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Title

Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning across tasks with different backgrounds and features

Authors

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Abstract

Using an actor-oriented perspective on transfer, we report a case of a student's transfer of covariational reasoning across tasks involving different backgrounds and features. In this study, we investigated the research question: How might a student's covariational reasoning on Ferris wheel tasks, involving attributes of distance, width, and height, influence a student's covariational reasoning on filling bottle tasks, involving attributes of volume and height? The student transferred covariational reasoning that she employed on Ferris wheel tasks to filling bottle tasks; yet, her covariational reasoning on filling bottle tasks was less advanced. When designing a sequence of tasks intended to engender students' covariational reasoning, we recommend that researchers begin by using situations consisting of "simpler" attributes, such as height and distance, which students may more readily conceive of as being possible to measure. By taking into account students' conceptions of task features, researchers can promote transfer of complex forms of mathematical reasoning, such as covariational reasoning.

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Covariational reasoning, entailing individuals' conceptions of change and variation, is a critical form of mathematical reasoning for secondary students to use when studying the gatekeeping concepts of rate and function (e.g., Confrey & Smith, 1995; Thompson & Carlson, 2017). By covariational reasoning, we mean a way of thinking involving two key components: conceiving of individual attributes as capable of varying, then conceiving of those attributes as varying concurrently (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson & Carlson, 2017). Although covariational reasoning can be challenging even for successful university mathematics students (e.g., Carlson et al., 2002; Stalvey & Vidakovic, 2015), secondary students can engage in covariational reasoning (e.g., Castillo-Garsow, 2012; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2012; Saldanha & Thompson, 1998). Furthermore, secondary students using covariational reasoning advanced their conceptions of exponential functions (Ellis, Özgür, Kulow, Dogan, & Amidon, 2016) and quantified variation in change occurring in a single direction (e.g., an "increasing" increase) (Ellis et al., 2015; Johnson, 2015a).

To promote students' covariational reasoning, researchers have designed tasks, often involving dynamic computer environments linking animations and graphs (e.g., Ellis et al., 2016; Johnson, 2012, 2013, 2015b; Kaput & Roschelle, 1999; Saldanha & Thompson, 1998; Thompson, 2002). Yet, despite task designers' intentions, students working on such tasks may not demonstrate the intended reasoning (e.g., Carlson et al., 2002; Johnson, 2013; Moore & Carlson, 2012). Researchers have shown that students' conceptions of task features impacted their work on tasks intended to engender covariational reasoning (Moore & Carlson, 2012). For example, a student's conception of volume may impact her use of covariational reasoning when working on a task involving changing volume, such as the well-known filling bottle task (Shell

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Centre for Mathematical Education (University of Nottingham), 1985). Yet, researchers investigating students' covariational reasoning have done little to problematize how students might conceive of the different types of attributes used in tasks intended to engender students' covariational reasoning. Because students' covariational reasoning involves their conceptions of attributes, we argue that the types of attributes used in tasks might mitigate students' engagement in covariational reasoning.

Watson (2016) recommended that researchers engaging in task design take into account theories of different grain sizes. To inform our design across a set of tasks intended to engender students' covariational reasoning, we drew on Marton's (2015) variation theory. To inform our selection of attributes to include in tasks, we drew on Thompson's theory of quantitative reasoning (1993, 1994, 2002, 2011). Because we place a focus on students' conceptions of attributes as being possible to measure and capable of varying, we selected an actor-oriented perspective on transfer (an AOT perspective). From an AOT perspective, researchers base claims of transfer on students' conceptions of similarities between situations, rather than on researchers' conceptions of structural similarities of tasks (Lobato, 2003). Consequently, an AOT perspective is useful for studying students' transfer of complex cognitive processes, such as covariational reasoning (Lobato, 2003, 2012; Thompson, 2011).

We report an instrumental case (Stake, 2005) of a student's transfer of covariational reasoning. We investigated the research question: How might a student's covariational reasoning on Ferris wheel tasks, involving attributes of distance, width, and height, influence a student's covariational reasoning on filling bottle tasks, involving attributes of volume and height? We

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discuss implications for the design of tasks intended to engender students' transfer of complex forms of mathematical reasoning, such as covariational reasoning.

Conceptual and Theoretical Perspective

Actor-Oriented Transfer (AOT)

Lobato (2003) defined an AOT perspective as "the personal construction of relations of similarity across activities, (i.e., seeing situations as the same)" (p. 20). Researchers using an AOT perspective focus on how individuals make meaning of situations, not assuming that individuals will notice structural similarities between situations that researchers might construe. Furthermore, individuals can still engage in transfer without necessarily noticing structural similarities identified by researchers. Therefore, from an AOT perspective, transfer occurs when an individual construes that she can treat a different situation as an instance of something about which she has already thought (Lobato, 2012).

An AOT perspective is particularly useful for researchers investigating transfer involving individuals' conceptions, rather than an individual's transfer of a procedure or skill (Lobato, 2012). Thompson (2011) addressed the utility of the AOT perspective for investigating students' quantitative reasoning, drawing on an earlier case study (Thompson, 1994) to contrast the use of AOT and traditional perspectives on transfer. Thompson (2011) explained that from a traditional perspective, the student provided no evidence of transfer because she used different solution strategies on the two problems. In contrast, Thompson (2011) posited that from an AOT perspective, the student provided evidence of transfer; across both problems, she demonstrated that she conceived of measuring distance in a similar way.

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Students' Conceptions of Attributes

Drawing on Thompson's theory of quantitative reasoning (1993, 1994, 2002, 2011), we focus on students' quantitative operations, which they can enact in thought as well as action. Conceiving of the possibility of measurement is one kind of quantitative operation. We claim that a student conceives of an attribute, such as height of liquid in a bottle, as a *quantity* if the student demonstrates that she can conceive of the possibility of measuring the height (a quantitative operation). When students enact multiple quantitative operations, they can engage in more complex conceptualizations. Thompson (2011) defined the process of quantification in terms of three related quantitative operations: conceiving of some attribute that is possible to measure, a unit of measure with which to measure the attribute, and a relationship between the measure of the attribute and the unit. Consequently, a student may conceive of some attribute as a quantity, yet not engage in the process of quantifying. For example, a student may conceive of the possibility of measuring the height or volume of liquid in a bottle, but be uncertain about a type of unit to use to measure the height or volume of the liquid or have difficulty forming a relationship between a unit and the measure of the height or volume of the liquid.

When we use the term image, we mean the "dynamics of mental operations," (Thompson, 1994, p. 231). For example, an individual can conceive of the possibility of measuring the height of liquid in a bottle without actually engaging in an observable action of measuring height. By images of change, we mean students' conceptualizations of variation (Castillo-Garsow, Johnson, & Moore, 2013). Castillo-Garsow et al. (2013) posited two contrasting images of change: chunky and smooth. A smooth image of change refers to a conception of change as occurring in progress. A chunky image of change refers to a conception of change as having occurred in

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particular increments. Castillo-Garsow et al. (2013) use the examples of an individual conceiving of a bottle being filled continually, as if from a dispenser, and a bottle being filled incrementally, as if pouring in liquid in cups, to distinguish between smooth and chunky images of change, respectively. In empirical studies with university and secondary students, researchers have demonstrated the utility of students' use of smooth images of change (e.g., Castillo-Garsow et al., 2013; Johnson, 2012; Moore, 2014). Furthermore, Thompson and Carlson (2017) used students' smooth images of change to make distinctions between levels of students' variational and covariational reasoning.

