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Johnson, H. L. (2015). Together yet separate: Students' associating amounts of change in quantities involved in rate of change. *Educational Studies in Mathematics, 89*(1), 89-110.

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Together yet separate: Students' associating amounts of change in quantities involved in rate of change

This paper extends work about quantitative reasoning related to covarying quantities involved in rate of change. It reports a multiple case study of three students' reasoning about quantities involved in rate of change when working on tasks incorporating multiple representations of covarying quantities. When interpreting relationships between associated amounts, students identified sections (e.g., an interval on a graph) in which they could make comparisons between amounts of change in quantities. Although such reasoning is useful for interpreting a Cartesian graph as a representation of covarying quantities, it does not foster attention to variation in the intensity of change in covarying quantities (e.g., a decreasing increase). Focusing on the kinds of relationships students make between amounts of change in covarying quantities might provide further insight into how students could develop a robust understanding of rate of change.

Quantitative Reasoning

Rate of change

Quantity

Covariational Reasoning

Cartesian graphs

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1 Introduction

More than two decades ago, the mathematics of change, including rate of change, was identified as one of five interwoven strands of mathematics that could be developed from informal childhood experiences through formal school experiences at the elementary, secondary, and college levels (Steen, 1990). Since then researchers have recommended supporting students' work with rate of change through problems involving contexts extending beyond motion (Wilhelm & Confrey, 2003), qualitative accounts of change (Stroup, 2002), and dynamic technological environments (Roschelle, Kaput, & Stroup, 2000; Stroup, 2005). Yet, rate of change remains a difficult to understand concept for both secondary students (e.g., Herbert & Pierce, 2012; Lobato, Ellis, & Muñoz, 2003; Stump, 2001) and university students (e.g., Bezuidenhout, 1998; Cantoral & Farfán, 1998; Ubuz, 2007; Zandieh & Knapp, 2006). The research reported in this paper investigates the following question: How do students reason about quantities involved in rate of change when working on tasks involving representations of covarying quantities? This research explores how students might form and interpret relationships between quantities involved in rate of change and how that might afford or constrain their rate-related reasoning.

Secondary students can draw on intuitive notions of rate of change to make sense of situations involving quantities that change together (e.g., Johnson, 2012; Monk & Nemirovsky, 1994; Saldanha & Thompson, 1998; Stroup, 2002). Such work can involve descriptions of gradations of change. Monk and Nemirovsky (1994) reported on a clinical interview with a high school student, Dan, who worked with an air flow device connected to a computer that produced graphs of flow rate vs. time for any operation of the device. Dan had attempted unsuccessfully to produce a desired graph of flow rate vs. time. After observing a successful demonstration by the researcher, Dan remarked: "Well-oh-It's just the amount of increase is less and less. I see." (p. 156). Stroup (2002) reported on eighth grade students' work with a graphing calculator program that produced, in real time, a position vs. time graph representing the position of a cursor moving from left to right at the bottom of the screen. Students could connect different portions of the position vs. time graph with different directions (right/left) and speeds (slow/fast) of movement of a cursor (e.g., "right slow"). When students use only verbal descriptions, the object about which students are reasoning may be difficult to discern. For instance, it could be argued that the eighth grade students were describing the motion of a cursor rather than a rate of change. Although descriptions of gradations of change can provide some indication of a student's attention to rate-related ideas, such descriptions may not be sufficient to suggest that a student would be reasoning about or making relationships between covarying quantities.

Researchers (Johnson, 2012; Saldanha & Thompson, 1998) have conducted fine-grained studies investigating how individual secondary students formed and interpreted relationships between covarying quantities. Saldanha and Thompson (1998) found that an eighth grade student, Shawn, using dynamic geometry software was able to make claims about changes in a car's distance from two fixed points as a car moved along a path. Although the distances varied simultaneously, Shawn worked with each of those changing distances individually such that the changing distances were each varying, but not necessarily varying together. Johnson (2012) found that a tenth grade student, Hannah, interpreting a graph of volume vs. height for a bottle being filled at a constant rate, could attend to variation in amounts of increase in volume with respect to the changing height. Unlike Shawn, Hannah related the covarying quantities in a way that supported their varying together. Further, Hannah reasoned about the quantities as varying such that a change in one quantity depended on a change in another quantity. Together, these studies highlight complexities in the ways that students can make or conceive of relationships between covarying quantities.

