Changes in a Middle School Teacher's Teaching of Fractions:
Goal Selection and Questioning Based on the French Fry Activity

by

Sean Paul Damon

University of Colorado Denver

B.S., University of Northern Colorado 2011

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This thesis for the Master of Science in Mathematics Education Degree by

Sean Paul Damon

has been approved by:

Ron Tzur

Heather Johnson

Carolyn Haug

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Thesis directed by Professor Ron Tzur

Abstract

This thesis examines how changes in a teacher's conceptual understanding of fractions led to changes in his selection of goals and questions to better promote student learning. The study focuses on the teacher's understanding of unit fractions as multiplicative relationships, and the impact of this understanding on his selection of goals for students' learning and questions to promote that learning within the context of an activity—French Fry—designed to allow students construct the concept of unit fractions for themselves. The researcher used studies on bridging, reflective abstraction, and fine grain assessment to analyze and better understand student thinking. Using the ideas of grounded theory the researcher recorded, transcribed, and analyzed teacher discourse over three lessons (class days). The analysis of these data shows the importance of teacher understanding of unit fractions to promote student learning. The researcher then reflected on his questioning of students, and how it changed, in order to help students construct their own learning. The results of this research are presented in a way to help teachers address their thinking and improve their practices of teaching mathematics.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Approved By: Ron Tzur

Dedication

I dedicate this thesis to my late step-mother Renae Kindler, who passed away this year (2017). Without her I would not have found the desire to always move forward no matter what obstacles lay in in my path. Also, to my partner, Richard, for being my support through everything, including times at which I might have not deserved that support.

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Chapter I

Introduction

This thesis addresses the problem of how a middle school teacher, myself, considers and designs learning experiences conducive for students' conceptualization of unit fractions. Specifically, it examines how changes in a teacher's conceptual understanding of fractions led to changes in his selection of goals and questions to better promote student learning. In most middle school classrooms, teachers will continue to teach grade-level mathematics even though many of their students do not understand basic mathematical concepts. One of the main concepts that most students do not truly understand is fractions, and many middle school teachers do not adequately address this problem. This research thesis examines the teacher's understanding of fractions and how it impacts his design and implementation of lessons that help bring students to conceptual understanding of unit fractions—the foundational concept for all fraction knowledge (Tzur & Hunt, 2015).

Identifying the Problem

It is well known that many middle school students struggle with fractions, even though fractions are a crucial part of middle school standards (Brown & Quinn 2006 and Hackenberg 2013). In a quantitative study by Drake (2011), 162 students were presented with two fraction tasks, only 8% of those students could answer those tasks correctly. Drake stated this is because these students did not have basic fractional understanding. One approach to this problem that many middle school teachers take is to focus on the procedural knowledge they are expected to teach, solving equations for example, and leave out the fractions. My experience suggests this may turn into a major problem.

When these students reach high school, if this problem is not addressed, they may not be successful as fractions are foundational to understanding high school mathematics (Brown, and Quinn 2006). For example, in algebra students must have a conceptual understanding of fractions to understand how the slope of a given linear equation could be 4/5, and what does this number mean. Without fractional understanding, students will struggle to understand what 4/5 means as a quantification of the steepness of a line. Brown and Quinn (2006) stated that, "Capable high school students often complain that they cannot do fractions" (p. 38). Moreover, students themselves often attest they do not understand how to calculate fractions. Therefore, in leaving fractions out, teachers are doing a disservice to their students. Conversely, if teachers do not leave fractions out and continue to teach their students the same way they have been taught, namely, procedurally—many of their students are likely to be left behind. This issue needs to be addressed from many different angles, and is one that starts in middle and elementary school.

As a middle school mathematics teacher, I have come across this issue quite often, where I have struggled to teach my students a concept that is related to fractions. Often times, in order to continue to teach students the grade level material, teachers will just "leave out the fractions", but as I mentioned before, this is not beneficial to our students. This is a common issue, which many middle school teachers seem to face every year. Often, middle school students struggle with operations on rational numbers, because they do not understand fractions as a multiplicative relationship (Tzur 2000, 2004). For example, understanding fractions as a multiplicative relationship entails 1/5 is not one-out-of-five-equal-parts, but rather a quantity that is five times smaller than the whole. Said differently, it takes five of the 1/5 fraction to make the whole unit, and thus the whole is five times as much as the unit fraction 1/5. My experience as a teacher indicates the majority of students do not have this understanding. It is our responsibility as middle school mathematics teachers to prepare our students for high school mathematics, and leaving students without a conceptual understanding of fractions does not meet this challenge. Therefore, typical methods of teaching fractions need to change, for example shifting away from teaching fractions as part-to-whole (Drake 2011).

Addressing the Issues

There are many elements in our teaching that need to be addressed, the first of which is finding out what our students know conceptually (Tzur et al. 2013). All teachers must have a plan to understand where their students are conceptually, and be able to address their students' current conceptions about fractions. This, in turn, would lead to selecting goals that fit and extend the students' available conceptions, and to asking questions that are likely to promote such a shift. In order to do this, teachers also need to have a clear conceptual understanding of fractions (Tzur 2004 and Tzur & Hunt (2015). At issue is the way teachers understand and teach fractions to their students—using the idea that fractions are a part-to-whole relationship. This way of conceptualizing fractions seems to hurt students more than it is helping them (Drake, 2011; Tzur, Hodkowski & Uribe, 2016). If we, as teachers, do not have a clear conceptual understanding of fractions, we are setting our students up to be unsuccessful in their future mathematics education. Therefore, as teachers, we need to first improve our own understanding as a means to help our students learn fractions better. The understanding of a part-of-whole relationship becomes confusing when students move onto fractions that are no longer part to whole, such as taking 1/3 of ¼, let alone taking ¼ five times.

The Approach

There are many resources out there for promoting fractional understanding in elementary students, but not many have looked at the importance of teaching fractions in middle school. The majority of the tasks that are presented in these articles are elementary level tasks. Although, these tasks could be used with middle school students, they are generally at a lower maturity level. Some of these tasks can be used for middle school students, but others need to be adapted to fit the needs and available conceptions of students. One example of an activity that is designed for elementary students, but can be adapted for middle school students is the French Fry activity (Tzur & Hunt 2015). This research thesis follows the direction of Tzur, Hodkowski & Uribe (2016), in the sense that we turn to research focused on elementary students if there is not much research available on this content at the upper levels (middle/high school).

Previous research in mathematics education unearthed various reasons why fractional understanding is important and difficult for middle schoolers. For example, Bailey et al. (2012) stated that increase in fractional understanding in 6^{th} grade predicted gains in 7^{th} grade mathematics. This study showed that one solution for helping students have higher achievement is to teach students fractions in a way that they can truly have conceptual understanding. This idea starts with the teacher and is why I, as the teacher, am the focus of this study.

Focusing on The teacher

As teachers, we must have a plan and method to figure out what our students' current conceptions are, in fractions like in any topic. As part of this plan, there is a teaching practice that teachers can use, which will be discussed in more detail in Chapter 2, bridging. Bridging is a teaching practice that starts with mathematical ideas that students have full conceptual understanding of, in order to build into topics that students do not yet know (Jin & Tzur 2011). In this thesis, I will be focusing on how a teacher can consider, design, and adapt instructional materials and situations to help foster students' understanding. The questions that this research will be focusing on are:

How does the teacher's understanding of unit fractions as multiplicative relations to a whole, and the inverse relationships among unit fractions, impact:

1) The teacher's selection of goals for student learning and the activities that prompt student learning?
2) Changes in the teacher's questioning to better adapt instruction to where students are conceptually?