Covariational Reasoning

Drawing on distinctions between chunky and smooth images of change, Thompson and Carlson (2017) developed two frameworks explicating levels of students' variational and covariational reasoning, placing reasoning involving smooth images of change at the highest levels of each framework. To distinguish higher levels *(*e.g., *smooth continuous variation)* from lower levels *(*e.g., *gross variation)* of variational reasoning, students should demonstrate that they could conceive of variation in values of a quantity, rather than just a more general variation in some quantity (Thompson & Carlson, 2017). For example, conceiving of how the values of height could vary as a bottle continually fills *(smooth continuous variation)* with liquid would be more advanced than just conceiving of the height as increasing, without attending to values for which it might be increasing (*gross variation)*.

Students at more advanced levels of covariational reasoning can conceive of a relationship between the values of individual quantities as a *multiplicative object*, which entails transforming individual quantities to create a new, joint, quantity that comprises the individual quantities

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(Thompson & Carlson, 2017). For example, students could transform individual quantities of height and volume of liquid in a filling bottle to create a new, joint quantity that comprises the two quantities. Researchers have shown that conceiving of a relationship between values of individual quantities is a challenging endeavor, even for university Calculus students (e.g., Stalvey & Vidakovic, 2015).

At the highest level of the covariation framework, Thompson and Carlson (2017) place *smooth continuous covariation*, which entails conceiving of quantities varying "smoothly and continuously" within any interval (Thompson & Carlson, 2017). At a lower level of the covariation framework, Thompson and Carlson (2017) place *gross covariation*, which entails conceiving of variation in each of the related quantities. A key distinction between these levels rests in a student's conception of a relationship between quantities as a *multiplicative object*. For example, a student might be able to claim that both the height and volume in a filling bottle would increase (sufficient for *gross covariation*) without necessarily forming a relationship between the two quantities that relates the values of both height and volume in any interval (necessary for *smooth continuous covariation*).

Once students engage in covariational reasoning, it is useful for them not only to conceive of variation in related quantities, but also to conceive of variation in change occurring in a single direction, or variation in unidirectional change, in one of the related quantities (e.g., Carlson et al., 2002; Ellis et al., 2015; Johnson, 2012; Oehrtman, Carlson, & Thompson, 2008; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015). For example, in the well-known filling bottle problem, both the height and volume of liquid increase, but can have different magnitudes of increase. However, even successful university students have difficulty using covariational reasoning when working on tasks involving variation in unidirectional change (e.g., Carlson et al., 2002; Oehrtman et al., 2008; Stalvey & Vidakovic, 2015).

To address students' difficulties with covariational reasoning, researchers recommended varying different task features (e.g., Moore, Paoletti, & Musgrave, 2013; Moore, Silverman, Paoletti, & LaForest, 2014; Stalvey & Vidakovic, 2015). Stalvey and Vidakovic (2015) recommended that, for tasks drawing on the filling bottle problem, researchers incorporate variation in related features, such as the flow of the water (e.g., filling or draining at different rates). Moore and colleagues (2013; 2014) recommended that researchers "break conventions" by providing students opportunities to interpret and represent quantities using different axes on a Cartesian plane. The recommendations of Stalvey and Vidakovic and Moore and colleagues provide guidance as to types of variation and invariance to incorporate when designing tasks to engender students' covariational reasoning.

Designing Dynamic Computer Environments and Tasks

By mathematical task, we mean not only a problem designed for a particular set of students (e.g., Sierpinska, 2004), but also the students' conceptions of the problem to be solved (Johnson, 2014). Drawing on Marton's Variation theory (2015), Johnson designed dynamic computer environments and tasks to engender students' covariational reasoning. The tasks contained two different backgrounds: a filling bottle and a turning Ferris wheel. The Ferris wheel task sequence incorporated the following attributes: *distance* traveled around one revolution of the Ferris wheel (circumference), *height* from the ground (vertical distance), and *width* from the center (horizontal distance) shown in Figs. 1 and 2, respectively. The filling bottle tasks incorporated the attributes of *height* and *volume* of filling liquid.

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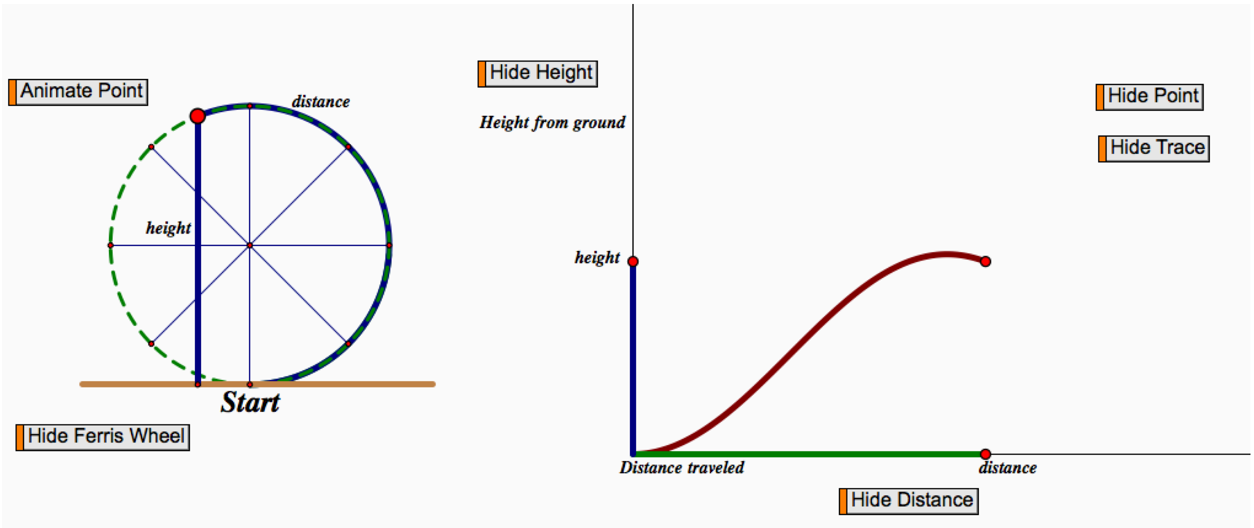

Fig. 1. Dynamic Ferris wheel computer environment, distance and height

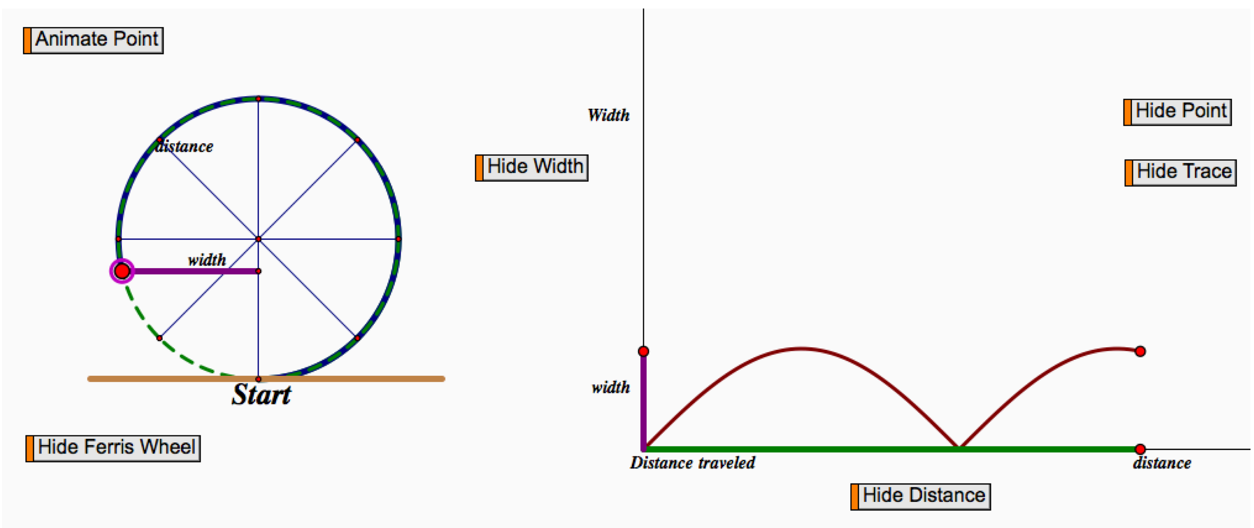

Fig. 2. Dynamic Ferris wheel computer environment, width and height

The Ferris wheel tasks included dynamic computer environments, which Johnson designed using Geometer's Sketchpad software (see Figs. 1 and 2). In the dynamic computer environments, Johnson used a circle to represent a Ferris wheel. The circle contained an active point, which represented a car on the Ferris wheel. Key design elements of the Ferris wheel

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environment included graphs with dynamic segments on the vertical and horizontal axes to represent change in individual attributes (see also Johnson, 2015b). Students could predict and observe related changes in the dynamic segments before and after clicking and dragging or animating the active point on the Ferris wheel. Furthermore, students could use keyboard commands to change the speed at which the Ferris wheel turned.

