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TITLE

A TEACHER'S CONCEPTION OF DEFINITION AND USE OF EXAMPLES WHEN DOING AND TEACHING MATHEMATICS

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Abstract

To contribute to an understanding of the nature of teachers' mathematical knowledge and its role in teaching, the case study reported in this article investigated a teacher's conception of a metamathematical concept, definition, and her use of examples in her doing of mathematics and her teaching of mathematics. Using an enactivist perspective on mathematical knowledge, the authors give an account of the case of Lily, a prospective, then beginning teacher who conceived of mathematical definition as an object with particular form and function and engaged in purposeful, specialized use of examples when doing and teaching mathematics. Lily's case illustrates how a teacher's (a) interpretation of examples (as exemplifications or as single instances) and (b) conception of the form and function of definitions can influence her doing of mathematics and teaching of mathematics. An implication of this case is that teacher preparation should foster teachers' abilities to make purposeful use of examples to provide students with rich opportunities to engage in mathematical processes such as defining.

Teachers' mathematical knowledge extends beyond content knowledge (Conference Board of the Mathematical Sciences, 2001, 2012); it includes what teachers know and understand about mathematics and how they establish and use mathematical results. As such, it is insufficient to characterize teachers' mathematical knowledge by examining how they engage with a single mathematical topic. Employing an enactivist perspective on mathematical knowledge (Varela, Thompson, & Rosch, 1991), we investigated a teacher's conception of definition and use of examples when doing and teaching mathematics. The purpose of this article is to argue that an important aspect of teachers' mathematical knowledge includes their conceptions of the form and function of the metamathematical concept (e.g., Pimm, 1993, Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008), *mathematical definition*.

Researchers have theorized about the nature and features of mathematical definitions (van Dormolen & Zaslavsky, 2003), investigated how individuals conceive of mathematical definitions (Edwards, 1997; Edwards & Ward, 2008; Levenson, 2012; Tall & Vinner, 1981; Zaslavksy & Shir, 2005), and addressed roles definitions might play in teaching and learning mathematics (Vinner, 1991). Van Dormolen and Zaslavsky (2003) addressed aspects of mathematical definitions, including minimality and axiomatization, which are valued by mathematicians. If an individual were to reason in ways that were purely deductive and axiomatic, then that individual's concept image (Tall & Vinner, 1981) would play a limited role in the construction of mathematical definitions (Vinner, 1991). However, individuals working with definitions do not necessarily conceive of definitions in the same way as mathematicians might, even if they have completed quite

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extensive amounts of mathematics coursework (Edwards & Ward, 2008). Studying three experienced junior high school teachers, Levenson (2012) reported that none of the teachers claimed that $a^0 = 1$ was a definition. Levenson hypothesized that teachers' lack of understanding of definitions could negatively impact how their students viewed and used definitions in mathematics. Contributing to this corpus of research, we argue that a teacher's conception of mathematical definition might influence the way in which that teacher engages in—and might engage students in—the process of mathematical defining.

Researchers have reported on the scope of activity involved in the process of mathematical defining, some requiring the construction of a definition (Leikin & Winicki-Landman, 2001) and others involving a wider range of activity, including "formulating, negotiating, and revising a definition" (Zandieh & Rasmussen, 2010, p. 59). Leikin and Winicki-Landman (2001) characterized mathematical defining as "the process of choosing an approach, construction, and formulation of a definition" (p. 64). Taking a broader view, Zandieh and Rasmussen (2010) suggested mathematical defining not be limited to the construction of definitions but also include activities such as proving statements, making conjectures, or creating examples when done for the purpose of formulating, negotiating, or revising definitions. By mathematical defining, we mean a process involving activity related to the construction of mathematical definitions. Further, we distinguish the constructive process of *mathematical defining* from *working from a given mathematical definition* (cf. *acting on a representation* in Zbiek, Heid, & Blume, 2012). Working from a mathematical definition encompasses a variety of activities, including (but not limited to) refining definitions, determining whether or not a definition is a "good" definition, deciding whether definitions are equivalent, or providing examples (or nonexamples) of entities meeting (or not meeting) the requirements of a definition.

It is known that some university students (Zandieh & Rasmussen, 2010) and some secondary students (Borasi, 1991; Zaslavsky & Shir, 2005) use examples when engaging in mathematical defining (Borasi, 1991; Zandieh & Rasmussen, 2010) and when working from given mathematical definitions (Zaslavsky & Shir, 2005). Zandieh and Rasmussen found that the university students in their study used examples serving as "extreme cases" to test whether a definition was viable (e.g., determining whether a definition of *triangle* should allow a three-sided figure with collinear vertices to be a triangle). Zaslavsky and Shir (2005) found that 12th-grade students used examples to make sense of given definitions for unfamiliar concepts (e.g., local maximum). The secondary student participants in Borasi's (1991) research used examples to make sense of concepts when they were not sure of a definition (e.g., drawing an example of a polygon when unsure of a definition of a polygon). Results of these studies suggest that examples can play important roles in students' uses of mathematical definitions and in their mathematical defining.

Researchers (Edwards & Ward, 2004; Pimm, 1993; Zaslavsky & Shir, 2005; Zazkis & Leikin, 2008) distinguished the metamathematical concept of mathematical definition from definitions of particular mathematical concepts. Edwards and Ward (2004) argued "the special nature of mathematical definitions should be treated as a concept in its own right, one that should be understood at some level by all college mathematics students" (p. 419). We concur with Edwards and Ward, and we argue that knowing more about how teachers conceive of the "metamathematical" concept of mathematical definition could offer insight into how teachers use examples when engaging in mathematical defining and working from definitions when doing and teaching mathematics.

We investigated the following research questions: How does a prospective (then beginning) teacher (a) use examples when engaging in mathematical defining and working from mathematical definitions when doing[1] mathematics, (b) use examples when engaging students in mathematical defining and working from mathematical definitions, and (c) conceive of the form and function of mathematical definitions? The first and second questions refer to specific mathematical concepts being defined, whereas the third question refers to the metamathematical concept of mathematical definition. In this article we report the case of Lily,[2] a prospective, then beginning teacher whose conception of mathematical definition as an object with particular form and function supported her purposeful, specialized use of examples in her doing of mathematics and her teaching of mathematics.

Form and Function of Mathematical Definitions

Poincaré (1914) posited that "good" mathematical definitions depend on the audience for whom a definition is intended.