Chapter II

Review of Literature

In this section, I depict the conceptual framework to be used for studying the above research questions. I focus on constructivist ideas that can help teachers question students to better understand their students' thinking and/or foster further learning. I begin with explaining the key constructs of *Bridging (Jin & Tzur, 2011)*, *Zone of Actual Development*, and *Zone of Proximal development (Vygotsky, 1978)*. Next, I go into detail on Piaget's (1985) notion of *Reflective Abstraction*, because his work can help guide the questioning of our students. Lastly, I discuss the notion of *Scheme* as it was depicted by von Glasersfeld (1995). All these constructs help strengthen what we, as middle school teachers, need to know about mathematic and about our students in order to select learning goals and questions that foster students' conceptual understanding of fractions .

Building a Conceptual Understanding of Fractions

Few students have a strong conceptual understanding of fractions. In order to have a strong conceptual understanding of fractions, students need to have a strong foundational knowledge of number as a composite unit (Steffe 1991), equipartitioning (Steffe & Olive 2010; Tzur et al 2013), inverse relationships between the number and size of unit fractions (Tzur 2004), and multiplicative reasoning (Steffe and Cobb 1994; Tzur et al 2013). I begin with multiplicative reasoning, because it is an essential concept in understanding fractions.

Multiplicative Reasoning

Tzur et al. (2013) presented six schemes that children construct for reasoning multiplicatively, along with tasks that can help students to develop these schemes. Building on von Glasersfeld's (1995) notion of schemes, they described them as "conceptual structures and operations children construct and use for reasoning…" (p. 85). In that article, they laid out a framework that makes connections among the six schemes for children to be able to reason multiplicatively in a variety of situations. The leap that children must make in order to reason multiplicatively is a challenging task, which must be monitored closely by teachers to know what their children do understand. According to Tzur et al.:

> *In our constructivist framework, to solve a task, a child has to (a) assimilate it into the situation part of an existing scheme, (b) identify the quantities (mental objects) involved, (c) set a goal compatible with the question, (d) initiate mental activities on those quantities that (in the child's mind) correspond to the depicted relationships, and (e) constantly compare the actual effects of the activity to the goal to determine conclusion of the activity. (p. 87)*

Tzur et al. (2013) utilized a constructivist theory, which built on Piaget (1985) and von Glasersfeld's (1995) seminal works. Constructivist frameworks explain how students learn new concepts, and could thus help teachers to plan learning tasks to create active learning environments.

In order to construct multiplicative reasoning schemes, students need to understand numbers as a composite unit (Steffe 1991). A composite unit is the idea that numbers are made up of other numbers. For example, 5 can be made of five units of 1. It is also the idea that when I count to 10 I can count by 5's twice (Ulrich, 2016). When children are able to develop an understanding of numbers as a composite unit, they begin to make the transition from additive to multiplicative reasoning. Numbers as a composite unit is a key development in children's ability to move through the six-scheme framework (Tzur et al., 2013). In addition, children must have the ability to perform operations on composite units of units, and move away from part-to-whole decomposition.

Once students understand number as a composite unit, one way to build multiplicative reasoning is through the activity "Please Go and Bring for Me" (Tzur et al., 2013). "Please Go and Bring for Me" is a game that was developed by Tzur to promote multiplicative reasoning in children by having them participate in coordinated counting behaviors. This activity is played in partners with unifix cubes. Student A asks the Student B to go bring them one tower of 4. When B returns A asks how much we have, "one tower of four, which is 4". Then A asks B to go bring 2 more towers of 4. Once again A ask B how much do we have, "we have 3 towers of four, which is 12" and finds the total by using multiplicative double counting (e.g., 1-is-4, 2-is-8, 3-is-12).

These concepts are summed in Tzur et al.'s (2013) six-scheme developmental framework, which consists of the following: multiplicative double counting (MDC), same-unit coordination (SUC), unit differentiation and selection (UDS), mixed-unit coordination (MUC), quotitive division (QD), and partitive division (PD). The framework could provide teachers with understanding of ways in which children develop schemes in their multiplicative reasoning.

One purpose of Tzur et al.'s (2013) research was to present a framework for teachers to build tasks for their students to develop multiplicative reasoning. In doing so, the framework can help teachers be aware of where students are conceptually – the schemes that are given help to give a road map in order guide students to a better understanding. The reason these schemes are considered here is straightforward; in order to learn fractions students must have the concept of number and multiplicative reasoning (Ulrich, 2015, 2016). Without such concepts, students lack the conceptual foundations for understanding fractions. Therefore, if needed, when we our instruction based on what our students needs are it may have to include tasks such as the "Please Go and Bring Me" activity. Then students will be conceptually ready to learn about fractions.

<u>Understanding Unit Fractions</u>

For the purpose of this thesis the focus will be on unit fractions, that is, fractional quantities symbolized with 1 in the numerator (e.g., 1/2, 1/3, 1/4, …etc). Ulrich (2015, 2016) and Hackenberg and Tillema (2009) stressed that schemes of multiplicative reasoning were presented as a foundation for constructing the concept of unit fractions. In order to construct the concept of unit fractions, also known as the equipartitioning scheme (Steffe & Olive 2010; Tzur et al 2013), students must understand those units as (a) multiplicative relationships with another unit considered 'the whole' (or 1) and (b) inverse relationships between unit fractions (e.g., 1/5 > 1/7 because 5 < 7). The concept of equipartitioning is the idea that when fractions are created, the "*N*" size piece fits into the whole exactly *N* times. For example, consider the two rectangular shapes in Figures 1 and 2:

Figure 1:

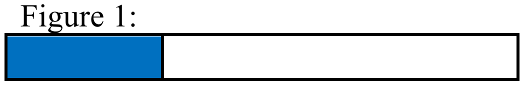
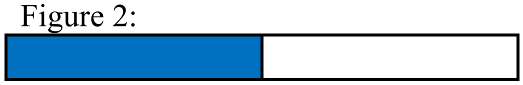


Figure 2:



In figure 1, many students will think the fraction is ½, because there are two parts and one of them is shaded. However, only Figure 2 shows ½, because the relationship between the shaded piece and the whole is 1-to-2 (one piece fits into the whole unit twice; the whole is twice as much as the shaded piece). The idea of an inverse relationship is directly related to equipartitioning (Tzur 2004). As we fit more and more pieces into a given whole (e.g., 3, 4, 5, etc.) – the pieces get smaller (1/3 > 1/4 > 1/5, etc.). This is the inverse relationship that exists in unit fractions. That is, 1/3 is greater than ¼ even though 3 is less than 4. For many students who struggle with understanding unit fractions this way, Tzur & Hunt (2015) have suggested to use the French Fry activity, which will be described further in Chapter 3. Once teachers have this understanding of fractions (the equipartitioning scheme (Tzur et al 2013)), they need to know how to design and implement tasks based on students' conceptual understanding.

