in pairs).

The benefits from making this shift were highly supportive of my teaching (and learning to teach) and for data collection and analysis. Starting at Lesson 3 and onward, I was able to transcribe much more easily and thoroughly than I was with the first two lessons. In my field notes after Lesson 3, I mentioned:

> Switching from playing the game with partners to being a teacher-directed game was very helpful. Today I was able to engage with all the students one-on-one in a meaningful way. I'm also confident that when I go to transcribe this lesson that I will be able to hear and understand much more of [the lesson] because there was usually only one person talking at a time today.

This data shows a key example of how I was learning to become an adaptive teacher. I was able to recognize a problem in how the lessons were being taught and develop a solution to benefit myself as the teacher/researcher, but also benefitted the students as well. It should be noted that, at the time, I used the term "teacher-directed." In hindsight, however, the key shift was from leaving the entire activity to the students and how they interacted, to involving the teacher in guiding their work. Thus, I could better adapt the units and numbers used in a problem to the level of abstraction each student seemed to have without compromising their active role in learning via reflection on one's own and on his peers' activities.

Aside from the aforementioned, conceptual benefit of the shift to including the teacher as the "Sender," two unexpected benefits evolved that supported students' reflective processes. First, the students became very helpful in reminding each other to answer the questions in full sentences, even when I forgot to remind them. This was important because it helped to reinforce the units being operated on. Second, the students heard each student answer each question and therefore reinforced what was being learned through providing ample opportunities for comparison Type-II (i.e., across instances of related activity-effects dyads). For example, each student responded in a full sentence and would hear the units being operated on (e.g., "I brought 5 cars with 4 wheels each"), which further emphasized the units on which he was operating—for oneself and for those who heard the full sentence said. This worked because the students usually were all working with cubes or all working with wheels. They could thus imagine (act mentally) using the same units.

This second kind of reinforcement became particularly useful when the students were working with cubes and towers and would often mix up the language and say, for example, "there are five towers in each cube." In this way, students were providing prompts to one another to help correct this language issue. The students were also able to learn via hearing their fellow classmates' counting. For example, if one student used fingers to keep track of how many composite units had been counted when not all of the manipulatives were visible, it could have prompted other students to try the same method. That is, via teacher-guided interactions each of the other students could learn to keep track while double counting without the teacher having to directly prompt them to develop a way to keep track. This benefit was echoed in my field notes:

> [A student] double counted while using his fingers to keep track of how many towers he had counted. This was the first student to keep

> track this way. All the others try to keep track in their head and more often than not they lose track. I was able to direct the students to pay extra attention to how [this student] was counting without explicitly pointing anything out.

This piece of data is important because it shows how the instruction was adjusted so the students were able to learn through prompting from their classmates and not just prompting from their teacher. Specifically, I learned how to use teacher-student interaction to guide individuals to demonstrate their ways of operating, and orient other students' attention to their peer's reasoning as a source for learning as opposed to teacher-direct demonstration of the intended procedure. Many of these in-class adjustments were made as a result of how I gauged the students' needs and understanding. How I was able to do this will be discussed in the next section.

Teacher Experience in Gauging Student Needs and Thinking

As a mathematics teacher, it can sometimes be difficult to determine a student's thought process or level of understanding. This can be because of a student's inability to articulately communicate his thinking or because the teacher is not picking up on cues from the student. In this section, I will discuss some of the triumphs and failures I experienced as I tried to gauge the students' level of understanding, process for solving problems, and reasons for confusion and incorrect answers.

Inferring into Student Mathematical Understandings

The students in this study used strategies different from what I expected them to use and different from strategies I have seen used when previously teaching multiplicative reasoning via the PGBM game. For example, when working with towers of size 5, the students frequently used recursive groupings into units of ten, and even of 15, to solve the problem, particularly when the towers were all or partially visible. The students would put the towers into groups of two and count by tens if there were an even number of towers or count by tens and then add an extra 5 cubes if there were an odd number of towers. The students also would group together 3 towers of 5 and count by 15s. For example, a student might count "15 and 15 is 30 and another 10 is 40, so 40 cubes." That is, with visible objects the students could operate at 3-levels of unit coordination (e.g., 2 units of 3 units of 5 = 30) However, when the towers were not physically in front of the students, like when I had the students draw the towers, they switched to counting by fives and not using groupings of 10s or 15s. While I noticed this shift in how the students operated with towers of size 5, I was unsure what, if anything, this shift indicated. For example, I reflected in my field notes after Lesson 5:

> Now that I'm having the boys operate a bit more abstractly by hiding some or all of their towers, they seem to have done away with their strategy of using groupings of 10s and 15s. I'm thinking that maybe it's too hard for them to keep track of how many towers they have counted with this strategy. But, they use other complex strategies when working abstractly, so I'm not really sure if that is the reason.

This data excerpt shows that the boys have more than one way of grouping and counting the cubes, but the level of abstraction at which the students are operating might dictate the strategy they use. For me as a teacher, it indicated a need to pay close attention to operations on units of units, and how such operations may be developmentally sensitive to the kind of unit (tangible, figural, or abstract) a child is using.

Another surprising solution strategy that I noticed being used by some students focused on using their knowledge of doubling to *recursively* employ this operation mentally. For example, to find the total number of cubes in 8 towers of 2, a student might count, “2 and 2 is 4, and then another 4 gives me 8 and another 8 gives me 16.” Students also employed this strategy on a more limited basis even when they could not use a doubling strategy to count all of the cubes. For example, if a student had 6 towers of 2, he might count “2 and 2 is 4, 4 and 4 is 8, and now I have two more towers, so 9, 10, 11, 12.” This recursive operation was done only with tangible objects or when working with composite units of 2. With composite units of 2, the students could think of units-of-units in the abstract. I reflected in my field notes from the fourth lesson:

> I had noticed that boys previously used a [recursive] doubling strategy to solve the problems when they had the towers in front of them. It seemed normal to me because they are really good at doubling numbers and they had the physical cubes in front of them to keep track of how many towers they have added. But, today some of the boys used this same strategy to count the cubes even when only one tower was in front of them. Its use surprised me, and I even had to have the boys repeat their process for getting the answer to make sure they actually got the answer correct and didn’t just get lucky [and guess the right answer].