> What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them; it is that

which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils. (p. 117)

Although it is important for a definition to be understood by students, indicating that a definition is a "good" mathematical definition by its function alone (e.g., being understandable) is insufficient. Appealing to the use of mathematical definition in textbooks and classrooms, Vinner (1991) posited underlying assumptions about mathematical definitions, including an assumption that mathematical definitions should be minimal.[3] In contrast to Poincaré's (1914) emphasis on the role of definitions as vehicles for communicating concepts, the assumption presented by Vinner (1991) suggests that what is considered a "good" definition is also related to attributes of the definitional statement. These contrasting assumptions foreground two complementary aspects of definitions: form and function.

Form

The form of a mathematical definition refers to attributes related to the presentation of that definition. Whereas the assumptions presented by Vinner (1991) emphasized minimality and elegance as important attributes of definitions, others have drawn attention to necessary attributes. Zaslavsky and Shir (2005) indicated two requirements of a mathematical definition: "a mathematical definition must be noncontradicting (i.e., all conditions of a definition should coexist) and unambiguous (i.e., its meaning should be uniquely interpreted)" (p. 319). Importantly, the form of a definition need not necessarily be minimal to meet these requirements (van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005). Minimal definitions may not be the most effective pedagogically, particularly when introducing new mathematical concepts (van Dormolen & Zaslavsky, 2003; Mariotti & Fischbein, 1997). In the Mariotti and Fischbein (1997) study investigating middle school students' construction of a mathematical definition to form the concept of a polygonal net, it was the researchers' goal for students to construct valid—rather than minimal—definitions. Practicing teachers who focused on students' understanding of definitions did not consider minimality to be an essential characteristic of a definition (Leikin & Winicki-Landman, 2000). Further, in a study investigating four high-achieving, twelfth-grade students' conceptions of definitions, Zaslavsky and Shir (2005) found that a student who initially thought minimality was a necessary criterion for a definition later accepted nonminimal statements as definitions. Taken together, results of these studies suggest that the use of a particular form of a definition can depend on the intended function of that definition.

Zazkis and Leikin (2008) used the term *barely-not-minimal* (p. 137) to refer to definitions that use appropriate terminology and contain necessary and sufficient conditions without being minimal. Zazkis and Leikin offered as an example a definition of *square* as *a rhombus with four right angles,* which would be barely-not-minimal because knowing that a rhombus has a single right angle would imply that all four angles are right angles. Although prospective secondary mathematics teachers participating in Zazkis and Leikin's study recognized that defining a square as a rhombus with four right angles would not be a minimal definition, they indicated a preference for the barely-not-minimal definition over a minimal definition (e.g., rhombus with one right angle) for classroom instruction because they thought it would make more sense to secondary students. Although minimal definitions may be elegant, other forms of mathematical definitions may serve different functions.

Function

Even when students use definitions with the types of forms valued by mathematicians, they may not be operating with a perspective on definition that is consistent with a mathematician's perspective. Edwards and Ward (2004) reported that university students' use of definitions suggested that they conceived of mathematical definitions as being *extracted* from situations in which a term or concept appeared, meaning that the function of a definition would be to "report usage" (p. 412) of the term or concept. This contrasts with a mathematician's conception of mathematical definitions as *stipulating* the kinds of situations in which the term or concept is applicable, meaning the function of a definition would be to "create usage" (Edwards & Ward, 2004, p. 412) for the term or concept. For example, given a definition of a circle as the locus of all points at a fixed distance from a given point, a student conceiving of the function of the definition from a definition-as-extracted perspective might envision other times he has encountered circles and drawing on those instances to determine properties of circles. In contrast, a student conceiving of the function of the given definition from a definition-as-stipulated perspective would attempt to envision the collection of objects stipulated by the definition as independent of ideas generated during times he may have encountered circles. The categories of stipulated and extracted definitions indicate two contrasting functions individuals may conceive definitions to have, regardless of the form of the definition.

Borasi (1991) proposed two functions for mathematical definitions: (a) to enable individuals to distinguish between "instances" and "noninstances" of a mathematical concept and (b) to characterize the "mathematical essence" of a concept (p. 18). Given a potential instance of a mathematical concept, a student could use a definition to determine whether the instance meets all of the requirements of the definition. Given an ellipse, a student could use the requirement *locus of all points a fixed distance from a given point* to argue that an ellipse is not a circle. Such use of a definition can be contrasted with that of a student who might claim that an ellipse is not a circle because it does not appear to be as round as a circle is. Although both students would have correctly indicated that an ellipse is not an instance of a circle, only the first student used the definition to make distinctions between instances and noninstances of a circle. If all properties of a concept could be concluded from a definition, then the definition would characterize the "mathematical essence" of a concept (Borasi, 1991). Although it is desirable for definitions to be both minimal and elegant (Vinner, 1991), barely-not-minimal definitions could be suitable for characterizing the "mathematical essence" of a concept.

Examples and Mathematical Definition

Freudenthal (1973) indicated that students should have the opportunity to invent mathematical definitions during classroom instruction. One way teachers can facilitate students' creation of mathematical definitions is by asking students to compare and contrast elements of a set containing examples of a mathematical object with elements of a set containing nonexamples of the mathematical object (Sowder, 1980). For instance, a teacher might ask students to construct a definition of *polygon* by providing examples and nonexamples of polygons. Alternatively, a teacher might use a specific example from which a student could deduce more general properties (Freudenthal, 1973), such as using a diagram of a nonisosceles oblique triangle to illustrate a theorem related to the medians of triangles (Watson & Mason, 2005). Further, a single example may be interpreted in many different ways. To illustrate, although both instances include the use of examples, the way in which a teacher might interpret the purpose of the examples can differ. Although the examples and nonexamples provide specific instances of polygons or nonpolygons, the nonisosceles oblique triangle provides a representation of something more general that could apply to triangles beyond this specific instance.

Theoretical Perspective: Enactivism and Exemplification

When we refer to mathematical knowledge, we use an enactivist perspective on knowledge (Davis, 1995; Sumara & Davis, 1997; Varela, Thompson & Rosch, 1991). From an enactivist perspective, an individual's knowledge and action are inseparable (Varela, Thompson, & Rosch, 1991). Consequently, we assume that a teacher's mathematical knowledge is intertwined with her work of doing mathematics, her teaching students mathematics, and any other related experience. When investigating teachers' mathematical knowledge, we focus on their actions, including those that occur when working from mathematical products (e.g., interpreting a given mathematical definition) and those that constitute mathematical processes involving intentional activity related to the creation of such products (e.g., constructing a mathematical definition) (Zbiek, Heid, & Blume, 2012).