This paper reports a multiple case study of three students' reasoning about quantities involved in rate of change when working on tasks incorporating multiple representations of covarying quantities. Focusing on the kinds of relationships students make between amounts of change in covarying quantities might provide further insight into how students could develop a robust understanding of rate of change.

2 Theoretical and Conceptual Background

2.1 Reasoning about quantities that change together

A quantity is an individual's conception of a measurable aspect of an object (e.g., Thompson, 1994). This means a quantity refers to more than just a measure used in the physical world or to a pairing of a unit and a number—it is an individual's conception. For example, an individual can conceive of volume as "an attribute that could be measured by the amount of space taken up by an object." Further, to conceive of volume an individual would not need to actually determine a numerical amount of volume. When considering how a quantity might change, an individual would engage in quantitative reasoning. Quantitative reasoning involves mental operations (Piaget, 1970) such as comparing or coordinating, that could afford the sense making of a quantitative situation (Thompson, 1994). For example, an individual might compare the volume of liquid that two different bottles could hold. By pouring liquid into the bottles, the individual might notice variation in how the height of the liquid changes. In doing so, the individual could begin to envision how quantities of height and volume change together.

When a student is reasoning about change in a single quantity, she is reasoning about variation (Clement, 1989). By forming or interpreting relationships between quantities that change together, a student would be reasoning about covariation (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Research-based distinctions between how individuals envision change as occurring include: (1) in completed chunks or (2) in a continuing process (Castillo-Garsow, 2010, 2012; Castillo-Garsow, Johnson, & Moore, 2013). Such envisioning can involve variation in one quantity or covariation between quantities. A student envisioning change as occurring in completed chunks might associate amounts of change in one quantity with amounts of change in another quantity (e.g., when the height increases by 1 inch, the volume increases by 2 ounces). In contrast, a student envisioning change as occurring in a continuing process might consider variation in the intensity of change in a quantity indicating a relationship between covarying quantities (e.g., as the height increases, the increases in volume begin to decrease). Although specific numerical values were

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used in the example of reasoning about variation as having occurred in completed chunks, the use of numerical values is not necessary to reason about variation in such a way. Students' envisioning of change as occurring (e.g., in completed chunks rather than as a continuing process) might help to explain in part students' difficulties with the concepts of ratio and rate.

2.2 Drawing parallels between ratio-related reasoning and rate-related reasoning

Both ratios and rates indicate multiplicative relationships between quantities (Thompson, 1994). However, students do not necessarily reason about ratio and rate as multiplicative relationships (e.g., Lamon, 2007; Lobato & Siebert, 2002; Thompson, 1994). The ways in which students form and interpret relationships between quantities involved in ratio and rate might suggest how they understand those concepts (e.g., Confrey & Smith, 1995; Thompson, 1994). Central to a robust conception of ratio or rate is the conception of ratio or rate as a single quantity (e.g., Simon, 2006; Thompson, 1994).

Researchers have posited two contrasting conceptions of ratio: *ratio as identical groups* (Heinz, 2000; Simon, 2006) and *ratio as measure* (Simon & Blume, 1994). A student with a ratio as identical groups conception would conceive of ratio as an association of amounts of quantities (or scalar multiples of those amounts): "the ratio is conceived of as two quantities that are being associated for a particular purpose—the ratio itself is not conceived of as a single quantity" (Heinz, 2000, p. 107). In contrast, a student having a ratio as measure conception would conceive of ratio as a single quantity involving a multiplicative relationship between quantities (Simon & Blume, 1994). A key distinction between a ratio as identical groups conception and a ratio as measure conception lies in an individual's conception of ratio: ratio as an association of quantities (identical groups) vs. ratio as a single quantity (ratio as measure).