What I found interesting about this strategy was that students employed it even when the physical manipulatives were not in front of them, unlike when using groupings of 10 and 15. In this sense, the adjustment made to teacher-student interaction seemed to lead to fruitful distinction on my part—I began noticing different capacities students seemed to have in relation to the type of unit and the number used in a problem. In other words, I realized that students who learn to reason multiplicatively do not develop the entire set of operations in one fell swoop, as this example of operating abstractly on recursive doubling (but not on other numbers) taught me.

There are some aspects of this strategy that seem simpler, but other parts of it seem much more complex, which is why its use surprised me. For example, the doubling part of this strategy seems like an easier strategy to use than just skip counting. The boys are very proficient at doubling numbers up to 15 as was shown by their ability to do so without much effort or thinking. However, to keep track of the number of towers (units of units) they have counted while using this doubling strategy seems more complex than just counting one tower at a time. When the boys used this doubling strategy, especially with 6 or more towers, I as the teacher had to spend more time thinking (on the spot) to make sure they were keeping track properly. This type of mathematical reasoning shows that these students have the ability to operate in mathematically complex ways, as it requires mental operation on units-of-units-of-units (Steffe, 1991). This is true even more so because they could use this doubling strategy while operating abstractly.

While the boys' strategies for determining the number of 1s (cubes or wheels) were additively complex, my teaching had to constantly be adapted to reach the learning goal of the students being able to reason multiplicatively with figural or abstract objects. For example, if I asked a student to tell me the product of 5 times 3, they could tell me the answer is 15. However, they did not make the connection that to figure out how many wheels there are on 5 tuk tuks with 3 wheels on each, all they had to do was to execute the same multiplication. In my field notes I reflected:

> These boys have all learned about multiplication, but none of them actually know what it means or how to use it. When I asked one of them 'What does it mean to multiply 5 times 3?,' he answered by saying '15'. Even after I tried to clarify the question, none of the boys could tell me what it means to multiply.

This piece of data indicates my growing sensitivity to the distinction between knowing (memorizing) the times tables and having developed the underlying conceptual understanding to make sense of a multiplicative situation. By the last lesson, some of the boys began to make the connection between the numbers they had memorized in the times tables, and the results they were getting from skip counting. For example, when I asked one student how he determined his answer to the number of wheels on 8 cars with 4 wheels each, he replied with a grin that he got 32 because 8 times 4 is 32. This important change in the student's thinking was, again, echoed in my field notes:

> Today one of the boys had a bit of a breakthrough. After he retrieved the cars, he didn't start skip counting like the rest of the boys. When I asked him why he wasn't doing his work, he told me he already knew the answer. Come to find out, he finally made the connection between multiplying and counting the total number of wheels and didn't need to count. He was even able to explain to me that all you have to do is take the number of cars and multiply it by the number of wheels on each car to get the answer.

To me as a teacher, this reflective segment indicates my growing understanding of this change in students' understanding. It is here, finally, that playing the PGBM and learning to coordinate the units involved in a multiplication problem could be linked to the procedural operation of multiplying two numbers. I realized how my teaching enabled students to understand that, for example, 8 times 4 equals 32 is more than a memorized sentence, as it was now associated with the coordination of units of cars, wheels per car, and wheels. There were, however, times when students confused the units, which impacted the way they expressed relationships in context. This is discussed in the next section.

Expressing Relationships in Context

I noticed a difficulty in how the students expressed their thinking depending on the familiarity of the context. For example, for a situation involving 3 cars with 4 wheels each they would properly say, “there are 4 wheels on each car.” But when the context was unfamiliar, such as when they began playing PGBM with cubes and towers, they would say, “there are 3 towers in each cube.” It took me a few lessons to pick up on how consistently the students were mixing up these words and its potential impact on the effectiveness of PGBM as a tool for developing multiplicative reasoning. For example, in my field notes after the third lesson, I reflected:

> I have noticed for a few days now that the boys sometimes mix up the units when they are using towers and cubes. …I’m starting to think that this is a deeper issue than just a minor language slip up because it is a mistake that I’m noticing quite frequently. It doesn’t seem like the word ‘cube’ and ‘tower’ actually hold any real meaning to these boys, and so they are just focused on getting a number answer and not concerned about the units, unlike when they are using the cars.

This piece of data indicates that proper multiplicative operations were available and could be expressed accordingly if the context was familiar, but expressing properly was not supported by the unfamiliar context. The inference I made is that the thinking (mental actions on units) was there, but expressing it depended on other variables. Said differently, my field notes indicated my noticing that the unfamiliar towers may have not been assimilated by students as composite units. PGBM can be useful in developing multiplicative reasoning if/when it engages the child in operating on, distinguishing, and relating the units used at each stage of the multiplication process. However, because the students were using the incorrect units – towers, towers per cube, and cubes – the usefulness of playing PGBM in the original context (cubes and towers) for developing an understanding of multiplication as a coordination of two units to produce a third unit was likely diminished.