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From our perspective, an individual is engaging in mathematical defining when she is working to construct a mathematical definition, even if her activity does not result in an explicitly stated mathematical definition. The process of mathematical defining includes activities such as distinguishing between examples and nonexamples of mathematical concepts and identifying necessary and sufficient conditions for membership in a class of mathematical objects. For instance, suppose an individual were to make distinctions between two collections of objects to determine a mathematical definition that could include all objects in one collection but not the other without successfully determining a mathematical definition. We would indicate that individual to be engaging in the process of mathematical defining, because that individual was engaging in activity that could lead to the construction of a mathematical definition. We refer to the individual's activity as *engaging in defining* rather than *using definition* to communicate the inseparability of knowledge and action. In doing so, we intend to communicate that we do not assume that the process of mathematical defining is something an individual would call on to *use* in a situation, rather an individual would enact the process of defining by *engaging in* actions in a given situation.

We use *example* in a broad sense "to stand for anything from which a learner might generalize" (Watson & Mason, 2005, p. 3). Acknowledging that individuals can interpret examples in myriad ways, we posit two contrasting interpretations. The first involves interpreting an example as a single instance of a more general idea (e.g., given a set of examples of parallelograms, a student could interpret one of those examples as a single instance of a parallelogram). The second involves interpreting an example as a

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specific instance used to illustrate a more general idea (e.g., a student could use a specific diagram to illustrate a general class of objects—parallelograms). A key distinction between these interpretations is the relationship between the particular and the general. The first interpretation requires one to envision a particular instance of something more general, whereas the second interpretation requires one to "see the general through the particular" (Watson & Mason, 2005, p. 4).

Watson and Mason (2005) refer to the second interpretation as *exemplification*, meaning that something specific is being used to illustrate a general class. We concur with Watson and Mason (2005) that exemplification is individual and situational, meaning:

> Exemplification is dependent on the knowledge, multiplicity of experience, and predisposition of the learning, but it is also framed by the wording of the prompt, who is making it, and under what circumstances; different learners may respond with different examples in the same learning environment, and the same learner may respond differently in different situations. (p. 44)

When we say that an individual conceives of an example as an exemplification, the individual's actions involve illustrating the general through the particular.

Method

Key aspects of enactivist methodology include taking "multiple views of a wide range of data" (Coles & Brown, 2001, p. 27) and "working from and with multiple perspectives" (Reid, 1996, p. 207). Consistent with enactivist methodology, to investigate teachers' mathematical knowledge, we examined their actions across multiple settings encompassing a range of experiences, including doing mathematics, teaching students mathematics, and reflecting on teaching. Although we used research methods designed to investigate particular experiences, we did not assume that data related to a particular experience would be limited to a particular setting. For instance, we used task-based interviews to investigate a teacher's doing of mathematics but did not assume that data relating to the doing of mathematics would only be found within the task-based interviews. Further, we acknowledge that a teacher might interact differently with different interviewers, and our design provided the opportunity for each teacher to interact with multiple interviewers. Our process of data analysis allowed for the vetting of multiple perspectives from different members of the research team. When analyzing data, we allowed for the possibility that a teacher might engage in a mathematical process (e.g., defining) in a coherent way that could differ from how we might engage in such a process, thereby affording us the opportunity to develop our own knowledge of the range of ways in which individuals engage in mathematical processes.

The case (Stake, 2005) detailed in this article is part of a larger, multiyear study that examined the mathematical knowledge of prospective, then beginning teachers. Essential to case study design is the choice of that which is to be studied (e.g., defining the case). We report the case of a prospective, then beginning teacher whose conception of mathematical definition as an object with particular form and function supported her purposeful, specialized use of examples when doing and teaching mathematics. Through this instrumental case (Stake, 2005), we intend to demonstrate that knowledge of metamathematical concepts, in particular of mathematical definition, is an important aspect of teachers' mathematical knowledge.

To investigate participants' doing of mathematics, we used individual task-based interviews (Goldin, 2000). By task, we mean a problem purposefully designed for a particular audience (e.g., Sierpinska, 2004). Consistent with task-based interview methods, interviewers presented tasks to participants, provided participants with ample time to work on the tasks, and used nonleading follow-up questions such as: "Could you tell me what you mean by that?" or "Why is that the case?" Interviewers used questions providing minimal prompting if participants did not respond to initial questions. Further, interviewers used exploratory questions asking participants to explain their thinking.

Data analysis incorporated three activities: description, analysis, and interpretation (Wolcott, 1994). In description, the researcher stays as close to the data as possible, focusing on making observations from the data. In analysis, the researcher identifies "essential features" and systematically describes relationships among those features. In interpretation, the researcher addresses issues of meaning that can be made from the data. We use the broader term *data analysis* to include all three activities.

Subject

The subject of our study, Lily, was a prospective teacher at the beginning of the study and a beginning teacher by the end of the study. During her 8-week early field placement and during her student teaching, Lily taught seventh-grade mathematics with the same mentor teacher in a small-town/rural school district. As a first-year teacher, Lily

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taught Honors Geometry and Algebra 1 in a block schedule at a suburban high school in a large school district.

Data Collection

Data collection took place during Lily's last 3 undergraduate semesters and continued into her first year of full-time teaching. During those last 3 semesters we conducted five task-based interviews with Lily, one classroom observation of a lesson that she taught during an early field placement, and five consecutive observations of lessons that she taught during her student teaching experience. During Lily's first year of full-time teaching, we conducted five consecutive classroom observations of lessons that she taught in an Honors Geometry course. A member of the research team involved in the larger study conducted preinterviews and postinterviews with each classroom observation.

Task-based interviews were video recorded and audio recorded, and annotated, verbatim transcripts of the task-based interviews were created. Preinterviews and postinterviews were audio recorded, and annotated, verbatim transcripts of the researcher–teacher conversations were created. Each observed lesson was audio recorded, and two observers took field notes, one capturing as much of Lily's and the students' utterances as possible and the other capturing observable aspects of the classroom activity (e.g., how students were grouped, materials that were distributed to students, information displayed or written by the teacher or students, the teacher's gestures, and the like). We used the audio recordings and field notes to create annotated, verbatim transcripts of the classroom lessons.