In each task in the Ferris wheel sequence, students first sketched a graph relating the given attributes (height and distance; or width and distance), then interacted with a dynamic computer environment, then compared their sketch to a computer-generated graph. In the filling bottle tasks, students viewed a video animation of a filling bottle, available at desmos.com (Fig. 3), but did not interact with a dynamic computer environment linking the animation to a dynamic graph. After viewing the video, students sketched a graph relating height and volume, with height represented on the horizontal axis and volume on the vertical axis.

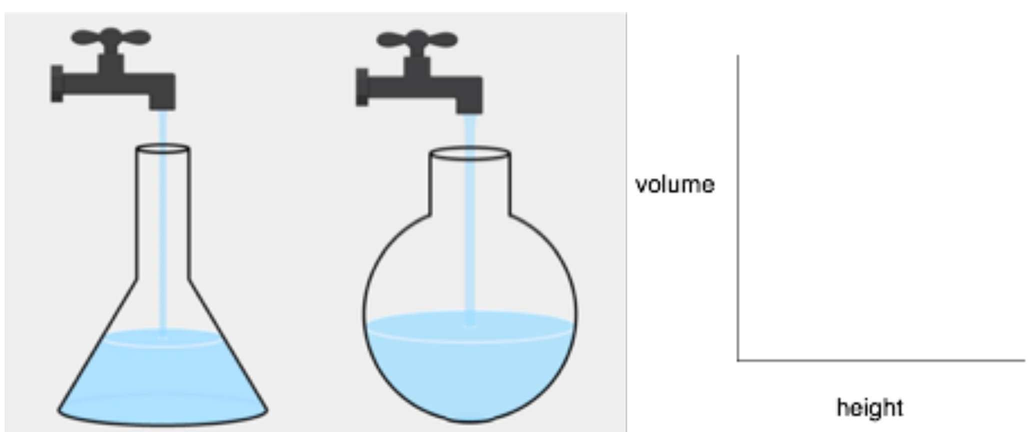

Fig. 3. Filling Bottle animations (https://teacher.desmos.com/waterline/walkthrough#Erlenmeyer; https://teacher.desmos.com/waterline/walkthrough#Evaporation)

Marton (2015) recommended that task designers develop task sequences incorporating patterns of variation, then invariance, for the purpose of fostering students' discernment of critical aspects. When those critical aspects contain components that are logically or functionally related, Marton (2015) recommended incorporating variation and invariance in each component, then variation in both components. We identified two critical aspects for students to discern, both of which contain components logically or functionally related: covariation in related quantities and variation in unidirectional change in one of the related quantities. In the Ferris wheel task sequence, we intended for students to discern covariation between height and distance (or width and distance) and variation in unidirectional change of height or width. In the filling bottle tasks, we intended for students to discern covariation between height and volume and variation in unidirectional change of height.

Method

Case Study

Yin (2006) identifies three steps in designing case studies: defining the case, justifying selection of single or multiple case studies, and explicitly articulating how theoretical perspectives inform (or deliberately do not inform) the case. We report a single case study (Stake, 2005) of a student's (Ana's) transfer of covariational reasoning from Ferris wheel tasks to a filling bottle task. We identify this single case as an instrumental case (Stake, 2005), through which we intend to advance what is known about secondary students' quantitative and covariational reasoning. We deliberately used our theoretical perspectives on reasoning and transfer to inform our design and analysis of this case study.

Setting and Participants

We conducted this study in a small neighborhood school that served students in middle through high school (~ages 11-18), located in an industrial region just north of a quickly gentrifying area of a large U.S. city. During the time of the study, at the school, 98% of students qualified for free and reduced lunch, and 99% of the students identified as nonwhite. The case study we report is part of a larger study (Johnson, 2015b) investigating 5 ninth grade students' (~age 15) quantitative and covariational reasoning. In the U.S., students typically take an introductory algebra course in either the eighth or ninth grade. All 5 ninth grade students who participated in the larger study were in the same introductory algebra course, taught by the same teacher.

Data Collection

Johnson led a sequence of 6 interviews, conducting one interview per week, just prior to the end of the school year. Typically, students participated in an interview with 1 or 2 other students. In the first three interviews, Johnson interviewed Ana with other students. In the last 3 interviews, Johnson interviewed Ana individually. Johnson reorganized the interviews in the midst of study for two reasons. First, Ana was absent due to illness on the day scheduled for the fourth interview. However, the two other students were available for the interview, so Johnson decided to interview the other students, so as to not create an undue delay in the project. Second, in earlier interviews, Johnson had observed that the other students would at times defer to Ana, or say that they thought in ways that were the same as or similar to Ana's thinking. Johnson conjectured that by interviewing Ana separately from the other two students, not only would she not need to delay the research, she might learn more about all three students' reasoning.

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Interview sequence

The interview sequence consisted of a PreInterview, four Interviews, and a PostInterview. Table 1 lists the different task backgrounds and features, by interview.

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Interview	Task Backgrounds	Task Features: Attributes	Task Features: Representation on Cartesian Plane
PreInterview	*Filling bottle*	*Height, volume*	*Height: horizontal axis* *Volume: vertical axis*
1	Ferris wheel	Height, distance	Distance: horizontal axis Height: vertical axis
2	Ferris wheel	Height, distance	Distance: horizontal axis Height: vertical axis
3	Ferris wheel	Width, distance	Distance: horizontal axis Width: vertical axis
4	Ferris wheel	Width, distance Height, distance	Height/Width: horizontal axis Distance: vertical axis
Post Interview	Ferris wheel	Height, distance	Height: horizontal axis opening left Distance: vertical axis
	Filling bottle	*Height, volume*	*Height: horizontal axis* *Volume: vertical axis*

Table 1. Task backgrounds and features, by interview

During the PreInterview and PostInterview, students viewed video animations of two filling bottles. The filling bottle videos in the PreInterview and PostInterview depicted a bottle that looked like an Erlenmeyer flask (Fig. 3, left) and a bottle containing a wide, spherical base and a narrow, cylindrical neck (Fig. 3, middle), respectively. Johnson designed the filling bottle tasks to provide opportunities for students to conceive of variation in individual components (height, volume), then variation in both components. After students viewed the videos, they discussed (1) how the height of the liquid in the bottle was changing as time elapsed, (2) how the volume of the liquid in the bottle was changing as time elapsed. Then, they had opportunities to relate variation in both components by sketching a graph relating height and volume (Fig. 3, right).