During this unit, I could easily determine when a student was confused due to a language issue, but I had a difficult time discerning when language and/or context issues came in the way of the students' mathematical learning. The most obvious way I determined if a student was confused was by noticing when a student did not follow directions properly. For example, if I asked a student to retrieve 3 towers with 5 cubes in each tower, one tower at a time, he might have retrieved one cube at a time and put the cubes into towers at his seat once he has retrieved 5 cubes. In my field notes after lesson two, I noted:

> [One of the students] seemed to have difficultly understanding the directions today when we made the shift to using cubes and towers. …One of the boys even explained the directions to him in Swahili, and I still had to get up and physically walk him over to the bag of cubes and show him what to do.

In a circumstance like this, it became obvious to me as a teacher that students might not understand directions due to insufficient linkage between the objects used and the units they may represent mentally. In my field notes, I also added:

> I'm not sure why [this student] was so confused because he seems to understand all of the other directions and questions I ask. There have been minor confusions with the other students over directions as well, but I think these were normal confusions I'd expect to see in a class full of native English-language speaker as well.

Reflecting back now, I realized that many of the confusions with directions came when we shifted to a slightly different activity and/or context/manipulative. For example, the directions seemed to confuse students when I had just shifted from using the bottle caps to using the cubes. This change in language and manipulatives may have contributed to the confusion.

I was also able to determine when students were not fully understanding the language I used by picking up on physical cues. There were a number of indications that students were confused. Sometimes the students would tilt their head in a certain way while other times they would stare down at the table as if to try and re-process what was said. In other times, the students would fiddle with the manipulatives and not respond to directions or questions in an appropriate manner. In my teacher notes after Lesson 2, I said:

> Overall, the students seem to understand most of what I say and they also seem to be able to communicate their thinking in English well. But, every once in awhile, I pick up on the boys not understanding something, but not because they asked a question or admitted they were confused. Usually, the look on their face gives it away. It's almost like I can see the wheels in their heads turning trying to reprocess what I said. I'm sure it's much the same look I get when I'm trying to process things said to me in Swahili.

This data excerpt shows how I began developing awareness and sensitivity to non-verbal cues, that is, activity-based indicators of the students' conceptual understandings (or lack thereof). This was a crucial development on my part, as the students rarely asked questions to clarify what was said. I will discuss possible reasons for this later.

To clarify points of confusion for the students, I used a variety of strategies. I would repeat the directions more slowly while using hand gestures to highlight important words, give the directions in Swahili, or have another student explain the directions in either English or Swahili. The students were very good about helping one another out, both with questions related to language and questions related to mathematics. Only once did I actually have to walk a student over to the cubes to physically show him the process I was expecting. In my field notes from Lesson 2, I reflected:

> The boys have been great about helping each other out. I'm thankful I can understand almost all of the Swahili they are speaking to one another because I know if directions are being explained properly. …I think the boys even appreciate when I try to explain or clarify points in Swahili because it's a time when they are teaching and correcting me, rather than me always teaching and correcting them.

The boys are used to communicating in a language other than their native language as there are many different tribal languages spoken in Kenya. It is not uncommon to hear conversations taking place in three languages as languages get blended together. As a teacher, I was able to adapt to meet the students' needs by finding another way of explaining myself. It was also helpful to have students used to communicating in multiple languages and not always understanding the languages spoken around them. Language was closely linked with the students' culture, which presented other issues requiring adaptation. These are discussed in the next section.

<u>Culture Issues</u>

As discussed in Chapter 2, in Kenyan culture hierarchies and positions of authority are very well respected. Often times it is considered unacceptable to question people in authority, especially in front of others. This makes having a classroom atmosphere of mutual respect and learning between teacher and students very difficult to foster. Students are taught not to question the teacher, but rather to submit to whatever a teacher says. While this might sound appealing from a discipline perspective, it can be problematic in developing students who are independent, critical thinkers. One way I saw this manifested during this unit was that students would generally agree with everything I said just because I was the teacher. This sometimes made it difficult to understand if the students fully understood what I was saying or if they were just agreeing with me because I was the teacher. For example, after the first lesson I mentioned in my field notes:

> Though the boys seemed to enjoy the learning activities from today, they seemed really hesitant to share their thinking and answer my questions. It's like they were really embarrassed to say something wrong, so they would just agree with whatever I said. And, they were hesitant to answer any questions because they didn't know what answer I expected them to say.

The data excerpt above shows how I became actively aware of the power/authority issue. This awareness, in turn, led to organizing lessons and activities within them so that students would overcome the cultural tendencies and learn to reason for themselves.

One strategy I began to use partway through the unit was to have a student explain back to me whatever point I was trying to make with them. This was the best strategy I employed, aside from observing their work, to determine how well students understood what I was trying to communicate. For example, when we first began playing PGBM with the matchbox cars, while I asked the boys how many wheels there were in all, they counted the four wheels on the first car even though they had just told me there were four wheels on each car. After a few rounds observing them doing this, I asked if it surprised them that they got four after counting the wheels. Students looked at me, seemed to be thinking about it, and said it was not surprising. So I asked, "Do you need to count those four wheels then? Can you just start counting with the second car?" Students thought about it and agreed with me. However, the body language indicated to me that this agreement may have fallen short of true understanding. So, I asked to explain what I just said back to me. I was pleasantly surprised when students were able to do so, and even more surprised when they corrected other students who were counting the first four wheels in the next round.

Additionally, I had to continually emphasize that it was okay for them to admit that they did not understand something and to disagree with what I was saying. To reinforce this point, I occasionally made purposeful mistakes, which I anticipated students would be able to pick up on—to see if they would correct me. For example, I was helping a student double count by 3s, and I counted, "3 and 3 is 5, plus another 3 is 8," only to have a student stop me and tell me that 3 plus 3 is actually 6, not 5. I even told them that I was purposefully going to make mistakes to trick them and encouraged them to try and point out those mistakes. I shared in my field notes after Lesson 3: "The boys seem to finally be getting comfortable with questioning me and admitting when they don't understand. I think it's been helpful that I'm making purposeful mistakes, which they are only too happy to correct." This was a very useful strategy for me to use because it led to students pointing out things they thought were mistakes, but were actually just points of confusion for the student. By the last lesson, the students were competing with one another to point out my mistakes. I mentioned in my field notes after the last lesson:

> The boys have really opened up in sharing their mathematical thinking and questioning me and their classmates in a friendly and helpful way. I've really appreciated that the boys' competitiveness hasn't impacted their willingness to help one another. They really seem to have taken responsibility for each other's learning in such a way that is very refreshing.