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Individual Task-Based Interviews

The five task-based interviews incorporated problems requiring participants to generalize patterns, find function rules, determine surface area and volume of a sequence of figures, sort three-dimensional figures, and evaluate definitions. The tasks were designed to provide opportunities for participants to engage in some or all of four specific mathematical processes (defining, representing, generalizing, and justifying) or in some collection of actions on mathematical products associated with those processes (e.g., definitions or generalizations). Data reported for this study involve only defining/definition, and come primarily from the fifth task-based interview. Table 1 describes the goals and mathematical content for the tasks in that interview. Goals marked with * are specifically addressed in this article, and Table 2 gives the tasks associated with those goals.

Classroom Observations and Accompanying Interviews

We observed Lily teaching lessons in three different contexts. During her early field placement we observed one seventh-grade mathematics lesson in which she was teaching the use of the addition and subtraction properties of equality to solve single-step linear equations. During her student-teaching semester we observed five consecutive seventh-grade mathematics lessons that were part of a unit designed to prepare students for the upcoming state assessment. Lessons addressed the topics of probability, geometric concepts, the use of patterns, and the coordinate plane. During her first year of teaching we observed five consecutive high school geometry lessons from a unit on polygons and quadrilaterals. Lessons addressed polygons and convexity and concavity of polygons, sums of interior and exterior angles of a polygon, parallelograms and properties of parallelograms, using properties of parallelograms, and rectangles and properties of rectangles.

The preobservation and postobservation interviews included, but were not limited to questions of three types—reflection, anticipation, and process-based. The first preobservation interview did not include reflection questions, because we had not yet observed any lessons. Observation interviews occurring between observed lessons served as a preinterview for the upcoming lesson and a postinterview for the lesson just observed. Reflection questions focused on Lily's clarification and assessment of events during the observed lesson. Anticipation questions focused on her planning for upcoming lessons. Process-based questions focused on her and her students' use of the four mathematical processes that were part of the larger study (defining, generalizing, justifying, and representing) and on their actions on the products of those processes (definitions, generalizations, justifications, and representations). These types of questions allowed interviewers to gain insight into Lily's perspective on events occurring in an observed class, her intentions for the next class, as well as Lily's attention to her students' mathematics.

Data Analysis

Our data analysis consisted of three phases: description, analysis, and interpretation. Primary data sources included Lily's verbal, gestural, and written communication during task-based interviews, preobservation interviews, and postobservation interviews, and her actions and communication with students during the classroom observations. Secondary data sources included lesson plans, classroom handouts, and other materials used in Lily's classroom instruction. During the first phase (coding/description), we coded transcripts of task-based interviews, preobservation and postobservation interviews, and classroom observations to identify episodes in which Lily's work during a task-based interview (her doing of mathematics) or her teaching students mathematics involved the process of mathematical defining or work with a mathematical definition. To distinguish between types of episodes, we determined whether an attempt was being made to construct a definition. If so, then we identified the episode as involving the process of mathematical defining. Otherwise, we identified the episode as work with mathematical definition. Individually, members of the research team coded data sources and wrote descriptions of each episode. During group research meetings, we vetted individuals' coding and identified episodes to analyze further. During the first phase of analysis we observed that Lily, when doing and teaching mathematics, frequently used examples in her mathematical defining and her work with definitions.

In the second phase (analysis), we focused on episodes involving Lily's use of examples. A goal of this phase was to develop explanations articulating a researcher's perspective of Lily's use of examples when doing and teaching mathematics. In doing so, we are not claiming that Lily would be aware of how she might be using examples (e.g., as examples or exemplifications). We analyzed episodes that occurred during task-based interviews separately from episodes that occurred during preobservation and postobservation interviews and classroom observations. We distinguished between

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instances in which—according to our analysis—Lily interpreted an example to be a specific instance of something more general (an example) and those in which she interpreted an example to represent a general class of objects (an exemplification). Then, we looked across episodes from both her doing of mathematics and her teaching of mathematics to compare and contrast Lily's use of examples in different settings. During the second phase of analysis we conjectured that Lily's use of examples could provide insight into how she conceived of the metamathematical concept, mathematical definition.

In the third phase (interpretation), we focused on episodes involving Lily's use of examples and episodes in which Lily spoke about mathematical definition. We identified three categories indicating aspects of Lily's conception of mathematical definition: the nature of a mathematical definition, the function of a mathematical definition, and what makes a "good" mathematical definition. We developed explanations of Lily's conception within each category. During research meetings, our team vetted emerging explanations to test their viability. We determined an explanation of Lily's conception of mathematical definition to be viable only after weighing alternative explanations and searching for potential disconfirming evidence.

Results

Constructing a Definition: A Duality of Interpretation of Examples

Lily's attempt to construct a definition during a task-based interview. During a task-based interview, Lily was given two sets of polyhedra, Alpha and Gamma (see Figure 1). Lily was asked to construct a definition for polyhedra in set Alpha such that

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the definition would exclude all polyhedra in set Gamma. The polyhedra in both sets Alpha and Gamma were prismatoids, meaning that all the vertices lie in one of two parallel planes. The polyhedra in set Gamma were also prisms, because the bases were polygonal and the faces were parallelograms.

Focusing on physical aspects of the polyhedra, Lily attempted to determine one "distinction" that would allow her to distinguish the polyhedra in set Alpha from the polyhedra in set Gamma. She began by considering whether polyhedra were convex or nonconvex. To make a determination, she used the locations of auxiliary lines passing through pairs of consecutive vertices, which she referred to as "points" on the polyhedra. Lily was trying to determine whether the line between the vertices would lie on an edge, which she referred to as a "side or face":

Lily: If it wasn't for this one [Polyhedron 22 in set Alpha, see Figure 1] I would have said the convex, nonconvex…. With all the other ones, if you were to draw a line between two points it would lie on the side or face, except for this one [Polyhedron 22]. That was my first thought but I see this one in here [Polyhedron 16 in set Gamma, see Figure 1] would be like this [Polyhedron 22].

As she said "a line between two points," Lily held Polyhedron 9 (see Figure 1), placed her thumb and forefinger on two vertices, and traced her forefinger along the edge connecting those vertices. Because Polyhedron 22 (set Alpha) and Polyhedron 16 (set Gamma) both have vertices for which a line drawn between them would not lie on an

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edge or face of the polyhedron, she discarded this "distinction." Next, Lily focused on the shape of the faces of the polyhedra:

Lily: I'm, I—have trouble distinguishing anything between these, [sets Alpha and Gamma] finding a difference—um, besides [pause 10 sec] well, can't say—something about triangles because this one [Polyhedron 26 in set Gamma, see Figure 1] doesn't have—a triangle—and twenty-five [Polyhedron 25 in set Gamma, see Figure 1] does. So that doesn't work.