During Interviews 1-4, students worked with the Ferris wheel dynamic computer environments (Figs. 1 and 2). Johnson structured the first part of Ferris wheel tasks similarly to the filling bottle tasks; students discussed how each component was changing as time elapsed, then sketched a graph relating the components. In addition, the Ferris wheel tasks incorporated

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repeated patterns of variation and invariance for different features. In Interviews 1 and 2, after sketching a graph, students used dynamic segments in a Ferris wheel dynamic computer environment to show (1) how the height was changing as time elapsed, (2) how the distance was changing as time elapsed, then to describe how the height and distance were changing together. In Interview 3, Johnson repeated the pattern for width and distance. In Interview 4, Johnson repeated the pattern for both height and distance, and width and distance, but varied their representation on the coordinate plane, placing distance on the vertical axis and the height or width on the horizontal axis (not shown). During the PostInterview, in addition to the filling bottle task, students worked on an adapted Ferris wheel task that provided one additional variation, a coordinate plane for which the positive direction opened to the left, rather than to the right (Fig. 4).

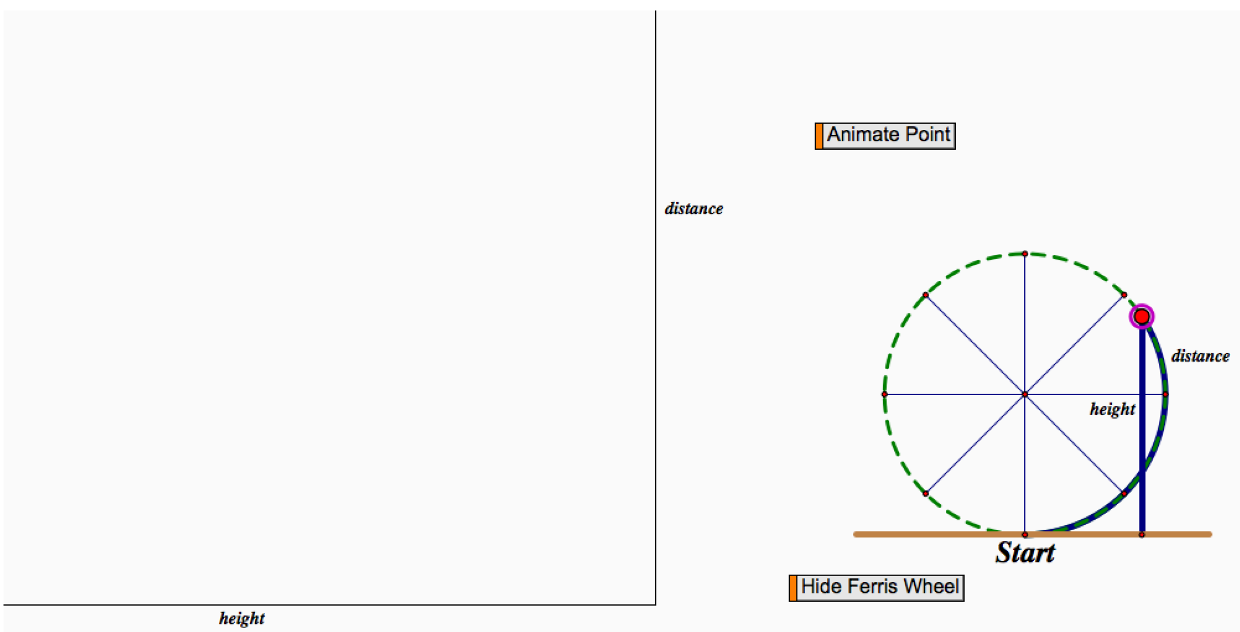

Fig. 4. PostInterview: Adapted Ferris wheel task

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Data Analysis

Lobato (2008) identified four elements to address to make claims regarding students' transfer: change in students' reasoning from PreInterview to PostInterview; students' demonstration of limited reasoning during PreInterview tasks; viable relationships between students' work during PostInterview and Interview tasks; and attribution of students' different reasoning to the Interview tasks rather than to spontaneous activity. Our data analysis methods worked to address each of these elements.

In the first pass of data analysis, we identified and coded portions of data across three interrelated dimensions, shown in Table 2. Despite beginning with codes in mind, our analysis incorporated a process of qualifying what each code meant in terms of students' quantitative operations. For example, when we coded Ana's conception of an attribute as being "measurable," we described the way in which she conceived of measuring that attribute, along with evidence supporting such a conception. For example, in the case of "distance traveled around the Ferris wheel," we described Ana's conception of distance as a length that she could measure with a linear unit, such as meters or feet.

Dimension	Codes
Conceptions of attribute(s)	Measurable Not measurable
Number of changing attributes	Variation (one changing attribute) Covariation (attributes changing together)
Types of change in attribute(s)	Different directions of change (e.g., increase vs. decrease) Variation in unidirectional change (e.g., variation in increases)

Table 2. Dimensions and codes

In subsequent passes, we used comparative analysis to trace changes in Ana's reasoning. Using each of the three dimensions, we compared Ana's reasoning for the filling Bottle tasks in the PreInterview and PostInterview and the Ferris wheel tasks in Interviews 1-4 and the PostInterview task. We worked from the descriptions qualifying what we meant by each code to identify relationships between Ana's work during Interviews 1-4 and the PostInterview. For example, we coded that Ana conceived of variation in unidirectional change when working on both the Ferris wheel and filling bottle tasks. We then looked at Ana's conceptions of attributes across those instances in which she conceived of variation in unidirectional change. Building from the relationships we identified, we worked to attribute Ana's changes in covariational reasoning to her work on the Interview tasks.

Results

We begin by situating Ana's case within the larger study. Next, we report Ana's work on the PreInterview filling bottle task, to demonstrate her initial reasoning. Then, we report Ana's work on the PostInterview Ferris wheel task, to demonstrate the extent of her covariational reasoning. Finally, we present Ana's work on the PostInterview filling bottle task, to demonstrate her change from her initial reasoning.

Situating Ana's case within the larger study

The larger study investigated 5 students' shifts from variational to covariational reasoning. In the Ferris wheel task sequence (Interviews 1-4), one student (Paola) demonstrated variational, but not covariational reasoning. Three students (Ana, Sofia, Lucia) demonstrated a shift from variational to covariational reasoning, and one student (Elisa) demonstrated covariational reasoning from the onset of the Ferris wheel task sequence. During the PostInterview filling bottle task, Ana was the only student to use a graph to represent covariation in height and volume

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of water in the filling bottle. In contrast, Elisa, Sofia, and Lucia used a graph to represent how the height of the water was changing as time elapsed. Furthermore, Paola demonstrated neither variational nor covariational reasoning in the PostInterview filling bottle task. We report Ana's case because she demonstrated a range of reasoning representative of all 5 students, and she was the only student to demonstrate transfer of covariational reasoning.

PreInterview: Filling Bottle Task

After viewing the video of the filling bottle shaped like an Erlenmeyer flask (Fig. 3, left), Ana sketched a graph relating the height and volume of the liquid in the filling bottle. Johnson provided Ana with a large pair of coordinate axes, with height labeled on the horizontal axis and volume labeled on the vertical axis (Fig. 3, right). Prior to sketching a graph, Ana correctly identified that the horizontal axis represented height in inches and the vertical axis represented volume in ounces. When graphing, Ana spontaneously made analogies to distance-time graphs, which she had drawn earlier in the year as part of her work for her Algebra I class. Rather than using the axes provided, Ana turned the paper to the blank side, drawing her own axes and sketching two graphs (Fig. 5, right).