This shift in my teaching shows an awareness of both the cognitive and social needs of the students and how I was able to adapt my teaching to those needs throughout the unit. Specifically, I seemed to realize that to reason and grow cognitively students would need to change their accepted ways of interacting. Similarly, changing their cognition allowed them to be more critical of incorrect reasoning.

Another cultural difference I had to overcome was to encourage the students to share their thinking even when they thought it was wrong. Unfortunately, there is a fear among students in Kenya that teachers will cane them if they give an incorrect response. While this used to be a regular practice in Kenya and it does happen on occasion in schools, it has never happened at Agape. Still, the fear of being wrong is a very real one for many students, including those in this study. In the beginning lessons, students were afraid to offer an answer if they were not completely certain, choosing to remain silent or even mumble an answer so quietly that no one could hear. I had to repeatedly encourage the students to share their answers and reasoning, whether correct or incorrect. In my field notes for the first lesson, I reflected: "It was a bit painful at times to try and get the boys to participate. I really need to work to get them more comfortable sharing their answers, whether they think they are right or wrong." I made a point of praising students for their effort on a problem rather than for getting a correct answer. It also helped the students to see me make mistakes, though mine were purposeful. By the end of the unit, the students seemed very open to share their answers. I mentioned in my field notes after the last lesson: "The boys have come a long way from the first lesson in a number of ways. …One of the most drastic changes I've noticed has been their willingness and ability to share their answers and reasoning." To illustrate this change, I consider an example of the boys racing to be the first ones to get their answer for the total number of cubes so that they could be the first to share their answer and reasoning. This indicates that it is possible for a teacher to help students in such a culture to overcome some norms by adapting to the needs and experiences of the students and reinforcing desirable behaviors.

I learned a lot about the way culture and language can impact how learning and teaching takes place. The next time I teach this unit, there are a number of things I would choose to keep the same, but there are also a number of things I would do differently. I will discuss those in the next chapter, in the section that focuses on the contribution of the study to the field in terms of a teacher learning via social-cultural immersion and exchange.

CHAPTER V

DISCUSSION

This chapter discusses ways in which the findings of this thesis study (reviewed in Chapter IV) contribute to the field of mathematics education, in terms of research and teaching practices. Special considerations for working with certain populations of students, such as street kids, need to be made in order to ensure their learning needs are being met. This research adds to the body of knowledge about: (1) becoming a student-adaptive teacher, (2) adapting a unit of instruction to address the needs of students, and (3) making adjustments during instruction. Each of these aspects of new knowledge is discussed (in turn) below.

Student-Adaptive Teacher

There were a number of instances during this unit of instruction in which I, as the teacher, needed to adapt to the specific needs of my students. Before I could consider how the unit needed to be adapted, both during planning and implementation, I needed to know the needs of the boys in this study. In this study, the process of being a student-adaptive teacher began with getting to know the boys at Agape. This meant I needed to spend time on the street, learning about the lives of street kids, as well as spending time with the boys on campus in formal and informal settings. It also meant visiting the homes of boys who have been successfully reintegrated into schools through Agape. For example, I learned that a number of the boys have been on the streets more than once. This means that they likely had gaps in their learning as they missed substantial school time. Most importantly, during my interactions with the boys I spent time watching and asking questions of them. I also watched how teachers interacted with their students, and how the house parents disciplined the boys. This helped me learn the social and cultural norms in the classroom. While I did not undertake these activities specifically in preparation for this research, these experiences have impacted my knowledge of Kenyan culture and more specifically the lives of street kids. In turn, these activities enabled making adjustment to contexts (e.g., from towers/cubes to tuk-tuks/wheels) and to numbers (from 5 or 10 per tower to 3 wheels per tuk-tuk).

I also learned more purposely about the boys in my class. I spoke with our counselors and social workers to learn about why these specific boys went to the street and what their homes were like. I also talked with their other teachers and reviewed the boys' assessments in mathematics, reading, and language. This provided me with invaluable information about my students' existing cognitive knowledge, both mathematically and linguistically. All of these efforts helped to develop a clear picture of my students' cognitive and social/cultural experiences as well as their prior knowledge, both formal and informal. As described in Chapter 4, this knowledge not only impacted my planning for the unit, but I had to keep my students' cognitive, social and cultural experiences in mind as I made "on the fly" adjustments during the implementation of the unit. For example, the adjustment from pair-work to teacher-led play of the (adapted) PGBM game was done partially in response to my realization that their need to be guided into more abstract ways of thinking would be better met by my control over *how* they play the game (while leaving to the intellectual work to them).

Becoming a student-adaptive teacher also required having a deep understanding of the mathematical concepts I intended to teach (here, multiplicative reasoning). Without this understanding, it would not matter how well I knew my students and their culture. In order to be a successful teacher of these students, I needed to be able to adapt the unit of instruction in such a way that was culturally appropriate *and* mathematically accurate. For example, if I had changed PGBM to be played with number of candy in a package (i.e. two peanut butter cups in a package) it is both culturally inappropriate (the boys rarely eat candy) and mathematically unhelpful (students would not actively construct operations on composite units). I also needed to be aware of minor changes in students' strategies and understanding as they worked on developing multiplicative reasoning. For example, I was able to pick up on a student who needed to learn to work with composite units of 5 when they were hidden from view. This shows my growing understanding of how multiplicative reasoning may develop, which teachers less familiar with this concept might not have picked up on.