By attending to physical aspects—considering each aspect individually—Lily was not able to construct a definition for polyhedra in set Alpha such that the definition would exclude all of those polyhedra in set Gamma. Importantly, when a particular "distinction" proved insufficient, it precluded that distinction from being part of the definition, even in conjunction with another distinction.

To continue to explore Lily's process of constructing a definition, the interviewer asked her what she was "looking for" as she was attempting to come up with a definition:

Lily: I'm looking for some kind of distinction that would let me—um classify these [Polyhedra in set Alpha, see Figure 1] and not include those [Polyhedra in set Gamma, see Figure 1]. But I don't see—a distinction. I don't see any, like, distinctions for all this set but not that set. So—I don't know.

For Lily, a "distinction" was a necessary and sufficient condition that would allow her to "classify" polyhedra in set Alpha and exclude polyhedra in set Gamma. For example, had Polyhedron 22 not been in set Alpha, convexity would have been a useful "distinction." However, Lily was unable to determine a "distinction" that would allow her to "classify" the polyhedra in set Alpha and exclude those polyhedra in set Gamma and therefore, she was not able to develop a definition. When engaging the process of constructing a definition, Lily conceived of each polyhedron as a specific instance of some more general class that she was unable to define.

Lily's implementation of a task requiring her students to construct a definition. We report an episode of a geometry lesson that took place during Lily's first year of teaching. This episode is typical of other episodes that we observed. During this episode, Lily asked students to look for "a difference between" examples and nonexamples (see Figure 2) to define *convex polygon*. In response to Lily's question, a student offered one difference between the sets: "It goes in, the angles." Lily affirmed the student's response, then told the students that all the nonexamples were "caved in," whereas none of the examples were "caved in" (see Figure 2). Next, she sketched auxiliary lines between adjacent vertices of a convex polygon (see Figure 3), telling students that that none of the auxiliary lines drawn on convex polygons were in the interior of the polygon:

Lily: Okay. None of these lines—no point on any of these lines is inside this polygon [see Figure 3]. So, if we draw a line on the sides of each polygon, none of the lines are seen to be in the interior of the polygon. Okay? If you draw the line here—[Lily draws and points to lines in Figure 4].

After drawing the auxillary lines for the concave polygon (see Figure 4), Lily gave students a definition for *convex polygon*:

Lily: Okay. This is concave, and this is convex, and for the definition of convex, we can say it's convex if the line containing—you'll want to write this down—write *convex*: Convex is the line containing each side, the line containing each side is drawn and it contains no points in the interior.

The definition that Lily gave students during the lesson was the same as the definition that she had written in her teacher notes (see Figure 5). Clearly, the definition that Lily planned to give to the students is not minimal and thus is not a "good" definition according to assumptions posited by Vinner (1991). Yet the definition could be used to distinguish between convex and nonconvex polygons.

When Lily asked her students to use examples and nonexamples to construct a definition for a convex polygon, she asked students to find "a difference between" elements in the different sets. She treated each of the elements in the sets as specific instances of some more general class that that she could define. Unlike her work during the task-based interview, in her classroom instruction, a noticeable "difference" was sufficient for students to use to define *convex polygon*.

Although Lily conceived of the physical objects used during the task-based interview and the diagrams used during her classroom instruction as specific instances of some more general class of objects, this was not always the case. On Lily's teacher notes (see Figure 5), she included what she labeled *Examples* below her definitions for convex and concave polygons. Unlike the *Examples of Convex Polygons* and *Non Examples* included at the top of Figure 5, these examples served as more than specific instances, as indicated by Lily's written comments next to each ("no points in the interior" and "some of the lines pass through the int [interior]").

Working From a Definition: Interpreting Specific Instances as Representing a More General Class of Objects

Lily's work from a definition during a task-based interview. Later during the same interview in which Lily was attempting to distinguish between polyhedra in sets Alpha and Gamma (see Figure 1), Lily was given the following definition of prismatoid: "A *prismatoid* is a polyhedron whose vertices all lie in one of two parallel planes." A prismatoid was a mathematical object unfamiliar to Lily, and after being given the definition, she expressed her uncertainty, saying that she had "no idea what that means on first read." Next, the interviewer provided Lily with a set of four statements (see Table 2), and asked Lily which she thought would be a good definition for set Alpha[4] (see Figure 1).

Lily: Oh, let me see. Well, I need to think about what a prismatoid is before I could say that. So I'm going to read the definition again.

Interviewer: Okay and tell me what you're thinking.

Lily: Whose vertices all lie in one of the two parallel planes. I don't, um, whose vertices all lie in one, of the two parallel planes. Oh, I can see that, okay. So these are all—

Interviewer: Tell me about—

Lily: Here, the first four vertices they lie on the plane, and then the other four lie in this plane.

Although she could read the words of the definition, Lily needed to "think about what a prismatoid is," before she could go further. Her work suggests that she conceived of the function of the given definition from a definition-as-extracted perspective, meaning that the definition would report what a prismatoid was. Once she could "see" what usage the definition was reporting, she could identify polyhedra as prismatoids.

Sweeping her hand across the top and bottom faces of Polyhedron 26, Lily developed what we call a *hand test* that she used to confirm whether polyhedra were prismatoids. Lily's hand test involved placing her hands on two faces of a polyhedron, and if each of the vertices was touching one of her hands, then the polyhedron was a prismatoid. Using her hand test, Lily correctly determined that Polyhedra 22, 7, 25, and 13 were prismatoids and incorrectly determined that Polyhedron 16 was not a prismatoid. When working with Polyhedron 16, Lily placed her hands on adjacent faces, forming a "V" shape with her hands. Although she could envision different spatial orientations for the same prismatoid, as evidenced when she described Polyhedron 13 as being a prismatoid "either way," she did not consider any other spatial orientations for Polyhedron 16. Because Lily used her hand test to determine whether objects were or were not prismatoids, the hand test served as a proxy for the given definition of prismatoid.

Lily went back to the definition only once after she developed the hand test. When working with Polyhedron 7, Lily questioned whether the vertices needed to lie in one or two planes.