Teaching for Conceptual Understanding

Bridging: Tasks to Foster Learning from Zones of Actual and Proximal Development:

Teaching for conceptual understanding requires using students' current conceptual understandings to plan and implement lessons to activate student learning (Jin and Tzur, 2011). These researchers described how Chinese teachers use a particular type of tasks, called bridging, as one of the key components of their everyday classroom teaching. The main idea behind bridging is bringing forth and capitalizing on concepts that are in every student's Zone of Actual development (Vygotsky, 1986), concepts that they already know – and use those concepts to launch students into learning a new concept. For example, the article presented information on a 7th grade classroom getting ready to learn about simplifying algebraic fractions, such as $\frac{8xy}{4x}$. In order to get students to the point they are ready to learn this concept, in other words making sure it is in their zone of proximal development, the teacher bridged students' learning with an elementary idea of simplifying numerical fractions, such as $-\frac{3}{6}$ and $\frac{13}{39}$. Then, using these basic concepts which the teachers knew all of their students already had in their Zone of Actual Development, the students can apply what they already know about simplifying numerical fractions to simplifying algebraic fractions like $\frac{8xy}{4x}$ and $\frac{4x^2}{2x}$. This process of building on what students already knew (ZAD) helped put students in a position to assimilate new concepts, which are in their Zone of Proximal Development (ZPD; see Vygotsky, 1986), because they could relate the new concepts to ones that they already have.

Jin and Tzur (2011) described another example of bridging that was unsuccessful, because it was outside of most students' Zone of Actual Development (ZAD). Since the task was not in every student's ZAD it became, at best, a learning task for those students instead of a bridging task. In order for the bridging task to have been successful, the task should have been in every student's ZAD. The unsuccessful task required students to use a concept, or scheme, they had not yet developed. Thus, only the most advanced students were able to make connections to their previous learning. Overall, bridging tasks, when used at an appropriate level, can reach all students, can help foster learning of new concepts by reactivating and building on what they already know (ZAD). It can also be an important component to planning and implementing successful learning tasks.

Jin and Tzur's (2011) study was done in 11 Chinese classrooms from two different cities in China. These classrooms were all 7th grade classrooms. Several excerpts from classroom situations were recorded and analyzed for student learning. Through their analysis, the researchers were able to conclude that one of the teachers was able to successfully use bridging; he was able to use what was in his students' ZAD to build up to the new concept which was in their ZPD. Therefore, bridging tasks, ZAD, and ZPD are constructs that help us, teachers, make a plan to build on students' conceptual understanding of fractions.

Reflective Abstraction:

When designing tasks for student conceptual understanding or questioning students we should know what learning may look like. One perspective on learning, presented by Piaget (1985), is known as Reflective Abstraction, which centers on the cognitive process of equilibration. Piaget explained that during this process, when we are presented with a new concept that we do not have current knowledge to understand, we encounter a perturbation. This perturbation, in turn, causes dis-equilibria in the available schemes. At this point in the learning process, the learner needs to adapt their current schemes: either by constructing a new scheme or adding on to existing schema that they already have. These changes will help the learner understand the new concept. Once the learner has internalized the new schemes that help them understand the new concept or concepts, they have re-equilibrated their mental system at a more advanced level.

As part of everyday teaching practices, teachers try to break down complex concepts into multiple, lower-level ideas. During each of these levels of learning, students may encounter perturbation. When perturbation happens, it causes dis-equilibration and can possibly, though not necessarily, lead to new learning. Students need to be continually construct and reconstruct their current schemes to move past the dis-equilibria to re-equilibration.

Piaget (1985) described two processes, which he considered to be fundamental components of cognitive equilibration, namely, accommodation and assimilation. The first process, assimilation, is when the student 'takes into' her or his available schemes some task/activity/problem situation. The other concept, accommodation, is where the student's current schemes are changed to form a new concept. Combined, these processes allow for the student to move past perturbation into re-equilibration and learning. Therefore, in order to completely understand a concept, students must assimilate a situation into their available schemes, the situation needs to create a perturbation for the students, and then bring about accommodation of those available schemes.

Piaget (1985) emphasized that re-equilibration does not always happen for all of our students. Unfortunately, when re-equilibration does not occur, we may leave those students behind. The re-equilibration may not happen for many reasons, which leads to students not understanding the concept that was taught. Piaget (1985) stated that, "It is obvious, therefore, that the real source of progress is re-equilibration" (p. 10). As teachers, this is something we often overlook. In order to ensure that our students understand a new concept we need to make sure they have accomplished re-equilibration at the intended, higher level of conceptualization. For example, in fractions, students need to abstract, reflectively, at each level of fractional understanding; *e.g.* unit fractions $\frac{1}{n}$, non-unit fractions $\frac{2}{m}$, operations with unit fractions $\frac{1}{n} \times \frac{1}{m}$, and so on. In re-equilibrating, students are able to reflect abstractly about the concept. Although some students are able to move past perturbations more quickly than others, most students have to work through their perturbation and develop those schemes of inverse relationships and multiplicative reasoning (Tzur 2004).

Reflective abstraction can be used to help teachers think about how their students are learning in their classrooms. When teaching a new lesson to students if teachers understand what processes are happening in their minds they can guide their students in the right direction. This idea can help teachers tailor tasks to fit the needs of their students. It also can help when planning what direction to go next within teaching those students. Therefore, teachers need to know when students can reflect abstractly about the concept and must have a way of assessing their progress.

Analyzing Student Understanding

<u>Fine Grain Assessment</u>

A key component of teaching is assessment, but many teachers are unaware of what to look for when students begin, go through, or have learned a new concept. Tzur (2007) discussed the idea of assessing student knowledge based on two learning stages, so a fine-grain assessment is achieved. The stages he described are the *participatory* and *anticipatory* stages in the construction of every new scheme. The participatory stage of learning represents a student's ability to answer questions, but only with assistance from an outside source. These students can 'participate' in the learning, but cannot anticipate the outcome without being prompted. For example, a given student might be able to partition a given unit into five equal parts to create a fraction of $\frac{1}{5}$. This could be something the student has done recently and briefly remembered what they did before in order to complete the task. It could also be that they receive some guidance or assistance from the teacher or a fellow classmate. Later, or on the next day, when the same student is given a different unit and asked to partition it into 8 parts, he/she will not able to complete the task without some sort of prompting. A student who has reached the anticipatory stage of learning can complete the new task without any kind of prompting or assistance from an outside source. As in this example indicates, the prompting may be something that is already in the student's mind, because they had possibly done a similar task shortly before the attempt of the new task. Therefore, Tzur (2007) noted that students should not be assessed on a concept right after they have solved a similar task. He explained that this may cause students that are in the participatory stage of learning to seem like they are in the anticipatory stage of learning, whereas they actually dependent on the activation of their participatory-stage scheme by being prompted.

Tzur's (2007) empirical study can help develop ways of analyzing students' thinking in their learning of a new concept. He demonstrated this by analyzing videos of lessons of twenty-eight 3rd grade students that had no previous instruction in fractions. The main focus of the analysis was to identify changes in conceptual understanding, and help teachers understand the extent to which their students meet the goals of learning that were set for them. It also helped to improve the analysis of student movement towards reflective abstraction. The task Tzur used in his study was the French Fry Activity. I will discuss this activity later, as one of the tasks that can be used to teach middle schools student's to conceptually understand fractions.