Ana: Because when, *[Flips paper over, sketches a small pair of coordinate axes, Fig. 5 upper right]* like we did it. Because he could be running fast, so it could be like very steep *[Beginning at origin, sketches a line segment very close to the vertical axis, Fig. 5 upper right]*. And he could be like slowing down, so it could go like this *[Beginning at the endpoint of the first line segment, sketches a much less steep line segment]*. And it goes with the shape, because *[Sketches a new pair of coordinate axes, Fig. 5*

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lower right] when the glass is like in the bottom, it's filling up really slowly. *[Beginning at origin, sketches a line segment very close to the horizontal axis, Fig. 5 lower right.]* But then it's getting higher, *[Extends the original line segment, Fig. 5 lower right.]* and then it's like, voom *[Beginning at the endpoint of the first line segment, sketches a much steeper line segment.]* because of, this is like the same. *[Points to neck of bottle, Fig. 5 left.]*

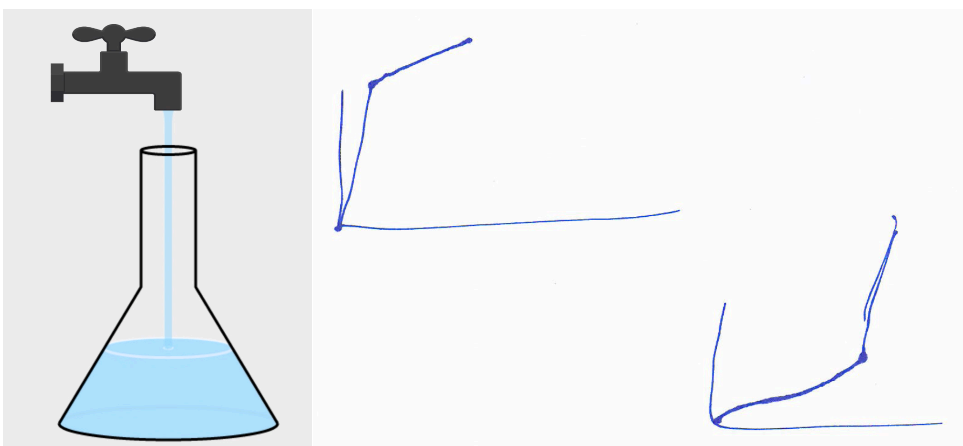

Fig. 5 Ana's graphs for the PreInterview filling bottle task

We coded for variation, because Ana provided evidence of conceiving of only one attribute as varying, the "it" that "goes with the shape." We interpret the "it" to be the level of the changing liquid, which extended across the bottle. We coded Ana's conception of the changing attribute as not measurable, because she used the shape of the bottle to define the "it" that she was representing, rather than separating the "it" from the bottle, then conceiving of the possibility of measuring that "it." We coded Ana as conceiving of variation in unidirectional change, because she made distinctions between slower and faster "filling" in sections of the bottle having different shapes, and she represented those differences in "filling" using two line segments having different steepness (Fig. 5, lower right).

Ana provided evidence of discerning variation in unidirectional change in the attribute she identified as "it." However, in this episode she did not demonstrate that she was conceiving of the height and volume of the water as varying concurrently. Hence, we claim that she was not engaging in covariational reasoning involving the attributes of height and volume. Furthermore, it is difficult to tell if Ana is discerning variation in unidirectional change in some observable attribute (e.g., the level of the water in the video) or an attribute that she can conceive of as possible to measure. Although Ana was yet demonstrate that she could conceive of height and volume as varying concurrently, her responses provide evidence of her smooth images of change (e.g., "filling up really slowly," "getting higher").

We interpret Ana's reasoning in this episode to be consistent with what Thompson and Carlson (2017) term *gross variation*, because she conceived of variation in the level of liquid, which depended on the shape of the bottle. Furthermore, Ana's reasoning demonstrates the extent of all 5 students' reasoning in the PreInterview filling bottle task. Only 2 students in the larger study, Ana and Elisa, engaged in variational reasoning in the PreInterview filling bottle task, both demonstrating variational reasoning at the gross variation level.

PostInterview part 1a: Adapted Ferris wheel task

Ana began the PostInterview by using the coordinate plane shown in Fig. 4 to sketch a graph relating distance and height. Without hesitation, Ana began at the origin, and sketching in one continual motion, drew the curved portion of the graph highlighted in yellow in Fig. 6. When Johnson asked Ana to explain, she stated: "Well the height increases *[draws segment along the*

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height axis], and to a certain point *[draws tangent segment at the maximum]* and then goes back down *[draws arrow near the graph]*, but the distance keeps increasing *[draws an arrow along the distance axis]*."

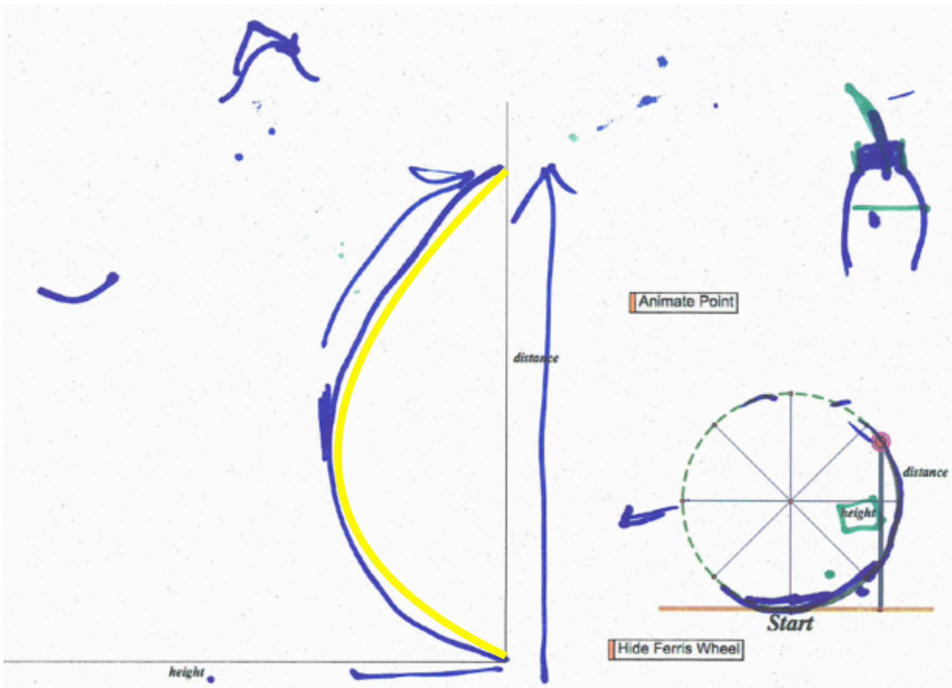

Fig. 6 Ana's graph for the PostInterview Ferris wheel task

We coded for covariation because Ana's graph demonstrated that she could conceive of height and distance as varying concurrently. She drew the marks to the right and below the vertical and horizontal axes, respectively, when representing the change in the individual attributes of distance and height. We coded Ana's conception of the attributes of distance and height as measurable, because she demonstrated that she could separate each attribute from the shape of the Ferris wheel. In addition, she showed that she could use a one-dimensional unit to measure each of those attributes, which provided evidence that she could quantify those attributes. We coded for variation in direction of change, because Ana focused on whether the quantities of height and distance were increasing or decreasing, not on variation within those increasing or decreasing sections.

In this episode, we interpret Ana to have transformed the individual quantities of distance and height to create a new joint quantity comprising those individual values. Put another way, Ana created what Thompson and Carlson (2017) term a *multiplicative object*. When Ana used segments/arrows to justify why she drew the graph, she represented variation in both the height and the distance. Furthermore, Ana demonstrated smooth images of change when sketching and explaining her graph. Not only did she sketch the graph using a continual motion, she described the variation in height and distance as continually occurring (e.g., "distance keeps increasing").

Johnson followed up to investigate if Ana might be able to use the attributes of height and distance to conceive of variation in unidirectional change for the height as occurring along with the changing distance. Johnson asked a question that varied the change in a related attribute—the speed at which the Ferris wheel is turning: "Would it still be slowly rising if I made it go really quick, if I just pulled it *[the dot representing the car, Fig. 6]* really fast?" The excerpt begins with Ana's attempt to relate distance and height to respond to Johnson's question.

Ana: Like you could be in the- perhaps you could say you're here, or well here *[Makes a dot on Ferris wheel to represent the car, Fig. 7 right]*, but that, it kind of seems like the same level above ground *[Darkens the horizontal segment under the Ferris wheel, Fig. 7, right]*. So the cart has already begun, like the little, like the spin, and you're here – you have already moved, but you're level, you're distance, or well height, it's still like the same as you started.