Lily: Because these two planes, all the vertices lie in those two planes [said while using her hand test for Polyhedron 7, see Figure 6.]. Two, yes, in one of two, oh wait. Oh hold on. All lie in the one of two parallel planes. So yes, one of two parallel planes. It can be both, right, doesn't have to be one. Okay. This [Polyhedron 25] is a prismatoid.

Once Lily had resolved the issue of how many planes in which the vertices needed to lie, she went back to using her hand test to determine whether polyhedra were prismatoids.

Lily's could use her hand test to represent conditions imposed by the definition of a prismatoid. Further, Lily's hand test was a specific test that she used to represent a general class of objects (prismatoids). Hence, her hand test was an exemplification of a prismatoid. A strength of her hand test is that it could indicate conditions not only for a specific prismatoid (e.g., Polyhedron 26) but also for a class of objects that are prismatoids (e.g., polyhedron such that all vertices are touching one of her two hands). A weakness of her hand test is that the test cannot be used to determine conditions included and excluded by the definition. For example, when Lily wondered whether all the vertices needed to be contained in one or both planes, she consulted the definition rather than relying on her hand test.

Lily's implementation of a task requiring her students to work from a definition. During a geometry class that occurred in her first year of teaching, Lily asked her students to work from a definition of parallelogram that she provided. Lily defined parallelogram as: "A quadrilateral is a parallelogram iff both pairs of opposite sides are parallel." Lily told her students that *iff* stood for "if and only if," using the definition of parallelogram to illustrate: "If both pairs of opposite sides are parallel, then you think parallelogram[5]." After providing students a definition of parallelogram, she instructed students to create a parallelogram using a ruler and a rectangular note card to investigate "the characteristics of a parallelogram." Students used opposite sides of the note card to sketch two parallel lines (see lines *j* and *k* in Figure 7). Next, students placed a ruler obliquely on the parallel lines they had just drawn and sketched another set of parallel lines to create a parallelogram (see Parallelogram ABCD in Figure 7).

After students created the parallelogram, Lily asked them to use the parallelogram they drew to determine what was "true" about parallelograms:

Lily: You just made a parallelogram. Okay. Now, looking at your parallelogram, I would like you to silently, by yourself, look at it and tell me if you can tell me, well, tell me other characteristics of a parallelogram. We know that the opposite sides are parallel. What else is true about a parallelogram?

Although Lily instructed students to create a specific instance of a parallelogram, her instructions to students suggest that she intended for that specific instance to represent a more general class of objects. This suggests that she conceived of the specific parallelograms that students drew as exemplifications of parallelograms. However, it is not clear whether students were conceiving of the parallelograms they had generated in a way that was compatible with the way in which Lily conceived of them.

The "characteristics" that Lily listed as being "true" about parallelograms (see Figure 8, top right) were sufficient conditions to identify a quadrilateral as a parallelogram. For example, one characteristic (both a necessary and sufficient condition) was "diagonals bisect each other." Some of these characteristics related specifically to specialized examples (see Figure 8, bottom left) that Lily included later during the lesson. In this lesson's postinterview, Lily referred to the examples in Figure 8 in a special way, calling them "defining examples." When the interviewer asked Lily what she meant by "defining example," Lily's response focused on how one could look at a "defining example" and then be certain that the object is a parallelogram.

Lily: If you see this, you know it has to be a parallelogram because if both pairs—this is what we are doing tomorrow—but if both pairs, the opposite sides are congruent, then it is a parallelogram because it is a theorem, too. But I guess that's what I mean by defining example. As soon as you see any of those four things, I guess, you would know, should know that's a parallelogram—defining.

Although Lily referred to the diagrams (see Figure 8, bottom left) as "examples," she intended for them to illustrate more than just specific instances of parallelograms. Each "defining example" represented a "characteristic" of a parallelogram.[6] The defining examples that Lily used in her classroom instruction paralleled her use of a hand test in her work during a task-based interview. Both the hand test and the defining examples served as exemplifications, representing a class of objects—prismatoids and parallelograms, respectively—by illustrating the sufficiency of a condition for objects in each class.

During this lesson, Lily provided her students with a minimal definition of parallelogram. Despite using a minimal definition, she engaged students in creating and interpreting examples of parallelograms to make sense of the definition. For example, opposite sides being parallel was a "characteristic" that an individual could "see" to "know that's a parallelogram." Lily's work suggests that she conceived of the function of the given definition from a definition-as-extracted perspective, meaning that the definition would report what a parallelogram was.

Aspects of Mathematical Definition: From Lily's Perspective

The form of a mathematical definition. When interpreting a definition during her work in a task-based interview, Lily considered it important to account for each word in the definition of adjacent angles. The interviewer provided Lily with the following definition of adjacent angles: "[Adjacent angles are] two angles in a plane that share a common vertex and a common side but do not overlap." When asked what she thought was meant by the definition, Lily parsed the definition, first considering what it would mean for two angles to share a common vertex, then adding the criterion of sharing a common side. When she reached the part of the definition specifying that the "angles do not overlap," she expressed uncertainty.

Lily: I don't think they can share a common side and overlap, but then I don't know why they would write that, if that's true.

Interviewer: All right. So you're not sure what's meant by the definition then?

Lily: I understand what adjacent angles are. I guess I just can't think of a nonexample. I can think of an example of angles that aren't adjacent, but I don't understand this *do not overlap* part. 'Cause I don't know what two angles which overlap would look like.

Because Lily thought angles could not simultaneously "share a common side and overlap," it did not make sense to her that both criteria would be included in the definition. To resolve this seeming inconsistency, Lily attempted to create a diagram that would allow her to distinguish between sharing a common side and not overlapping. However, she could not envision "two angles which overlap," and she was unable to reconcile the different parts of the definition. For Lily, definitions should not include potentially contradictory information (e.g., not include both "share a common side" and "overlap" if one precludes the other).

During the task-based interview in which Lily examined prismatoids, the interviewer prompted Lily to sort six ways (labeled *A*–*F*) of talking about parallelepipeds (see Table 2):

Interviewer: I want you to compare them and first to sort them into good descriptions, poor descriptions, and kind of in-between—sort of mediocre—descriptions.

Lily: Okay. Um, poor descriptions according to who, me?

Interviewer: Yes.

Lily's query regarding whose perspective to use to sort the descriptions suggests that, from her perspective, whether a description is good depends on the audience for whom the description is intended. Prior to asking Lily to sort the descriptions of parallelepipeds, the interviewer had asked Lily whether she had ever heard of a parallelepiped. Lily stated two things that came to mind: "3-D object, possibly" and "parallel sides." Further, she stated that she did not know "what that *piped* part is about," suggesting that parallelpiped was a new object for her.