The participatory and anticipatory stages of constructing a new scheme are also important when it comes to planning goals and tasks for students' learning. The foci of the tasks that are given to students should be helping students to internalize the concept, or reflective abstraction. Students will progress through the participatory stage and may move to the anticipatory stage of learning. Many students will reach the participatory stage (prompt-dependent) but never reach the anticipatory stage (prompt-independent). This can present a problem, when a teacher may think students have mastered a concept one day, but when reviewing the concept on the next day the students would seem to have forgotten it. However, it is likely they did not forget, as revealed when they remember with prompting (Tzur & Lambert, 2011; Tzur & Simon, 2004). Rather, these students may still be in the participatory stage of learning the new scheme. We also see this happening in classrooms when students know the information and can explain it while working in partners or in groups, but when they take a test they forget. Thus, when teachers are planning tasks for their students, it is important that they are aware of where their students are in their stages of learning. Overall, these stages of learning are helpful for teachers to understand students thinking, and improve their analysis of student thinking while trying to foster reflective abstraction of a new, intended mathematical concept.

Chapter III

Method

The purpose of this thesis is to examine how changes in a teacher's own understanding of unit fractions as multiplicative relationship impacts changes in his selection of goals and questions for promoting students' learning. To this end, serving as a teacher and a researcher, I prepared and taught a 3-day unit on unit fractions. Taught at the beginning of the school year, this unit could help students have a greater fractional understanding. The data that I analyze focus on the teacher's thinking before, during, and after a lesson on fractions. In the remaining of this chapter, I first describe the setting of the study, then the methods for collecting and analyzing data.

Context and Site

As a middle school mathematics teacher, I work with 7th grade students. It is my goal to bring students to a greater understanding of fractions. The school in which I work is a charter school, with over 70% of students receiving free or reduced lunch, and over 60% identified as ELL students. This school has a wide range of student ability. Incoming 7th graders may have as low as 1st or 2nd grade ability in mathematics (e.g., barely having constructed an initial stage of number as composite unit), and as high as high school level (e.g., understanding variables and functions).

In this setting, I assumed the dual role of students' teacher and the researcher. While teaching the planned intervention to about ninety (90) 7th graders, I collected and analyzed these data from my own contemplations about goals and activities for students' learning. For this study, the data came from my teaching of one classroom (~30 students).

Unit of Instruction

In order to teach fractions to my students, a concept-based activity that focuses on conceptual understanding of unit fractions is needed. The activity I chose to use is called *The French Fry Activity* (Tzur & Hunt 2015). This activity was designed to allow students to build their own conceptualization of unit fractions as multiplicative relationships. Here, I only provide a brief description of it; Tzur and Hunt's article provides the full explanation.

The French fry activity requires students have a concept of number as composite unit (Steffe 1991;Steffe & von Glassersfeld, 1985), and is more conducive to students' learning if they also developed schemes of multiplicative reasoning. The students are supposed to work in pairs, and are given yellow strips of paper and white strips of paper of different lengths. The teacher tells students to consider the yellow strips of paper as their French Fry, and the other as a tool to help them figure out how big their share of the French Fry is.

To begin the activity, students are told they must share the French Fry with one other person, or two people total. The first time, it is expected they will fold the yellow strip (French Fry) in half. Then they are asked to describe the piece that is their share of the French Fry. Students need to understand it takes two of the pieces that are the size of their share to make the whole.

The next step is to have the students figure out what their share is if they have to equally share the French Fry among three people. Before they do this, the teacher presents two important constraints: 1) telling students they are no longer allowed to fold the French Fry nor the other (white) piece and 2) telling them they are not allowed to use any measuring instrument with pre-made units, such as a ruler. When students attempt to share the whole French Fry equally, the teacher is supposed to ask them if they think their next attempt for sharing will be larger or smaller than the piece that they shared with one other person. This is where they start building the idea of the inverse relationship (Tzur 2004), because they need to understand that as they share the French fry with more people their share gets smaller. The same process of sharing, and the same questions asked by the teacher, continue for sharing a different, new French fry among 4, 5, and 6 people, or more (e.g., 11). To resolve the perturbation created by the constraints (no folding, no ruler), students typically come up with their own idea of iterating the share of one person. If not, after a few attempts this can come to be explained by another student who has come up with the idea (or by the teacher). "Showing" the iteration activity is fine, because students' familiarity with the activity is a means, while the goal for their learning is to construct the equipartitioning scheme (unit fractions as multiplicative relationships, and the inverse relation between size and number of parts (Steffe & Olive 2010; Tzur et al 2013)).

Data Collection

The research done for this thesis is similar to an ethnographical study (Anderson, Herr and Hihlen, 2007), because the focus of this research was on me, the teacher. The data was gathered throughout the implementation and planning of the French Fry activity. Another focus was on how the teacher's conceptual understanding of fractions affected the teaching as well as how well the ideas of Bridging, Schemes, ZAD, and ZPD were used to inform my design and implementation of the lessons. This study is considered action research and allows for "deep reflection that leads to individual professional growth" (p. 31).

To collect data, I acted as a participant observer (Anderson, Herr and Hihlen 2007) of my own teaching the 7th grade mathematics classes. I recorded three days of teaching using the French Fry activity. In each of these three days, I taught each class for an 80-minute period, or a 240 minutes total in each class. I collected data by audio recording my verbal interaction with my students during each lesson: both one on one and whole class interaction. In between classes, as well as at the end of each day, I also audio recorded short oral reflections about the lesson and how I should proceed in the next lesson. The data that was drawn from the recordings focused on my actions and what I said as a teacher; such as how I questioned students in order to drive their thinking. As important as the student learning is, that is not the focus of this research.

Data Analysis

To analyze the data, I guided my thinking using Glaser and Strauss' (1967) description of a methodological tool called Grounded Theory: "theory as process; that is, theory as an ever-developing entity" (p. 32). As teachers, we are constantly using previously developed theories and trying to make them our own, and therefore building our own theory to guide our teaching. This is the main purpose of this research: to challenge and drive teacher thinking to become better teachers. In order to do this, teachers may continually analyze their teaching practices and reflect on what they have learned and put new ideas into practice in our classrooms. Glaser and Strauss (1967) described generating theory as, "Joint collection, coding and analysis of data is the underlying operation. The generation of theory, coupled with the notion of theory as process, requires that all three operations be done together as much as possible" (p. 43).

The data gathered through audio recording (on a video camera) was then transcribed and combined with the field notes I took prior to the teaching of each lesson. This discourse was transcribed and coded (Glasser and Strauss, 1967) into three main events. The first event consisted of teaching moves made through verbal discourse that established the learning goals, adjustments I (teacher) made to those goals, and the adjustment of tasks based on those goals (e.g. moving from sharing with two people to sharing with three people). Specifically, I asked of the data: When the teacher adjusted the goals for a student and presented them with a new task, how did his questioning align with the overall goal of understanding unit fractions and building the reverse fraction scheme? Also, is it clear that the teacher understands these concepts and guides the students in a way that helps them reach the learning goal in the moment and the overall goals of the whole lesson? The second event focused on questions posed by the teacher that analyze student understanding of the given task and help discern whether students are in the anticipatory stage or the participatory stage of constructing the equipartitioning scheme (Tzur et al 2013). I analyzed these events in three separate categories: the establishing of learning goals, questioning of student learning, and differentiating of learning goals for students. I also included my own reflections to help strengthen the analysis.