Lastly, I had to regularly reflect on my teaching and student interactions in order to develop as a student-adaptive teacher. As part of this research, I actively reflected on each lesson taught (e.g., via my recording and thinking about my field notes). This continual process of reflection, which focused on specific questions (i.e., my research questions), encouraged me to consider plausible reasons for my successes and failures as a teacher—and ways to improve. Because these reflections took place immediately after each lesson, I was able to use them as a basis for making changes as necessary for the next day's lesson. For these reflections to be beneficial, I had to be honest about what was working and what was not working. For example, I admittedly noted in my reflections that I wish I had asked a student who was learning to work with composite units of 5 to consider a similar problem using wheels instead of cubes. Doing so would have helped me better understand the extent to which the student was able to operate with composite units. This reflection-born strategy will be helpful the next time I teach this unit.

In my teaching prior to this unit, I had not taken the time after every lesson I taught to reflect on the lesson. Due to this thesis study, I plan to use this practice going forward, as it proved to be a crucial piece in developing as a student-adaptive teacher and thus foster my students' learning to reason multiplicatively. Under time pressure, even taking five minutes to consider the successes and failures of each student during the lesson would be a worthwhile practice. This will help me to continue to refine my practice to meet the individual needs of my students.

Adaptations of the US Unit (PGBM Game)

The teacher considerations while adapting this unit, which was rooted in the Please Go and Bring for Me (PGBM) game developed in the US, can also contribute to mathematics education research and practice. The key aspects considered in the adaptation of this unit are the (1) objects, contexts, and numbers used in instruction, (2) language, and (3) cultural values (each is discussed below).

The design of the original PGBM platform game was filtered through the cultural, social, and mathematical lenses through which I "saw" the boys in this study. This led to a number of adaptations, while keeping some of the original game intact. For example, the structure of how and when students are to work with specific numbers was kept much the same. The students began playing PGBM while using composite units of 2, 3, 4, and 5 (as the unit rate). These easier numbers are also introduced first in the original PGBM game. Additionally, ways of pushing the students to work in progressively more abstract ways was also borrowed from the original PGBM game. For example, I hid some of the boys' towers as they were trying to figure out the total number of cubes and scaled back the level of abstraction as each individual student needed. Most importantly, the question and response form of the original PGBM game was kept the same. Mathematically, this is one of the most important aspects of PGBM, as it helps to orient the students' attention to and reflection on the coordination of units involved in multiplication.

This thesis study also suggests that using objects familiar to the children, and more specifically objects of interest, is a worthwhile endeavor. While the goal of PGBM, and any unit of instruction that uses manipulatives, is to progressively advance students to a more abstract way of operating mathematically, familiar manipulatives have shown to be useful while students are still operating at a participatory stage of a new (to them) scheme. For example, the students were highly engaged when using the wheels (bottle caps) in place of the cubes/towers, and thus seemed to make progress in their multiplicative reasoning. One implication of this for using PGBM with other students would be to come up with familiar manipulatives that particular students with whom one works can use while solving PGBM tasks that involve numbers larger than four. Indeed, such a shift in number is easy to make when using cubes and towers. However, if the cubes and towers create a problem in how a student can express relationships in context (e.g., language) and other manipulatives (e.g., beads) are not appropriate, the teacher needs to revisit students' needs and interests and adapt the PGBM game to use those objects instead. For example, the cars were engaging manipulatives and contributed to the boys' learning to reason multiplicatively when the unit rate is 4, but a change in context would have been needed to increase the unit-rate (e.g., a car towing a small cart with two additional wheels would make for the unit-rate of 6).

As seen in Chapter 4, adapting a unit of instruction also needs to include considerations of specific aspects of the students' language and culture. This need was evident, for example, in the confusion the boys faced with the language needed to describe the relationships between cubes and towers. Using the Unifix cubes was not my first choice of manipulative because I recognized prior to this unit that Kenyan street kids' everyday language does not include the language of cubes and towers. This suggests that the difficultly the street children in this study faced could have been due to the unit being taught in their second or third language, or due to a lack of familiarity with the manipulatives being used, or both. To avoid the language confusion seen in this study, and because the students were learning in their second or third language, it proved worthwhile to use manipulatives with which the students were familiar. For example, bikes, tuk tuks, and cars are objects street children in Kenya encounter on a daily basis and have readily accessible vocabulary for expressing relationships among these objects. Because of this familiarity, the students could make use of these manipulatives within the adapted PGBM game context to help increase their multiplicative reasoning skills and with a high level of engagement, while using knowledge they came into the unit already knowing (i.e. three wheels on a tuk tuk).

If the manipulatives being used are holding back a child from learning the intended mathematics (e.g., because of a language issue), then the manipulatives need to be changed. This idea is supported by the findings of this study, which indicated the boys did not have similar issues of say, confusing wheels and cars as they confused cubes and towers. The key point, and contribution to the field, is that while the boys' mathematical thinking seemed to be properly available, they experienced difficultly in expressing it when operating in an unfamiliar context. In mathematics, it is important for a student to be at an anticipatory stage of understanding on an intended concept before building further on that concept (Tzur, 2000; Tzur & Simon, 2004). For example, before moving on to missing second addend problems, a student should be at the anticipatory stage in counting-on. This thesis study indicates that the same may be true for the relationship between language and mathematics when multiplicative reasoning is at issue. The boys in this study were at a participatory stage, and therefore struggled when asked to use this participatory language knowledge to build up their learning of multiplicative reasoning while referring to cubes and towers. However, they could express their thinking when dealing with wheels and cars, and were thus able to successfully use this knowledge as a base from which to build up their learning of multiplicative reasoning (including, later, transferring it to the cubes/towers context).