In response to Lily's question regarding whose perspective to adopt, the interviewer asked Lily to sort the descriptions (see Table 2), first on the basis of what Lily would prefer, then on the basis of what Lily thought a seventh-grade student would prefer, and finally based on what Lily thought a mathematics professor would prefer. Although the interviewer referred to the six ways of talking about parallelepipeds as "descriptions," when sorting the descriptions, Lily referred to *A* (see Table 2) as "a pretty helpful definition." Lily's use of the word *definition* gives some indication that she may not distinguish between description and definition.

From Lily's perspective, the better descriptions had "simpler" terms—terms that she would know without thinking about "for a really long time" and that she could "immediately picture." Lily's sort for a seventh grader was nearly identical to her sort for herself (see Table 3). She switched only two descriptions based on which terms she thought seventh graders might know more readily, implying that a good description for a seventh grader would be similar to that which would be a good description for herself. When prompted by the interviewer to sort the descriptions in a way consistent with what a mathematics professor might do, Lily's sort changed drastically (see Table 3). She focused on the form of the description rather than on ease of interpretation. She moved *B*, which had been in the middle of her sort for herself and for a seventh grader to the top of the list, because it "seems like a pretty formal definition, specific, what can and can't be." As illustrated by the changing of her sorting, for Lily, the goodness of a definition varies depending upon the user of the definition.

Next, the interviewer specifically stated that the parallelpiped statements (*A*–*F* in Table 3) were descriptions, then asked Lily to determine which would be definitions:

Interviewer: Okay. Let's see. Now these are actually descriptions okay?

Lily: Okay.

Interviewer: Okay. Um, which if any of them would you call a definition?

Lily: I would call *B* a definition.

Interviewer: Uh huh.

Lily: This [Statement *A*] is definitely not a definition.

Interviewer: Okay. You said *A* is not.

Lily: No, yes, *A* is not.

After previously characterizing *A* as "a pretty helpful definition," Lily now said that *A* was "definitely not a definition." Lily identified *B*, *D*, and *E* as definitions. She stated that she was "on the fence" about *C*, but she did not explain why. She questioned *F* because it was "saying pretty much the same thing" as *E*. Next, the interviewer asked her what made statements definitions:

Interviewer: What makes them definitions again?

Lily: Um, I, I just, when I think of definition I think of, like, formalities, you know, like, they seem formal. They're kind of specific, to me.

Lily's response suggests that she is now referring to definitions for mathematics professors rather than for herself or for a seventh-grade student, because she is using the same criteria that she used to sort definitions for a mathematics professor (e.g., formal, specific).

Although the interviewer distinguished between descriptions and definitions, Lily's use of the word *definition* suggests that she does not distinguish between statements of description and statements of definition. Even though she altered her stance on A when the interviewer changed the prompt, Lily's earlier responses suggest that any of the descriptions would be suitable for her to use as definitions, but not necessarily suitable as definitions to a mathematics professor. For Lily, the form of a "good" definition can vary depending on the audience for whom the definition is intended.

The function of a mathematical definition. When considering Euclidean Geometry from an axiomatic-system perspective, *point*, *line*, and *plane* are undefined terms, and therefore would not have definitions. During a geometry lesson occurring during her student teaching, Lily provided students with descriptions of point, line, and plane that appeared in the textbook. For example, the description of line was: "a series of points that extend in two opposite directions without end." Although the textbook did not indicate the descriptions to be definitions, during the preobservation interview, Lily spoke about the descriptions as if they were definitions:

Interviewer: Talk a little bit about where, where this came from and how you went about deciding what to use here.

Lily: Yeah. Um, most of this was from the book, and once again it was just something I needed to be covered. These were two different sections that I combined, types of lines and congruent figures. Um, so I got these definitions from, directly from the book. I don't think I looked up anything on my own. And, most of these examples were also in the book.

Interviewer: You called these definitions, what, what makes them definitions?

Lily: Um, just like defining what, I mean, see I think students know what a line is, but they wouldn't necessarily know that it's a series of points that extend into opposite directions without end. You know, so, just maybe putting some words to the mental picture that they have in their head, or they probably don't know that it goes on forever.

Lily predicted that line was a familiar object for her students. By giving them what she conceived to be a definition of *line*, she was providing words that could augment how her students might be picturing a line. For example, it was important to Lily that her students knew that a line extended "into opposite directions without end." Lily's remark that a definition could "put words to the mental picture that they [students] have in their head" suggests that Lily conceived of the function of the description of *line* that she gave to her students from a definition-as-extracted perspective, because she intended to provide students with words that they could use to communicate about *line*.

Later during the interview, the interviewer spoke explicitly about an axiomatic system, asking Lily to identify the ideas of college-level mathematics toward which her teaching of seventh-grade mathematics might be working.

Interviewer: When we talk about geometry we often talk about a mathematical system that has undefined terms and definitions, axioms, and theorems. So,

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which, which things do they get out of this sort of college-level math ideas, when, when you're teaching seventh grade?

Lily: I guess, I would say these are definitions—geometric, you know. I—I don't see why they wouldn't be held up like in a college geometry book, you know—a line is a series of points that extend to opposite directions without end.

Although *line* is typically considered an undefined term in mathematics, Lily's response suggests that she does not have an issue with the object, line, having a definition, and this does not conflict with her view of collegiate mathematics. Lily's lack of conflict offers insight into how she conceives of the function of a mathematical definition. For Lily, definitions provide meaning for an object by characterizing the object as fully as possible. Therefore, having undefined objects would not make sense for Lily, because any object could be characterized, which would imply that any object might have a definition.

Discussion

It is well known that sets of examples and nonexamples can be used to construct definitions of mathematical concepts (e.g., Sowder, 1980). However, less is known about how teachers might interpret those examples or encourage students to interpret those examples. When Lily worked from a set of examples and nonexamples to engage in mathematical defining during a task-based interview, she interpreted those examples as single instances, rather than exemplifications of a more general concept. Therefore, when a "distinction" she noticed was not sufficient for her to "classify" elements of one set and exclude elements of another set, she discarded that distinction and searched for a new one. Lily's work with students paralleled her work during the task-based interview. When teaching secondary students, Lily fostered their interpretations of examples and nonexamples of convex polygons as specific instances of convex polygons. Lily's case demonstrates that the way in which a teacher interprets examples (as exemplifications or as single instances) can afford or constrain her engagement in or her engagement of students in mathematical defining.