The questions were then categorized even further. They were divided up into questions asked on each day, and then each of those questions were coded as to what type they seemed to indicate based on the classification proposed by Hunt and Tzur (2017). The question types used were 1) Assessment Questions, 2) Cause and Affect Relationship, and 3) Comparison. The assessment questions checked for student understanding, the cause and affect questions checked for student understanding of why something happened (e.g. if the size of my share is smaller, then it will fit more times), and comparison questions checked for student understanding of the different sizes in their share of the French Fry. Each of these question types were then identified for whether or not they showed a direct teaching question: a question that directed the student learning in a certain way. The reason for this was to identify whether or not the teacher was directing student learning or if the teacher was allowing them to construct their scheme of unit fractions. The results of this data collection and analysis processes are described in the next chapter.

Chapter IV

Results

This chapter presents the results of the study, including the selection of tasks and goals within the French fry activity as presented by the teacher (myself), how the teacher questioned students to understand where they are conceptually, and how the goals and tasks were changed based on that questioning. A list of sample questions and how they were coded is included in Appendix A.

Establishing the Goals

When I initially presented the French Fry Activity to my students, I allowed them time to struggle with the initial tasks, with minimal explanation from me. The first task engaged students in figuring out how much was their share of the French fry when equally sharing between two people. The reason for giving minimal direction was to allow students a chance to begin to figure out things independently, thus bringing forth their available schemes. This can be considered a bridging task (Jin & Tzur, 2011), as it helps students begin using unit iteration to create composite units (Steffe 1991) – in service of learning the new, intended conception (unit iteration to create a unit fraction). Also, when students can construct ideas through their activities, and struggle on their own, they understand them better. My first direction to them was:

> *These yellow strips are your French fry. Your first task is to figure out how big your share is if you were to share that French fry between you and your partner. Go.*

Solving this initial task only took students a few minutes, as they were allowed to fold the yellow strip of paper. Their solutions resonated with the teacher's expectation, while bringing forth conceptions of equal sharing implying a goal of all parts being of the same size. This I inferred based on the time that lapsed between presenting them with the first task (a few seconds) and coming back together to discuss what they had done and present the next task:

> *I noticed most [students], they did this – folded it in half. How do we know that this is the share of our French fry when we share it between our self and someone else? (Calls on a student)*
>
> *Ok, how do we know that it's half…Because it's split evenly? Yes, that's a good place to start, but this is something I really want us to think about during your next task.*
>
> *From this point forward, there is no more folding.*

At this point, students understood that if they fold a paper strip into two pieces, this has to be one person's share, but they could not tell me anything else about why this was their share among two people or why it constituted one-half of the whole strip. This was quite surprising to me. A reasonable explanation I expected to hear could have been that those pieces were equal because when they folded the whole French fry in half both pieces started and ended at the same point. This became part of my goal for their next learning and something I had them take notice of, because I would keep coming back to this idea. It also indicated to me the possible challenges ahead, when folding and using rulers would be disallowed.

As Tzur & Hunt (2015) explained, a key reason for introducing the constraint of "no folding" is to promote students' use of unit iteration. At this point, however, there should have been an additional direction, which I did not include in the directions to my students. I should have explained to them that they were not allowed to use rulers or anything with lines on it. The reason that they were supposed not to use rulers is similar to the "no folding" constraint, namely, to foster their use of iteration to create a unit fraction as a measuring unit, and hopefully come up with the repeat strategy. Initially, I forgot to address this with the entire class, but it did come up with some students—at which time I told the students they could not use any tool to measure.

I considered this omission in class, and recognition of it in retrospect, as an indication of my learning. In spite of the number of times I have been 'seeing' and experiencing the French Fry activity as a student in Dr. Tzur's classes, realizing the importance of this constraint only occurred to me in hindsight. Using the theory of learning presented in the Conceptual Framework, my learning occurred through reflecting on my goal-directed teaching activities, due to the perturbation in the form of discrepancy between my initial, anticipated actions (forgetting ruler) and hearing Dr. Tzur's response that it must be included. In real time, I actually moved students from sharing between two people to the following task – sharing a French fry equally among three people – just with the constraints of "no folding."

> *Ok, now…Your next step is to figure out what is your share of the French fry, when you share it with yourself and two other people. Remember that there is no folding, but you can use the white strips of paper.*

When we moved on to that next task, the overall learning goals could begin to be addressed under the "no folding" constraint. Students had to figure out whether or not this was their piece when the French fry when it is shared equally among three people, and how do they find one person's piece. As with the first task, drawing on the constructivist idea of perturbation (Piaget, 1985), I once again let the students struggle with the challenging task. After they spent ample time, none of my students were sure how to proceed. I explain this by their lack of idea that the first piece could be used to produce the share of the other two people. I inferred they had not yet connected their piece back to the whole (when solving for two people), by being able to show that it took two of them to make the whole French fry. This connection would have helped them continue on to sharing among three people. It was at this point I decided to redirect their thinking, by also pointing out the possible relationship between their sharing experiences, between two or among three people:

> *Let's stop for a second. So, we did a piece that was our share among two people. Now we need a piece that is our share when we are sharing among three people. So, is the next piece going to be bigger or smaller?*

I note that, usually, this question is supposed to be brought up later in the activity. At this point, since students were still struggling, I thought that addressing this question earlier on could help orient students towards iteration. My idea was that by thinking of the resulting piece, for which they targeted their activities, may open the way for them to use it, in iteration, to produce an equal sharing of the paper strip among three people. Most students replied, "Smaller," so I continued:

> *…. Smaller. Keeping that in mind, see if you can figure out how big the next piece is. You already know that it must be smaller than this (holds up a piece that is the ½).*

As the excerpt above shows, I repeated a student's answer, "Smaller." I repeated it to make sure students heard what was said and with the idea in mind that when students hear things more than once they tend to think about it more and therefore understand it more. Within this explanation, I intended to foster the students' initial thinking about the inverse relationship among unit fractions (Tzur 2004). The idea behind the inverse relationship is that as the number of pieces the whole is divided into gets larger, then the smaller each of those pieces have to be. This idea could be founded on the need to iterate the piece more times to fit within the whole (Tzur & Hunt, 2015). In the case of this activity, that number is implied by the number of people with whom they are sharing the French fry. As it turned out, beginning to understand the concept of inverse relations among unit fractions took most students the majority of the first day of the activity.

The other learning goal of this activity, which seemed to begin emerging at this point, was getting students to use iteration of one person's share, which is at the core of the repeat strategy. As students were working on figuring out their share among three people I asked the following questions to individual students and small groups:

> *Can you figure out how much of the French fry that piece is? How can you show that?*

I posed this question to orient students' thinking about how many times the piece fits into the whole French fry. In one of my (other) classes, after about 20 minutes of work, only one group could start engaging in the repeat strategy. Thus, in the class reported in this thesis, I allowed students more time to struggle with figuring this out, in hope more students would come up with the repeat strategy on their own. As I explained in the conceptual framework, this expectation is rooted in trusting students have a conception of number as a composite unit, which involves iteration of 1s to create a given number.