Lastly, I considered the culture of my students in developing and adapting this unit. The boys are very competitive and active, so I knew that playing a game that allowed the boys to get out of their seats and race back and forth would be engaging for them. I was also expecting to deal with issues of hierarchy with my students. The key here, as pointed above, is the close familiarity of the teacher with students' social-cultural needs and interests, which can inform adaptations to a "borrowed" activity that would be conducive to their learning.

This thesis study has thus highlighted the important process of planning for a unit of instruction and implementing it. Considerations I made as a teacher and researcher in this study need to go into every teacher's planning for a unit, though the specifics of these considerations are likely to vary with different populations. Clearly, the students I was considering in this study are different from the average student in Kenya. Similarly, every teacher needs to consider what will get her or his students interested, what manipulatives will aid in developing the intended conceptual understanding, how they will handle students having difficulty, and so forth. As expected, even the best planning needs to be further adapted to the needs of the students in any given lesson. This means adapting in-the-teaching-moment to better help the students through confusions or to push them along in the learning process.

In-Class Adaptations

This thesis study helped show the process and thinking behind in-class adaptations made by the teacher. Adaptations during the implementation of a unit need to be made because of mathematical needs and/or language/cultural difficulties. If my ongoing evaluation indicated the students were operating at a level of abstraction that was too easy or difficult, I had to adapt the instruction in real time to ensure the students' learning was fostered while operating at a level that is not too far beyond their current understandings. For example, I did not ask students to begin playing PGBM using towers of size 7. I was aware of my students' abilities and knew that this task would not be conducive to learning multiplicative double counting. The students needed to first experience the game and begin developing multiplicative double counting with smaller numbers. Then, if I tried to change the number to a more "difficult" one, and the students were confused, it provided me with further assessment of their progress, while also indicating the possible need to postpone using that number.

Many other in-class adjustments were a result of language and culture issues, of which I attempted to be constantly aware. This required that I pay close attention to verbal and non-verbal cues children provided, whether or not they knew those were used by me as hints. For example, a student with a blank look on his face is likely experiencing some sort of confusion (or perturbation). In my real-time teaching, I then had to learn to infer the plausible source of confusion and clarify the issue. Many times, to clarify the issue, I turned to other students to help explain directions or important points of instruction. For example, at times I would have a student explain the directions in Swahili. If this was not successful, then I had to adapt further.

I expected to face classroom culture issues going into this unit, and I handled them through in-class adjustments. For example, when the students in the class were afraid of participating or answering incorrectly, I began to make purposeful mistakes while communicating to them the expectation they should correct me. By the end of the unit, the boys were competing with one another to point out my mistakes. I also began to praise students for their effort rather than for correct answers only. These adaptations illustrate my awareness of cultural issues surrounding the students' lack of participation, and the ongoing effort to change the classroom culture so it fosters learning.

Not surprisingly, this thesis study demonstrated that teacher learning to make in-class adjustments successfully requires a trial and error process. This trial and error process is not random or arbitrary, however; rather, it can be greatly informed by the theory of learning she or he uses to explain how and what students are (or not) learning. For example, when I had a student who did not understand directions, I first tried re-explaining them using manipulatives and hand motions (i.e., changing from abstract/verbal communication to action-based communication). When this did not work, I had another student explain in Swahili. When this finally did not work, I physically walked the student through the directions I was explaining. The crucial point this research has taught me was that not all adaptations work for every student. A teacher needs to be constantly attuned to students' experiences in order to properly adapt the plan to their evolving understandings.

Applications of this Research

While this research focused on how I, as the teacher, adapted, reflected on, and adjusted to the specific learning needs of a small group of Kenyan street kids, teachers in other contexts can learn from this process and use it for their own purposes**.** First, this research highlighted the importance of identifying and using students' existing experiences and understandings to develop new concepts, beyond just their mathematical knowledge. By adapting the unit to be played with manipulatives other than cubes and towers, the students were highly engaged while developing multiplicative reasoning using manipulatives that allowed the students to properly express relationships in context. It was only through using a variety of manipulatives in this research that this key result was discovered. Thus, for teachers working with any group of students, it would behoove the teacher to test and reflect on the use of various manipulatives and how they contribute to or distract from the learning goals.

Secondly, in diverse classrooms, both culturally and ability-wise, a teacher needs to learn how to identify confusions and develop plans for helping students overcome them. This requires teachers to not only have a deep understanding of the intended learning concept, but also of the cultural influences (including language) that impact how a student experiences a learning situation and is able to express their understanding. These two key understandings both highlight the need for teachers to develop classrooms that are adaptive to the needs of the students, rather than expecting students to adapt to the strategies of the teacher. This is done through a rigorous process of learning about the students (academically, socially, and culturally), adapting to their knowledge and abilities, reflecting on the success and failures of teaching methods, and adjusting instruction as necessary to meet the evolving needs of the students.

While this research was useful in learning about the process of planning, adapting, implementing and reflecting, it was not without its limitations. These limitations are discussed in the next section.

Limitations

A major limitation of this thesis study is its scope. In this research, I taught one unit of instruction at one school, in one specific culture, and with one class of five students. This research allowed pointing to some trends about teaching considerations in adapting a unit on multiplicative reasoning in this particular setting, but these trends are limited in the generalizations that can be made. It would be interesting to replicate the study with teachers working with a similar population of students to teach this same unit. The process of becoming a student-adaptive teacher is continual and needs to be developed in wider settings.