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Johnson: So if you wanted to show that on the graph, where the distance went around, but the height was almost the same as you started, how would you show that?

Ana: Like this–you started to go up. *[Sketches leftmost portion of graph, contained within the dotted oval, Fig. 7 left.]*

Johnson: Why like that?

Ana: Because distance is going, but the height is still like zero. *[Darkens horizontal, then vertical axis, marking on the axis, with marks contained within the dotted oval, Fig. 7 left.]*

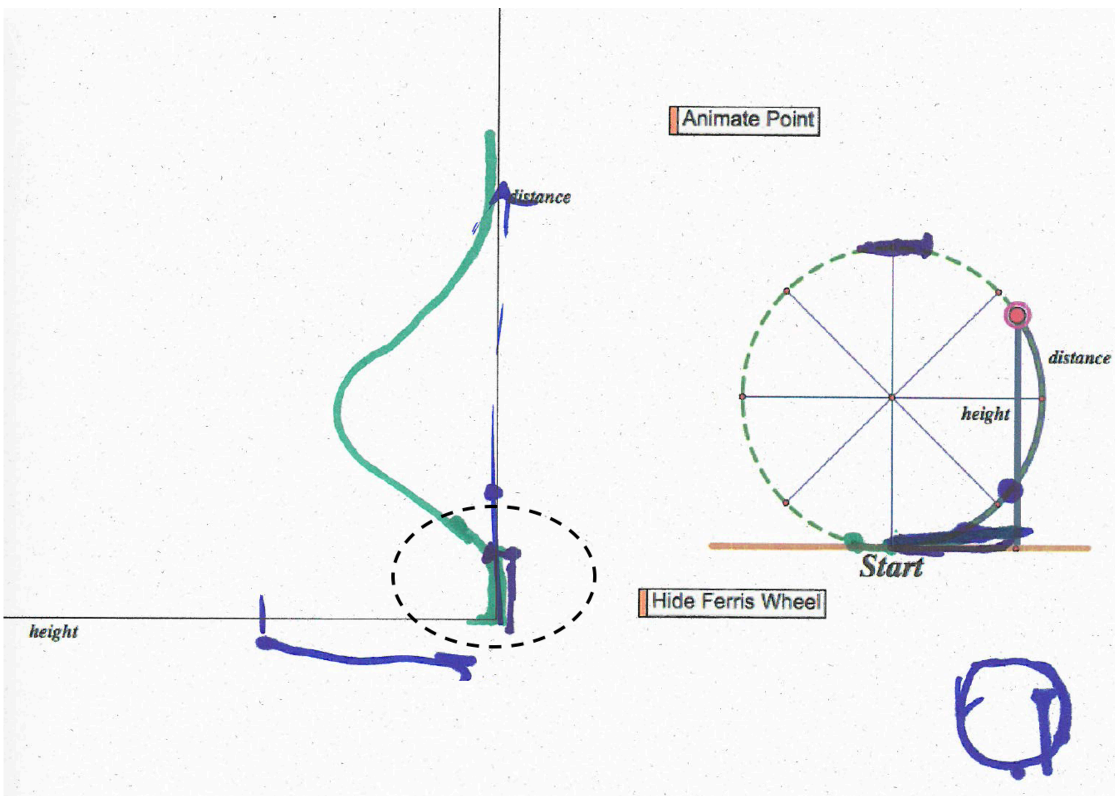

Fig. 7 Ana's revised graph for the PostInterview Ferris wheel task

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Fig. 7 shows the completed graph that Ana drew. At this point, Ana had only made the green markings on the graph contained within the dotted oval. She added the blue markings later, to represent changes in the distance and height for one revolution of the Ferris wheel. We interpret Ana's reasoning in this episode to be consistent with what Thompson and Carlson (2017) term *smooth continuous covariation*, because she conceived of height and distance as varying together within different intervals. Furthermore, in this episode, we now coded for variation in unidirectional change, because Ana attempted to represent a relationship between the attributes of distance and height, such that distance changed by a large amount, but the height did not change by very much. We contrast representing variation in unidirectional change *within* an interval from representing variation in unidirectional change *across* intervals. Only 2 students in the larger study, Ana and Elisa, represented variation in unidirectional change within an interval, rather across intervals.

PostInterview part 1b: Relating the Ferris wheel to the filling bottle

When working with the Ferris wheel tasks, on three occasions, in three different interviews, Ana spontaneously mentioned how she thought the Ferris wheel was like the filling bottle. On the first two occasions, which occurred during Interviews 3 and 4, Ana distinguished between the speed at which water flowed into the bottle and the speed at which the bottle appeared to be filling. To illustrate, we provide an excerpt of Ana's response from Interview 3. When working with a Ferris wheel task, Ana made an observation about the width: "it looks like it's getting smaller faster." When Johnson asked Ana to explain how something could "get smaller faster," Ana referred to a water bottle: "it filled up faster, but the water came out, came going at it, kept going at the same speed." The final occasion occurred during the first part of the PostInterview,

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before she encountered the second filling bottle task. In this occasion, Ana drew an analogy between the speed of the turning Ferris wheel and the speed of the water pouring into the bottle.

After Ana drew the entire graph shown in Fig. 7, she spontaneously conjectured about what the "pouring" water "would represent" in the Ferris wheel situation.

Ana:	It could also be like the water bottle again, the water keeps pouring, but then, I'm not sure what the water would represent in here. *[Points to the Ferris wheel]*
Johnson:	Oh like the water pouring-
Ana:	Like at the same rate.
Johnson:	If you had to guess?
Ana:	It would be like the height. The height or the speed or how fast?
Johnson:	What do you think? Well what are the things that are changing in the water bottle?
Ana:	The amount of water getting filled up, and the shape. I think the water would represent speed.
Johnson:	And what does speed mean for this Ferris wheel?
Ana:	Like how fast the ride is going.

Ana's response demonstrates that she conceived of additional attributes of the Ferris wheel and the filling bottle, the water that "keeps pouring" and the "speed" of the Ferris wheel, which she had not directly represented on one of the coordinate axes. We interpret Ana's statement to mean that she could conceive of different situations as having analogous measurable attributes.

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To investigate Ana's reasoning further, Johnson asked if changing how "fast the ride is going" would change the graph that she had drawn for the Ferris wheel.

Johnson: If I change the speed of the ride, if I change how fast the ride is going, would it change the graph?

Ana: Mmm, I don't think so, because we're not graphing the amount of time that it takes.

Johnson: What are we graphing?

Ana: The height and the distance, and it doesn't change, because if they had a change, the, the, how big the Ferris wheel would change as well.

Johnson: Ah, so can you give me an example?

Ana: So like the distance and the height would only change if it was a smaller Ferris wheel, because if it was a smaller Ferris wheel, the distance would be smaller as well, and the height would be like smaller as well because it wouldn't have to go as tall or long. *[Draws Ferris wheel shape, lower right, Fig. 7.]*

By claiming that the Ferris wheel itself would need to change size to result in a change in her graph (e.g., "if it was a smaller Ferris wheel), Ana demonstrated that she could separate the attribute of the speed of the Ferris wheel from the attributes of distance and height, which she represented in her graphs. In fact, only after conceiving of variation in unidirectional change for the height, occurring along with the changing distance, did she separate the speed at which the Ferris wheel was turning from the attributes of changing height and distance.

Ana was the only student in the larger study who attempted to draw analogies between the speed of the turning Ferris wheel and the speed of the water pouring into the filling bottle. One other student, Elisa, demonstrated covariational reasoning compatible to Ana's when working on the Ferris wheel tasks. However, in the PostInterview filling bottle task, Elisa did not demonstrate discernment of covariation between height and volume. We conjecture that Ana's efforts to draw analogies between the speed at which the Ferris wheel was turning and speed at which the water was pouring contributed to her discernment of covariation between height and volume on the PostInterview filling bottle task.