Interpreting examples as exemplifications provides individuals the opportunity to attend to the general by examining the particular (Watson & Mason, 2005). Using examples to engage students in a careful examination of properties of familiar objects is a productive way to support students' construction of mathematical definitions (Freudenthal, 1973). When Lily was using examples when working from a given definition in a task-based interview, she interpreted examples as exemplifications, such that single instances represented a more general class of objects (e.g., her hand test for a prismatoid). Further, once Lily had created an exemplification, she used it as a proxy for a definition. When teaching secondary students, she encouraged them to interpret examples in a similar way (e.g., "defining examples" of parallelograms). Although exemplifications are powerful, they should support, not replace mathematical definitions. Having students generate or interpret exemplifications, such as Lily's "defining examples" of parallelograms, could have been a powerful way for students to conceive of the general class of objects, parallelogram. However, allowing exemplifications to serve as a proxy for a mathematical definition strips a mathematical definition of its power to stipulate conditions that specify a mathematical object.

Although it is known that prospective teachers preferred barely-not-minimal definitions to minimal definitions (Zazkis & Leikin, 2008), it is unclear how those prospective teachers conceived of the function of the definitions. When Lily expressed a preference for a definition of a particular form, her preference was inseparable from her perspective of the function of a definition (e.g., when a definition was functioning for classroom instruction for seventh graders, Lily expressed a preference for definitions with "simpler" words that she could "immediately picture"). Even when she used a minimal definition in classroom instruction (e.g., a quadrilateral is a parallelogram if and only if both pairs of opposite sides are parallel), that definition contained familiar words for students, and she engaged students in activity that could provide them the opportunity to create an example of such a quadrilateral. Although Lily's work with students involved using examples as exemplifications of more general objects (e.g., parallelograms) having mathematical definitions, she conceived of the function of a mathematical definition from a definition-as-extracted perspective, meaning that the definition would characterize what an object was. Therefore, students' work with mathematical definitions involved activity related to knowing what properties of an object were, rather than deductively organizing those properties. Lily's case demonstrates the interrelationship between a teacher's conception of the form and function of mathematical definition.

Referring to mathematical definitions as metamathematical concepts having a particular function, Freudenthal (1973) posited that mathematical definitions have "special meaning:"

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> In mathematics a definition does not just serve to explain to people what is meant by a certain word. In mathematics definitions are links in deductive chains, but how can you forge such a link unless you know in which chain it could fit? (p. 416)

Notably, the function that Freudenthal posited (links in deductive chains) differs from the function that Lily conceived mathematical definitions to have (characterizing what an object is). Although it is possible for a teacher to operate in an axiomatic system in her own mathematical work and choose to not require students to operate in an axiomatic system, making such a conscious choice would involve awareness of the special function metamathematical concepts (e.g., definition) play in that system. Vinner's (1991) assumptions regarding definitions can be separated into assumptions related to their form (e.g., "Definitions should be minimal") and their function (e.g., "Definitions are arbitrary"[7]). Although Lily used minimal definitions to support students' acquiring mathematical concepts, the function that she conceived the metamathematical concept, *definition*, to have impacted her use of minimal definitions. Lily's case illustrates the inseparability of the assumptions posited by Vinner (1991). If a teacher conceives of the function of a mathematical definition from a definition-as-extracted perspective, she may use definitions of particular forms (e.g., minimal definitions), but the function of those definitions would be different from what it would be for an individual conceiving of a mathematical definition from a definition-as-stipulated perspective. Further, if the function of a mathematical definition is to characterize what an object is, then having an undefined term would be illogical, because it would make little sense to leave a term (e.g., line) uncharacterized.

Conclusion

Lily's case illustrates that a teacher's knowledge of metamathematical concepts, such as definition, can influence her doing and teaching mathematics. Future research investigating how teachers might conceive of other metamathematical concepts, such as generalization, could expand the scope of this work. Although this research investigated Lily's conception of mathematical definition, we did not investigate students' conceptions of mathematical definition. A future study could investigate how a teacher's conception of a metamathematical concept, such as definition, might impact her students' conceptions of mathematical definitions. In addition, little is known about how prospective and novice teachers' knowledge of metamathematical concepts might differ from more experienced teachers' knowledge or develop as they refine their teaching practice. Including teachers with different levels of experience could provide further insight how teachers enact their mathematical knowledge of metamathematical concepts through processes (e.g., defining) and actions on products of processes (e.g., definition). Further, this research has implications for teacher preparation. Teachers' purposeful use of examples serving as specific instances and examples serving as exemplifications in their classroom could provide students with rich opportunities to engage in mathematical processes such as defining.

As illustrated by Lily's case, the way in which a teacher conceives of the function of a metamathematical concept, definition, can impact both her doing of mathematics and

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her teaching of mathematics. Further, a teacher's conception of both the form and function of a metamathematical concept are inextricably linked. Consequently, we argue that it is important for teachers to develop robust knowledge of the form and function of metamathematical concepts, such as definition. Such knowledge could support teachers' ability to effectively engage students in mathematical defining and work from definition, so that definitions can accomplish multiple purposes, including, as advocated by Poincaré (1914) in the previously quoted passage, being ones that students can understand.

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[1] Doing mathematics includes a teacher's use of mathematical processes (e.g., defining) and work with mathematical products (e.g., definition). A teacher could engage in doing mathematics in a variety of settings.
[2] Lily is a pseudonym.
[3] For a definition to be minimal, it "should not contain parts which can be mathematically inferred from other parts" (Vinner, 1991, p. 65) of the definition. As an example, Vinner explained that defining *rectangle* as a quadrilateral with three right angles would be preferred to a definition mentioning four right angles, because a fourth right angle is a logical consequence of three right angles.
[4] Although Lily had been working with the polyhedra in sets Alpha and Gamma previously during the interview, she neither recognized them as being prismatoids nor noticed that the vertices of the polyhedra in both sets lie on one of two parallel planes.
[5] Although Lily used "iff," her instruction did not directly address the biconditional nature of iff.
[6] Although the defining example in which markings were used to indicate the congruence of opposite sides (see Figure 8) did not correspond with a listed characteristic, in the lesson's postinterview Lily stated that, "both pairs of opposites sides are congruent" should have been included as a characteristic.
[7] Assuming a definition to be arbitrary could be likened to conceiving of the function of a definition from a definition-as-stipulated perspective.