Another limitation of this research is the short duration of the unit. Only 6 lessons were taught due to attendance issues. Due to the fluid nature of the population of students in this study, no more than two of the five students were available to participate in the class beyond the sixth lesson. This potential drawback was known before the unit of instruction began, and thus the unit was planned for six lessons only. If this unit of instruction were to be taught again, I would increase the number of lessons in order to continue teaching the boys more advanced schemes on the 6-scheme framework of multiplicative reasoning, which would provide further analysis of the adaptation process over a longer period of time. Would the boys get bored with using bottle caps and cubes? Would the language issue with cubes and towers have eventually worked itself out the more the students used those manipulatives? As the teacher, I would have to learn to adapt to new situations arising as the students progressed in their development of multiplicative reasoning.

Lastly, because beads could not be used as a manipulative in this study, their usefulness was left unexamined. Would using beads to teach this specific population of students, in place of using cubes and towers that seemed to hamper their learning, be of benefit for learning to reason multiplicatively? In the future, beads might be a viable manipulative for use with larger numbers, but this research could not be used to determine that. Coming up with a few options of manipulatives that could be used for working with numbers larger than four, and then testing those manipulatives to determine how useful and supportive they are for the students' learning would be important. Further research can be conducted to determine what manipulatives are effective to both engage the students and support their learning mathematically.

Implications for Further Research

As noted, studying how a teacher learns to adapt teaching this same unit to a different population of students would serve to further substantiate, or potentially contradict, the findings of this research. It would be interesting to see how students of a different culture would respond to the unit and what considerations would be taken on the teacher's part. For example, how would Kenyan students who have never lived in the street respond to this unit? What about using the same unit while adapting it to a group of Kenyan street girls? Would this unit be successful with street children in India or Guatemala? Teaching this unit of instruction to other groups of students would help to determine how widely applicable the adaptations made to this unit are, and what teaching considerations seem to stay invariant across settings. For example, what form would the adaptation of using bikes, cars and tuk tuks take in other countries/cultures? Would (and why) such adaptations enhance the learning for students in the US, for whom this unit was originally developed?

Another issue for further research would be to figure out the impact of teaching the unit in Swahili, or even in the students' first, tribal, language. This would allow determining if, for example, the problem of working with cubes and towers depended on using English as the language or on a general lack of familiarity with the manipulatives used. If this unit was ever taught in Swahili, it would be worthwhile to do so in Tanzania with street kids, where schools are allowed to teach in Swahili (the law in Kenya requires teaching in English). Some of Agape's boys come from Tanzania, which could provide a comparable research site.

Finally, future research can focus on teaching considerations required for adapting units on other mathematical topics. Multiplicative reasoning is but one of many concepts students need to learn in primary school. Therefore, it would be interesting to conduct similar studies with populations of street children in which other successful units of instruction can be adapted for teaching this population of students. For example, many of the students at Agape struggle with counting-on (i.e., they need to develop the concept of number as composite unit), and teachers struggle to help students get over this conceptual hurdle. It would thus be useful to take a successful unit of instruction for counting-on and adapt it in order to build on the street kids' existing knowledge and understandings. Doing so would allow further refining of processes in becoming a student-adaptive teacher.

Concluding Remarks

My motivation to conduct this research grew out of my desire to see the street boys at Agape become engaged learners in the mathematics classroom. I wanted to meet students wherever they were at, both mathematically and culturally, in order to help them develop multiplicative reasoning. I have experienced too many lessons where students just became "robots" repeating after the teacher and not being encouraged to think independently. Likewise, I have watched too many well-intentioned teachers take units of instruction and entire curricula from one culture and broadly and inappropriately apply them in another culture (e.g., US teachers recent adoption of the Singapore curriculum "as is"). This unit was my attempt to take a successful unit of instruction and proactively adapt it to be engaging for the students, culturally relevant, and mathematically appropriate.

As an educator and life-long learner, I am glad to have gone through this rigorous process of research, which required adapting, implementing, reflecting on, and adjusting a unit of instruction for use with this unique group of students. This process has helped me to see what might work with these students and what does not work with these students, along with how I think about the adaptations needed. I have developed strategies for making in-class adjustments and reflecting on their benefits for student learning. This research-based experience and knowledge will help continue refining my practice as an educator of street kids in Kenya, or other at-risk populations in different countries. I will continue to reflect on my planning and teaching in order to continue learning and improving. I believe that a teacher who does so on a regular basis not only increases her knowledge but also becomes better equipped to help her students do the same.

Undoubtedly, fostering learning in the five boys who participated was the highlight of my journey. These boys all had conceptual gaps coming into this unit of instruction. Each of them seemed to complete the unit with a meaningful understanding of multiplication. As these boys all move on from Agape to be reintegrated with family members and attend their local Kenyan school, what they were able to learn in this study will put them on a more level playing field with their peers than if they had not participated. It is my hope that the study provided the boys with experience and knowledge that can make them more successful in school and therefore settle well into their new lives at home. Most importantly, this research will benefit other boys who will be studying at the Agape campus. As I am sharing what I have learned with the other mathematics teachers at Agape, our boys will begin to experience a learning atmosphere adaptive to their unique needs, leading to their future success in school and family reintegration.

REFERENCES

Atkinson, P., & Hammersley, M. (1994). Ethnography and Participant Observation. In N. K. Denzin, & Y. S. Lincoln, *Handbook of Qualitative Research* (pp. 248-261). Thousand Oaks, CA: Sage Publications, Inc.

Bishop, A. J. (1988). *Mathematical enculturation: a cultural perspective on mathematics education*. Dordrecht, The Netherlands: Kluwer Academic.

Bishop, A. J. & Abreu, G. de (1991). Children's use of outside school knowledge to solve mathematical problems at school. In F. Furinchetti (Ed.) *Proceedings of the* 15^{th} *International Conference for the Psychology of Mathematics Education, Assisi, Italy, 1,* 128-135.