PostInterview part 2: Filling bottle task

In the second part of the PostInterview Ana worked on a filling bottle task. After viewing a video of the filling bottle with a seemingly spherical base (Fig. 7), Johnson asked Ana to sketch a graph relating the height and volume of the liquid in the filling bottle. Prior to sketching the graph, Johnson asked Ana what kinds of units she might use to measure the height and volume of the water in the filling bottle. Ana selected centimeters to measure the height, which she described as "how tall it is," and milliliters to measure the volume, which she described as "the amount of water that's inside." In the excerpt the follows, Ana explained her thinking while sketching a graph.

Johnson: If you had to sketch me a graph that would relate the volume of water in the bottle and height of the water that's in the bottle-

Ana: I think it'd be like a linear graph, linear

Johnson: Show me

Ana: Because the height, if we were talking about like individuals, the height would still increase *[Sketches segment on height axis, extending from origin, Fig. 8]*, unless you drank it *[Smiles]*, and the volume would still increase *[Sketches segment on volume axis, extending from origin, Fig. 8]* because it's like getting filled up, so you want more, so that would be this *[Sketches a linear, monotonically increasing graph, Fig. 8]*

Johnson: And can you tell me what this graph means in terms of the bottle?

Ana: The amount of water that's in it-

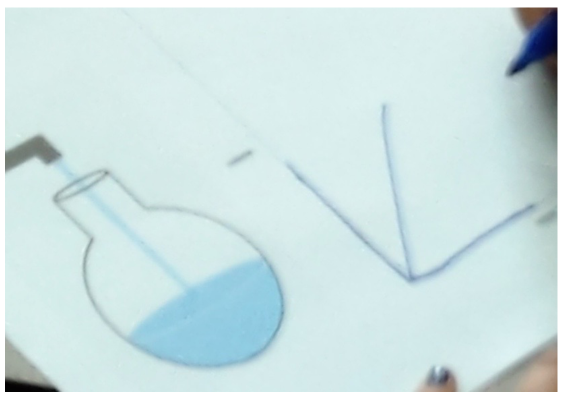

Fig. 8 Ana's graph for the new filling bottle

We coded for covariation because Ana conceived of both height and volume as varying individually, then represented a relationship between those varying attributes. She drew segments along the horizontal and vertical axes, respectively, when representing the change in the individual attributes of volume and height. We coded Ana's conception of the attributes of volume and height as measurable, because she described what each attribute could measure, and identified an appropriate unit that she could measure.

On the Ferris wheel tasks during Interviews 1-4 and the PostInterview, Ana conceived of the attributes of distance and height or distance and width as each varying individually, then represented a relationship between those varying attributes. During the PostInterview, she conceived of change in each of the individual attributes of height and volume, then represented a relationship between those changing attributes. Overall, the covariational reasoning that Ana transferred involved conceiving of change as progressing in individual attributes, then associating changes in those attributes. We interpret Ana's reasoning in this episode to be consistent with what Thompson and Carlson (2017) term *gross covariation*, because she conceived of the height and volume as varying together, but it is not yet clear that she is forming a new relationship that represents a joint quantity relating height and volume (what Thompson and Carlson (2017) term a *multiplicative object*).

Johnson followed up to investigate if Ana might be able to use the attributes of height and volume to conceive of variation in unidirectional change for the Filling bottle task, as she had just done with the Ferris wheel task. However, when Johnson asked how her graph showed the "faster" and "slower," Ana shifted from focusing on change in the individual quantities of height and volume, and went back to the "shape" that she had discussed in earlier interviews.

Johnson: And so, can you show me in your graph, like when we have the water that's changing, can you show me where on the graph you see the faster and the slower? And how you know?

Ana: Ooh, it could also be one of them graphs where where they're like, he was walking slowly and then faster, and then he stopped *[Sketches small graph with unlabeled axes, Fig. 9 upper middle]*.

Johnson: So what would that be? So tell me when you say it's going to be like one

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of-

Ana: The graphs we were practicing on, the water is filling up very fast, and then it's like slower, and then it's really fast. *[Sketches piecewise linear graph, associating segments of different steepness with faster and slower, Fig. 9 upper right.]* Really fast would happen here *[Boxes in the neck of the bottle, Fig. 9 lower left]*, and kind of fast here *[Boxes in the bottom of the bottle, Fig. 9 lower left]*, and slow here *[Writes "slow" on widest part of the bottle, Fig. 9 lower left.]*.

Johnson: If you had to label the axes, what would you label them?

Ana: *[Writes volume on vertical axis and height on horizontal axis, Fig. 9 upper right.]* Well because they're both increasing. *[Pause]*

Johnson: So can you have two different graphs for the same thing? *[Points to graph in upper right corner and graph on given axes, Fig. 9.]*

Ana: *[Pause]* I'm not sure.

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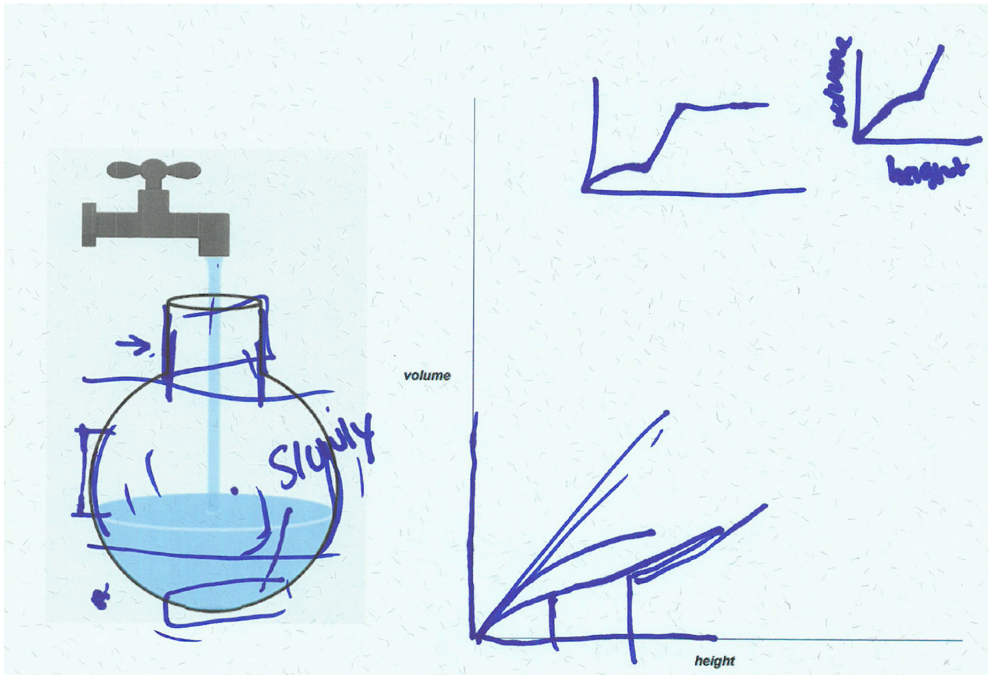

Fig. 9 Ana's continued work for the filling bottle, when asked to show faster and slower increases

When Ana attempted to represent variation in unidirectional change, she no longer demonstrated evidence that she was conceiving of change in the individual attributes of volume and height. Rather than talking about how the height and volume were changing, as she had just done previously, Ana referred to how the "water" was changing, stating "the water is filling up very fast, and then it's like slower, and then it's really fast." Although Ana labeled the axes "volume" and "height," which she had not done during the PreInterview, she added her labels after sketching the graph, rather than conceiving of change in each attribute, then representing a relationship between the changing attributes.