When I felt students had come to a point where their learning was no longer benefitting from this struggle (e.g., they quit trying, or engaged in other activities), I had one of those who used the repeat strategy explain to the whole class what they did. After hearing that explanation, I paraphrased it, in order to insure everyone clearly heard and understood the repeat strategy – so that they could use it as a tool to help them construct the equipartitioning scheme (see next Excerpt). Showing how the repeat strategy works and making sure students can use it is crucial. I now recognize that this is something to be explained much sooner during this task the next time I teach it:

> *[Following the presentation of strategy] What can we call that? Yes, we did divide into three pieces, but that's not what we attempted to do. We took this piece and we repeated. We repeated it how many times? Yes, 3 times. Notice that this is our share when we are sharing among three people, 'cause this piece fits three times to*

make the whole French fry. Ok, is anyone confused? … Ok, if you don't have this piece yet, you need to make this piece that is your share among three people. Once you can show me, and tell me why that is your share among three people, then I will allow you to go on to the next one.

The excerpt above indicates my inference of and goal for students' conceptualization. For me, at issue was not the successful solution to the task. Rather, I tried to orient students' attention to key aspects of the activity: estimating the size of just one person's share, repeating (iterating) it to produce the other shares while keeping all of them equal, and having to adjust just one piece (the original) and try again. I also intended to bring students back to productive work on the task, as they will move to larger number of people who share a single French fry.

Once students were shown the repeat strategy, they all began to use it. I inferred that this was due to their bringing forth of a conception of number as composite unit—in which iteration of the unit of 1 underlies production of larger numbers (Steffe 1991; Steffe & von Glasersfeld, 1985). A few groups still needed some guidance, such as being reminded to mark on the French fry as they repeated it. The need to mark the French fry could help students see how many times they repeated that piece to get the whole French fry, that is, to keep track of *their* unit iteration activity. It also could help students be more accurate when they were using the repeat strategy, thus creating a basis for the need to adjust the pieces – which could lead to conceptualizing also the intended inverse relation (Tzur 2004). This last point is important not only, or mainly, for students' success in solving the task. Rather, it is a central part of their reflection process that could lead to conceptualizing unit fractions as unique multiplicative relations: For each given whole and number of sharers, they need to understand there could be only one piece that fits exactly so many time in the whole. Thinking about how many times it fits in the whole could then lead to thinking that the whole is as much as that piece and then to students abstracting the intended conception of unit fraction as multiplicative relation (e.g., 1/3 is the fractional relation between any whole and another piece that fits 3 times in that whole).

All in all, at this point in the lesson it seemed to me a key learning activity has been established; developing the repeat strategy as a tool to construct the equipartitioning scheme (Steffe & Olive 2010; Tzur et al 2013). To recap, the equipartitioning scheme consists of two sub-concepts: unit fraction as a multiplicative relationship and the inverse relationship among fractions (Steffe & Olive, 2010; Tzur, 2007). I considered all students to have established the repeat strategy, and thus being able to solve more challenging tasks (e.g., equally sharing the French fry among 4, 5, or 6 people) as a means to construct those two sub-concepts that would constitute their equipartitioning scheme. As students progressed though the activity, I began to question them in order to further infer their understanding of these two sub-concepts. Accordingly, the questions I asked of individual students differed, and sometimes I slightly changed the goals for their learning and/or tasks they were supposed to solve. The selection of goals and tasks needed to be adjusted in order to meet where students are conceptually. The next section examines this further.

Questioning of Student Learning

Within any lesson, it is important to have a method for understanding student thinking and their ability to apply the concepts they are learning or have learned. This section describes the questioning of students that took place in order to understand where the students are conceptually. Appendix A contains a list of sample questions from the three days this activity was presented, along with categorization of those questions according to the framework proposed by Hunt and Tzur (2015). In order to assess the extent to which my students met the goals of this activity, I gauged how they were able to answer questions. These questions were specifically aimed at conceptual understanding so I could understand their thought processes, particularly if they were at the participatory or anticipatory stages of learning. The main two questions I used to assess student knowledge were:

1. "How do you know this is your share among *n* people?" (utilizing the repeat strategy in order to foster construction of the multiplicative relationship between the piece and the whole)
2. "Is your next piece going to be bigger or smaller than the one we just made?" (utilizing connection between attempts to foster construction of the inverse relationship)

These were generic questions I used throughout the activity, where *n* changed based on the individual student's task (e.g., were they sharing among three people, then *n*=3). The way students answered these questions helped me determine how well they understood the sub-concepts, as I explain below.

There were two types of answers I was looking for when asking students these questions. The first was whether students could answer those questions on their own without any further questioning from me. If they could, it would indicate they might have been at an anticipatory stage of learning the sub-concepts and had fully constructed the sub-concepts of the equipartitioning scheme. An example of my utterances to figure out if a student is at or approaching the anticipatory stage is:

> *Ok, do you have it? Can you show me that piece? Ok, tell me about that piece. …Yes, it's our share among people because we can repeat it 3 times to get the whole French fry.*

If a student was able to immediately explain to me how she knew (anticipated) this was their share among three people – it would serve as evidence for the anticipatory stage of the equipartitioning scheme (Steffe & Olive 2010; Tzur et al 2013). This is the stage of understanding was where I was trying to get all students to. This showed they understood how to utilize the repeat strategy, and therefore would be able to use it to help them construct equipartitioning. On the first day, it was difficult for me to determine their level of understanding. The reason for this seemed to be that most students were still struggling to understand how the repeat strategy worked. Those struggling students, on the second day, had to have it re-explained to them and practice it before they began to truly understand how to use this tool.

The second type of answer I was looking for was one which required me to ask them additional questions to help them to put together their explanation (Hunt & Tzur, 2017; Tzur & Hunt, 2015). In other words, my students seemed to be at ā participatory stage of constructing the equipartitioning scheme; meaning that they knew how to act to solve the questions, but had to be prompted further so that they could recall the relationship between the number of iterations and the fractional unit. Below, I present an example of how I prompted when thinking a student might be at the participatory stage of connecting the repeat strategy to understanding why the next piece had to be smaller when sharing among more people (all lines with student responses are omitted – only my utterances are included):

> *Now, I would like you to figure out how big your share is when sharing among 4 people. Compared to the one that you just found [sharing among 3], is the next one [person's share] going to be bigger? Or smaller [than one person's share for 3]?*
>
> *Yes, it's going to be smaller, why is it going to be smaller?*
>
> *Yes, but what are you going to have to do? What did you do before with the piece that was your share among three people?*

Yes, you repeated; how many times?

Yes, and now how many times do you have to be able to repeat this one?

Good, 4 times, give it a try and I'll be back.

Based on this way of questioning, I could infer if a student was starting to understand the idea that their share needed to get smaller when more sharers were considered. In order for them to remember this, I might have had to question students multiple times, to get them to produce the idea of making their piece smaller so it could fit more times into the whole French fry. Once students successfully solved the task of sharing a fry among 3 people, I would have them try sharing among four people so that they could go through repeat strategy and thinking through all of it again. This would help them reach that anticipatory stage by reflecting across the instances of their experiences in two similar, yet separate tasks.