Bishop, A. J. & Pompeu, G. (1991). Influences of an ethnomathematical approach on teacher attitudes to mathematics education. In F. Furinchetti (Ed.) *Proceedings of the* 15^{th} *International Conference for the Psychology of Mathematics Education, Assisi, Italy, 1,* 136-143.

Boaler, J. (2006). How a detracked mathematics approach promoted respect, responsibility, and high achievement. *Theory Into Practice, 45(1)*, 40-46.

Bogdan, R. C., & Biklen, S. K. (1992). *Qualitative Research for education: An introduction to theory and methods*. Boston, MA: Allyn & Bacon.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology , 3*, 21-19.

Clandinin, D. J., & Connelly, F. M. (1994). Peronsal Experience Methods. In N. K. Denzin, & Y. S. Lincoln, *Handbook of Qualitative Research* (pp. 413-427). Thousand Oaks, CA: Sage Publications, Inc.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 12* (1), 27-39.

Cobb, P. & Yackel, E. (1995). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds.), *Proceedings of the* 17^{th} *Annual Meeting of the North American Chapter of the International Group for the Psycology of Mathematics Education* (pp. 3-29). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Cresswell, J. W. (2005). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research*. Upper Saddle River, New Jersey: Pearson Merrill Prentice Hall.

Dewey, J. (1938). *Experience and education*. New York: Collier.

Gagne, R. M. (1965). *The conditions of learning.* New York: Holt, Rinehart, and Winston

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. New York: Aldine de Gruyter.

Glewwe, P., Kremer, M., & Moulin, S. (2009). Many Children Left Behind? Textbooks and Test Scores in Kenya. *American Economic Journal: Applied Economics , 1* (1), 112-135.

Guthrie, E. R. (1942). Conditioning: A theory of learning in terms of stimulus, response, and association. In N. B. Henry (Ed.), *The Forty-First Yearbook of the National Society for the Study of Education (Part II – The Psychology of Learning)* (pp. 17-60). Chicago: The National Society for the Study of Education.

Hardman, F. J.-K. (2008). Pedagogical renewal: Improving the quality of classroom interaction in Nigerian primary schools. *Internation Journal of Education Development , 28*, 55-69.

Jin, Z. (2001). *Chinese middle school mathematics teachers' practices and perspectives viewed through a western lens*.

Masingila, J. O., Muthwii, S. M., & Kimani, P. M. (2011). Understanding students' out-of-school mathematics and science practice. *International Journal of Science and Mathematics Education , 9*, 89-108.

Ngware, M. W., Mutisya, M., & Oketch, M. (2012). Patterns of teaching style and active teaching: do they differ across subjects in low and high performing primary schools in Kenya? *London Review of Education , 10* (1), 35-54.

Nickson, M. (1992). The culture of the mathematics classroom: An unknown quantity? In D. A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning* (pp. 101-114). New York: Macmillan.

Pascual-Leone, J. (1976). A view of cognition from a formalist's perspective. In K. F. Riegel & J. A. Meacham (Eds.), *The developing individual in a changing world: Vol. 1 Historical and cultural issues* (pp. 89-110). The Hague, The Netherlands: Mouton.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(2-3), 165-190.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*(4), 405-435

Smith, L. M. (1994). Biographical Method. In N. K. Denzin, & Y. S. Lincoln, *Handbook of Qualitative Research* (pp. 286-305). Thousand Oaks, CA: Sage Publications, Inc.

Steffe, L. P., (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 119-140). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1990). Adaptive mathematics teaching. In T. J. Cooney & C. R. Hirsch, (Eds.), *Teaching and leaning mathematics in the 1990s* (pp. 41-51). Reston, VA: National Council of Teachers of Mathematics.

Steffe, L. P. (1991). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (3rd ed., Vol. 2, pp. 119-140). Reston, VA: Lawrence Erlbaum; National Council of Teachers of Mathematics.

Steffe, L. P. (1991). The learning paradox: a plausible counterexample. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26-44). New York: Springer-Verlag.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P., & Cobb, P. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics, 20*(1), 45-62.

Steffe, L. P., & von Glasersfeld, E. (1985). Helping children to conceive of number. *Recherches en Didactique des Mathematiques, 6*(2/3), 269-303.

Stenhouse, L. (1967). *Culture and education*. New York: Weybright & Talley.

Tzur, R. (2000). An integrated research on children's construction of meaningful, symbolic, partitioning-related conceptions, and the teacher's role in fostering that learning. *Journal of Mathematical Behavior, 18*(2), 123-147.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics, 66*(3), 273-291.

Tzur, R. (2008). Profound awareness of the learning paradox (PALP): A journey towards epistemologically regulated pedagogy in mathematics teaching and teacher education. In B. Jaworski & T. Wood (Eds.), *The International Handbook of Mathematics Teacher Education: The Mathematics Teacher Educator's as a Developing Professional* (Vol. 4, pp. 137-156). Rotterdam, the Netherlands: Sense.

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., et al. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA , 7* (3), 85-101.

Tzur, R., Johnson, H., McClintock, E., Xin, Y. P., Si, L., Kenney, R., et al. (2012). Children's development of multiplicative reasoning: A schemes and tasks framework. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155-162). Taipei, Taiwan: National Taiwan Normal University.

Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as Zone of Proximal Development correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. *Journal for Research in Mathematics Education, 42*(5), 418-450.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2,* 287-304.

Tzur, R., Xin, Y. P., Si, L., Kenney, R., & Guebert, A. (2010). *Students with learning disability in math are left behind in multiplicative reasoning? Number as abstract composite unit is a likely "culprit".* Paper presented at the American Educational Research Association. Retrieved from http://www.eric.ed.gov/.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer.

Wager, A. A. (2012). Incorporating out-of-school mathematics: from cultural context to embedded practice. *Journal of Mathematics Teacher Education , 15*, 9-23.