In the PostInterview filling bottle task, Ana transferred covariational reasoning involving related attributes she conceived of as measurable (quantities). However, when attempting to conceive of variation in unidirectional change, Ana reverted back to variational reasoning.

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Furthermore, when Ana reverted back to variational reasoning, she no longer demonstrated conceiving of change in an attribute that she conceived of as measurable. We posit that volume is a more complex attribute for students to conceive of measuring, and this complexity mitigated Ana's transfer of her covariational reasoning. Furthermore, Ana's difficulties suggest that when students encounter new situations with different types of attributes, they may need to re-progress through forms of reasoning that they already have demonstrated.

Discussion

Ana's reasoning provides further evidence to support the usefulness of smooth images of change for fostering students' quantitative and covariational reasoning (Castillo-Garsow et al., 2013; Johnson, 2012; Moore, 2014; Thompson & Carlson, 2017). Specifically, Ana's reasoning illustrates both the challenge and utility of conceiving of quantities varying *within*, rather than *across* intervals. When students compare different kinds of increases in attributes *across* different intervals, they may not reason covariationally, or reason about quantities. For instance, a student may discern that a bottle fills at different speeds, but that student may just be attending to variation in a single attribute. Furthermore the student may conceive of the "filling" as a physical, rather than a measurable attribute. Consequently, we recommend that students have opportunities to discern variation in unidirectional change in related attributes in conjunction with opportunities to separate the attributes from the objects, conceive of the possibility of measuring those attributes, and conceive of those attributes as changing continually.

After examining the variation and covariation frameworks presented by Thompson and Carlson (2017), we chose to explicitly distinguish the critical aspect of covariation from the critical aspect of variation in unidirectional change. Separating these critical aspects helped to target our analysis of Ana's reasoning. Ana demonstrated covariational reasoning at a *gross covariation* level on the filling bottle task in the PostInterview. However, when attempting to conceive of variation in unidirectional change in one of the related quantities, Ana reverted back to variational reasoning at a *gross variation* level, which she demonstrated during the PreInterview. In contrast, on the Ferris wheel task in the PostInterview, Ana demonstrated covariational reasoning at a *smooth continuous covariation* level, and when attempting to conceive of variation in unidirectional change in one of the related quantities, she continued to engage in smooth continuous covariational reasoning. Based on Ana's work, we conjecture that the level of a student's covariational reasoning may impact a student's conceiving of variation in unidirectional change in one of the related quantities. We cannot claim that smooth continuous covariational reasoning is a necessary co-requisite for students to conceive of variation in unidirectional change in one of the related quantities. Yet, Ana's smooth continuous covariational reasoning fostered her conceiving of variation in unidirectional change in the height, occurring in conjunction with continuing change in distance.

Thompson (2011) argued that the AOT perspective could prove to be a valuable lens for investigating students' quantitative and covariational reasoning, and we concur. One valuable insight involved students' conceptions of analogous attributes across different task situations. We did not explicitly design our study to investigate whether students might conceive of analogous attributes across situations such as the Ferris wheel and filling bottle. However, Ana's spontaneous response provided us with insights into how students might construe such relationships. Notably, Ana identified the speed of the Ferris wheel to be analogous to the speed at which the water was pouring into the filling bottle after she demonstrated that she could conceive of variation in unidirectional change when forming relationships between the quantities of distance and height. With Ana's reasoning in mind, we return to the recommendation of Stalvey and Vidakovic (2015)—that filling bottle tasks should incorporate variation in the rate of flow. Ana's case suggests that it would be useful for teacher/researchers to implement tasks incorporating variation in the rate of flow in conjunction with opportunities for students to separate the flow rate from other attributes in the situation. We think that future research should investigate secondary students' covariational reasoning in situations involving filling (or emptying) bottles with variable flow rates, Ferris wheels with variable speeds, or other analogous settings.

Implications/Conclusions

Our focus on the student perspective informed our design of tasks to foster students' engagement in and transfer of domain specific mathematical reasoning. Thompson and Carlson (2017) used students' conceptions of a new, joint quantity relating individual quantities (a multiplicative object) to distinguish between levels of covariational reasoning. Results of our study point to conditions under which students may develop such conceptions. Specifically, incorporating tasks involving "simpler" attributes, such as height and distance, which students may more readily conceive of as being possible to measure, holds promise for fostering students' covariational reasoning. To promote transfer of complex forms of mathematical reasoning, such as covariational reasoning, it is useful to take into account students' conceptions of task features. For example, for bottles such as those shown in Fig. 3, students may perceive a physical feature, the bottle "filling" with liquid, yet have difficulty conceiving of that feature as an attribute possible to measure. In our task design, we intended to engineer opportunities for students to conceive of physical features as attributes possible to measure (quantities), use smooth images of change to conceive of variation in each quantity, and form and interpret relationships between those quantities. If students only view features of dynamic computer environments and related tasks as physical features, rather than attributes possible to measure, it can inhibit their development of more complex forms of mathematical reasoning.

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References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 55-73). Laramie, WY: University of Wyoming College of Education

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31-37.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Ellis, A. B., Özgür, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning, 18*(3), 151-181.

Ellis, A. B., Özgür, Z., Kulow, T., Williams, C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *Journal of Mathematical Behavior, 39*, 131-155.

Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313-330.

Johnson, H. L. (2013). Designing covariation tasks to support students reasoning about quantities involved in rate of change. In C. Margolinas (Ed.), *Task design in Mathematics Education. Proceedings of ICMI Study 22* (Vol. 1, pp. 213-222). Oxford

Johnson, H. L. (2014). A role of context in constructivist model building: What problem is the learner solving? *Constructivist Foundations, 9*(3), 339-341.
Johnson, H. L. (2015a). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64-90.
Johnson, H. L. (2015b). Task design: Fostering secondary students' shifts from variational to covariational reasoning. In K. Beswick, T. Muir & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Hobart, Tasmania: University of Tasmania
Kaput, J. J., & Roschelle, J. (1999). The mathematics of change and variation from a millenial perspective: New content, new context. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 155-170). London: Falmer Press
Lobato, J. (2003). How design experiments can inform a rethinking of transfer and vice versa. *Educational Researcher, 32*(1), 17-20.
Lobato, J. (2008). When students don't apply the knowledge that you think they have, rethink your assumptions about transfer. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 289-304). Washington, DC: Mathematical Association of America
Lobato, J. (2012). The actor-oriented transfer perspective and its contributions to educational research and practice. *Educational Psychologist, 47*(3), 232-247.
Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.
Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138.
Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*(1), 48-59.
Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior, 32*(3), 461-473.
Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. *Mathematics Teacher Education, 2*(2), 141-157.
Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, DC: Mathematical Association of America
Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson, K. R. Dawkins, M. Blanton, W. N. Coloumbe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the 20th annual meeting of the Psychology of Mathematics Education North American Chapter* (Vol. 1, pp. 298-303). Raleigh, NC: North Carolina State University
Shell Centre for Mathematical Education (University of Nottingham). (1985). *The language of functions and graphs: An examination module for secondary schools*. Nottingham, UK: JMB/Shell Centre for Mathematical Education.
Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

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Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (pp. 443-466). Thousand Oaks, CA: Sage Publications, Inc.

Stalvey, H. E., & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *Journal of Mathematical Behavior, 40*, 192-210.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*, 165-208.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Lehrer, B. van Oers & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education*. Dordrecht, The Netherlands: Kluwer

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlain & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for wisdom^e* (Vol. 1, pp. 33-56). Laramie, WY: University of Wyoming College of Education

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics

Watson, A. (2016). *Parameters for practice and research in task design in mathematics education*. Paper presented at the 13th International Congress on Mathematical Education, Hamburg, Germany.

Yin, R. K. (2006). Case study methods. In J. L. Green, G. Camilli & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 111-122). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.