Overall, throughout the activity students were advancing at different paces. This is typical of any learning environment, and is something that, as a teacher, I need to be able to address by changing learning goals and/or activities (e.g., prompts) for individual students and groups. For example, some students may still be working on their shares among 4 and 5 people, whereas other groups might already be working on 8 and 9. How I adapted their learning goals is discussed in the following section.

Differentiating Learning Goals for Students

With each of the situations described in the previous section, I changed students' task as they completed them based on where they seemed to be conceptually – as indicated by their responses to my questions. Initially, all students were engaged in figuring out the size of one share among three people and then moving on to four people. However, as they moved onto figuring out how big their share was among more people, their individual/group learning goals began to change. For example, a student may begin to apply the equipartitioning scheme to solve another task, and hence be ready to focus on learning a new idea. On the other hand, another student could still be developing the initial (participatory) stages of a multiplicative (Steffe and Cobb 1994; Tzur et al 2013) and/or an inverse relationship (Tzur 2004), which would lead me to keep them on the task they were working on while paying attention to my prompting. This was evident in how I was questioning students, and how quickly or slowly I moved them to the next task (Hunt & Tzur, 2017).

At the beginning of both the second and third days, I started all students with the same goal and the same task. In doing so it allowed me to re-assess student understanding and progression through the tasks as I had described in the previous section. It also allowed me to see if those groups had advanced in reaching the learning goals. I recorded this in my reflection at the end of the first day:

> *I feel like most of the students are still at a participatory stage of understanding what their share is among n people. There's a couple that are nearing the anticipatory stage and are able to explain why their piece is their share, without further prompting. Tomorrow, I will start again [with bridging] at sharing among 3 people to see if they can get it more quickly and work on them being able to explain how they know that their piece is their share.*

By bringing all students to work on the same task, I could also see what students had retained from what they had learned the previous day. The students who did, were able to move on more quickly, because for them the task was, in fact, a bridging task (Jin & Tzur, 2011). For other students, with whom I actually needed to work, the task proved to be of value as another opportunity to construct the intended equipartitioning scheme (Steffe & Olive 2010; Tzur et al 2013).

Based on the students' work during the second-day lesson, on the third day I decided to let those who seemed more advanced work on the task of sharing a whole fry in the least amount of trials. This challenge is intended to foster students' attention to how a shortage or overage in their use of repeat strategy could be divided into the given number of sharers (Hunt, Tzur, & Westenskow, 2016). Students needed to figure out how to share a French fry, say, among 4 people, with the least amount of trials. That is, for such students I changed the goal to being able to apply the repeat strategy to construct the sub-concept of how much to increase or decrease the size of just one part they used in iteration. Below, I present my utterances that indicate realization of students' progress along with change of goals (all student lines were omitted, only the teacher's lines are presented):

> *Ok, you've got it? How many adjustments did you have to make?*
> *Just one? Awesome! Now let's try 11, does that piece need to be bigger or smaller?*
> *Yes; [but] a lot smaller or a little smaller?*
> *Good; just a little smaller. Now try to get that piece in as few tries as possible.*

The exchanges above indicate my attempt to orient the student's attention onto the size of piece to be reduced in adjusting the size of one person's share. Just like in the case studies of Lia and Ana (Hunt, Tzur, & Westenskow, 2016), my questions indicate a goal for the student to conceptualize the relationship between number of sharers and magnitude of adjustment to one person's share. This goal was rooted in my inference that the adjustment, as seen by the student, should not consist of the entire overage.

Unlike the above situation of focusing students onto the amount of change needed, goals for other students still focused on the direction of change needed. This showed me that they were at a participatory stage of understanding of the nature of change needed. Further guidance and prompting was thus required. I noticed that, as expected by the designers of the French Fry activity (Tzur & Hunt, 2015), after students/groups were successfully completed the task of sharing among 3 and 4 people, progress came fairly quick.

About halfway through day-three lesson, I made another adaptation in learning goals for some of the students. This change was my attempt to capitalize on a solution neither they nor I expected, while working to share a fry equally among 8 people:

> *Ok, so what piece is this?*
>
> *Your share among 9 people, even though you were trying for 8?*
>
> *How do you know?*
>
> *Good, so what do you have to do now?*
>
> *Yes, make it a little bit bigger. So, think about the piece that you are trying to find compared to your share among 7 people. Originally you made it too small, so how much smaller do you need to make it?*
>
> *Yes, it has to between your share among 9 and among 7. So now you have to think, do I need to make it a little bit smaller or a lot smaller?*

During this interaction, I had to change the approach of my own thinking and my questioning. This was because the question, and the goal for learning, changed from "is it bigger or smaller?" to "how much bigger or how much smaller?" (Hunt & Tzur, 2015). In this case, the next piece (for 8), I wanted the students to notice the next attempt needed to be in between the two pieces that were made (for 7 and for 9). This proved a non-trivial task, which required longer time than I expected. This change in goals required a lot of deeper thinking strategies, because there was no longer a simple answer. Eventually, I could support some students' construction of this "in-between" idea, as evidenced in their independent use of it for sharing different numbers. Using what I had learned about my students, I would be able to plan additional activities that could focus on their need.

Summary of Chapter IV

Through the activities of data collection and analysis of my own teaching, I learned to relate my teaching with student thinking. I realized that it is important to take the time and really listen to students as they explain themselves. This allows the teacher to understand where students are at in their learning and thinking. It also entails encouraging students to explain their thinking as well as showing it through some actions (Hunt & Tzur, 2017). Also, as a teacher, it is important to allow students to work at a pace that is comfortable for them to generate their own understanding. The time it took students to construct the equipartitioning differed greatly, including a longer period spent in working at the participatory stage. It was also important for me, as the teacher, to have a deeper understanding of the mathematical concepts myself, in order to be able to teach them to my students. Clearly, I now understand equipartitioning better, after having taught it through the French fry activity. I now appreciate much more how this deeper teacher understanding is crucial for real student understanding. I discuss these realizations further in the next chapter.

Chapter IV

Discussion

This study examined my own teaching in order to understand how I considered ways to support students' learning of fractions and how my own, growing understanding of fractions, changed through this work. It examined how a teacher's questions could be posed to help guide student learning with tasks that are based upon student's current conceptions. To this end, I explored student thinking using the conceptual lens of participatory and anticipatory stages of learning, and examined my approach, questioning, and my thought processes throughout the French Fry activity (Tzur & Hunt, 2013).

A major focus of this study is the importance of the teacher's questioning when attempting to promote students' construction of intended mathematics (Hunt & Tzur, 2017). Analysis of the questioning, as shown in Appendix A, indicated that at times I questioned my students without leading them too much. At other times, however, my questioning was much more direct, for example, when I introduced the repeat strategy instead of letting them struggle through discovering it on their own. As the chart in Appendix A shows, during days 1 and 2 I used more direct teaching questions, whereas in day 3 I did not use such questions. It was evident that throughout teaching this activity, my questioning for student thinking and learning about fractions improved. It is important to keep in mind that questioning does not imply directly instructing students in a specific manner (Hunt and Tzur 2017). This is a point with which I struggled and I conjecture it would be relevant for many teachers. Through this study, I was able to examine and reflect upon how I ask questions of students not just as a tool to assess the extent of their mathematical learning but also to foster their advance. As my data and analysis showed, at times I fell into a mode of attempting to over direct student learning. In particular, I seemed to have asked direct, leading questions, which attempted to "show" student the intended thinking instead of allowing them to think for themselves.

These direct-teaching attempts reflected my teacher desire to help students learn, which turned into pushing them to learn it in certain ways. This seems a commonplace issue, as such attempts overlook the diverse ways in which students learn and how they see the world. This study showed that a teacher could learn to recognize this problem and thus begin changing his (or her) instructional practice. Moreover, students cannot all be forced to "see the mathematics" in exactly the same way that a teacher does. Instead, a teacher needs to help students construct the intended mathematics for themselves. Accordingly, it is important to not interact with all students in the same way, but rather to differentiate questions so they are tailored to promoting students' progress at different entry points (Hunt & Tzur, 2017). The approach must meet what each individual student knows and be designed to help them learn the way that is best for them. My study showed how such an approach could be accomplished gradually by a teacher.

Implications of the Study for Teaching

This study has two important implications for other teachers. First and foremost, my case showed the crucial role the teacher's own understanding of the mathematics he is teaching plays in students' learning. In order to ensure student understanding, teachers must constantly deepen their own conceptual understandings. Quite often, teachers teach a topic that they are not too familiar with themselves. My case clearly illustrates this point, as I began the work with an understanding of fractions as part-of-whole, and later changed it to multiplicative relationship, in turn, the lack of proper and deep conceptual understanding could preclude the teacher from knowing when students have truly understood the concept themselves. Through this understanding I was able to select goals and tasks that prompted student learning and encouraged them to construct the concepts for themselves.

Additionally, inadequate teacher's conceptual understanding is likely to cause students to develop procedural knowledge and/or inappropriate conceptual knowledge. My study highlights the growing awareness of a teacher to the distinction between students' ability to execute a procedure and conceptual understanding of why or how it actually works. When this happens, they are not likely to readily apply ("transfer") it to other situations and/or concepts. These foci on procedures seem all too common in today's classrooms.

In doing this research, I constantly had to address my own thinking and understanding about unit fractions. Before learning about and using the French Fry activity to teach unit fractions, I did not have as clear an understanding of fractions as I thought I did. I have really been able to expand my own thinking and understanding about fractions. This happened through the literature that I read to build upon my own understanding, and through closely watching students work on the tasks as they constructed this thinking for themselves. In this sense, my case study highlights both a plausible source of students' inadequate learning and a way forward through turning the teacher's attention onto his (or her) mathematical understanding.

The second implication of my study is the need for a teacher to take the time to reflect on his teaching practices, including one's perspective on student thinking and understanding. This is how teachers can improve their teaching. My case study illustrated this point in relation to using the learning stages—participatory and anticipatory (Tzur & Simon, 2004). This was shown by my reflections after each lesson. When I, as the teacher, took the time to deeply reflect on student understandings, I could better prepare myself for how to begin the next lesson. Through reflecting on and understanding where students are conceptually, and helping them create an understanding for themselves at the participatory stage of constructing a scheme of unit fractions as multiplicative relations, I seemed to open a way for them to advance to the anticipatory stage of that scheme.

This second implication is directly linked to the teacher's understanding of how to question students in a way that promotes their understanding, and thus is also related to the teacher's own mathematical understanding. In using the French Fry activity, I had to both think about the mathematics myself and about how I was questioning students: Was I doing it in a way that was helping them build their own understanding, or was I doing the thinking for them? As my analysis showed, if I was not careful in my questioning, I was not helping students reach that anticipatory stage of learning. At times, I was giving them too much information, and too direct, which seemed to contribute to them persisting at the participatory stage. In reflecting on my own teaching, I have been able to improve how I question students, not only in this activity but in other areas of my teaching.

Limitations and Implications for Further Research

There were two main limitations to my study. The first was that I did not collect student data. For this thesis, I wanted to strictly focus on teacher thinking and understanding. Therefore, this research became a lens through which to look at my teaching from an outside perspective. Quantitative and qualitative data (of students' work) could have shown to what extent they learned during the activity, which I feel would be the next step in continuing this research. It would show how much the activity and the teaching moves that I did affected student learning. Secondly, the time constraints were an issue due to my teaching schedule. Given more time for this research activity and looking at how the students could continue to develop a deeper understanding of fractions by applying their understanding to non-unit fractions could show how much they have learned. I think spending as much time as necessary to ensure student learning would have been greatly beneficial, but is unfortunately not always an option.

Concluding Remarks

As a teacher and a lifelong learner, I feel that this research helped me to reevaluate how I have been teaching and how I approach student learning—in general and particularly when teaching unit fractions. I think that teachers often get stuck in ways they have been doing things and sometimes need to step back and really take a look at things from an outsider's perspective. In doing this research, I had to look at my teaching through a different lens and think deeply about how I can improve my approach to teaching. Overall, I am a better teacher because of it, and my students are able to learn and understand fractions more than they did before.

Appendix A

Type of questions:

1. Assess
2. Cause and Effect Relationship
3. Comparison

D is added to a question type if the question is directing student learning.

Examples Questions on Each day of French Fry Activity	Question Type
Day One Questions "How do we know that this is the share of our French fry when we share it between our self and someone else?"	2
"Ok, how do we know that it's half….Because it's split evenly? Yes, that's a good place to start, but this is something I really want us to think about during your next task."	2D
"So, is the next piece going to be bigger or smaller?"	2, 3
"Can you figure out how much of the French fry that piece is?"	1, 2
"How can you show that?"	1
".... Smaller. Keeping that in mind, see if you can figure out how big the next piece is. You already know that it must be smaller than this (holds up a piece that is the ½)."	2D, 3
"Ok, you're a little short, so do you need to make it a little bit longer, or a little bit shorter. "	2D, 3
Day Two Questions "Now tell me about this piece, tell me how you know that it's your share among 3 people. …not because your sharing it with three people, but because this piece fits 3 times. Repeating this 3 times, makes the whole French fry."	1D, 2D
"Compared to the one that you just found [sharing among 3], is the next one [person's share] going to be bigger? Or smaller [than one person's share for 3]?	3 ☺
"Yes, but what are you going to have to do?"	
"What did you do before with the piece that was your share among three people?"	1, 3
"Yes, you repeated, how many times?"	1, 3
"Yes, and how many times do you have to be able to repeat this one?"	1D

“Now, I would like you to figure out how big your share is when sharing among 4 people. Compared to the one that you just found [sharing among 3], is the next one [person’s share] going to be bigger? Or smaller [than one person’s share for 3]?”	1, 3 1, 2, and 3
Day Three Questions “Is it bigger or smaller than … ?” “How much bigger or smaller?” “Ok, so what piece is this?” “Your share among 9 people, even though you were trying for 8?” “Good, so what do you have to do now?” “So how much smaller do you need to make it? Why?” “So now, you have to think, do I need to make it a little bit smaller or a lot smaller?” “Ok, so you got your one for three times? Good, in how many trials? Just one? Nice job, now glue that one down on the front side and try to find your share among 4 people on the back. So, does this one need to be bigger or smaller? Yes smaller, why? Very good. Keep going. “	 3 3 1 1, 3 2, 3 1, 2 2, 3 1, 2, and 3

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