To illustrate distinctions between conceptions of ratio, consider the measurable attribute of a rectangle, *squareness* (e.g., Simon & Blume, 1994), such that *squareness* measures a multiplicative relationship between the length and width of a rectangle. A student with an identical groups conception would conceive of *squareness* in terms of associations between amounts of lengths and widths of rectangles. For example, a rectangle with 2 cm of length for every 1 cm of width would be less "square" than a rectangle with 1.5 cm of length for every 1 cm of width. In contrast, a student with a ratio as measure conception would conceive of *squareness* as a single quantity. For example, a rectangle having a *squareness* measure of 2 would be less "square" than a rectangle having a *squareness* measure of 1.5. As illustrated by the *squareness* example, a ratio as measure conception is more powerful than an identical groups conception of ratio, because it involves a conception of ratio as a single quantity, and thereby supports students' deep understanding of other mathematical concepts, such as slope (Simon, 2006).

Parallels exist between identical groups conceptions of ratio and impoverished conceptions of rate. Investigating high school students' conceptions of slope, Lobato, Ellis, and Muñoz (2003) found that although the teacher had defined slope as a constant rate of change, the students did not use ratio-based reasoning to make sense of slope. Even when students used the slope formula correctly, they interpreted the result by associating an amount of change in y-values with an amount of change in x-values. The students in Lobato et al.'s study were neither making multiplicative relationships between quantities involved in slope nor reasoning about slope as a single quantity.

Thompson (1994) defined rate as a "reflectively abstracted constant ratio" (p. 192). Working from Thompson's definition, a student with a ratio as identical groups conception might conceive of rate as an association of amounts of related quantities (e.g., "ounces per inch" as an association of ounces of volume and inches of height). In contrast, a student with a ratio as measure conception might conceive of rate as a single quantity that indicates a multiplicative relationship between two varying quantities (e.g., "ounces per inch" as a single quantity measuring a multiplicative relationship between volume and height). As suggested by these examples, a student's conception of ratio might influence her conceptions of rate.

2.3 Quantifying relationships between rate-related quantities

Recently, researchers have articulated empirically based, educationally critical aspects (ECA) of rate for secondary students (Herbert & Pierce, 2012). Key to the ECA of rate is a conception of rate as a relationship between changing quantities, such that the relationship can vary. Importantly, Herbert and Pierce address the centrality of quantifying rate, which would involve understanding rate as "expressing a measure of the relationship between changes in two quantities" (p. 99). Claiming that quantification could be addressed when rate is expressed numerically, Herbert and Pierce drew distinctions between expressing a rate with descriptive words or numerically. Although expressing a relationship numerically might provide evidence of students' attending to rate as a quantity, using descriptive words to express a relationship also may indicate a student's conceiving of rate quantitatively. Further, the ways in which students make relationships between quantities to arrive at a numerical amount could indicate differences in how students conceive of rate. For example, two students could express a relationship in a way that could appear to be the same numerically (e.g., 2 m/s). One student could have envisioned the 2m/s as indicating associations of 2-meter distances and 1-second units of time (this could be likened to a ratio as identical groups conception extended to rate). Alternatively, another student could have envisioned the 2 m/s as measuring the coordination of meters and seconds in a way that does not depend on accruals of 2-meter distances and 1-second units of time (this could be likened to a ratio as measure conception extended to rate). Further investigation of students' interpreting and forming of relationships between rate-related quantities might help to explain why some students' conceptions of rate may be so impoverished, even when using numerical amounts in their oral or written expressions.

2.4 Using Cartesian graphs to represent relationships between covarying quantities

Researchers (e.g., Herbert & Pierce, 2012; Stroup, 2002) have recommended that instruction related to rate of change include situations involving varying as well as constant rates of change. Such situations could be modeled by Cartesian graphs. Researchers have identified many difficulties that students may have when creating and interpreting Cartesian graphs (e.g., Clement, 1989; Janvier, 1998; Leinhardt, Zaslavsky, & Stein, 1990). One difficulty involves interpreting a Cartesian graph as a picture of a situation (Clement, 1989; Leinhardt et al., 1990). Another difficulty involves focusing on the shape of a graph, rather than on objects undergoing change (Sierpinska, 1992) or on relationships between quantities (Moore, Paoletti, & Musgrave, 2013). A further difficulty involves students' interpretation of relationships between quantities, as students may envision relationships from a static perspective, such that a value of one quantity is paired with a value of another quantity (Clement, 1989). In contrast, when students consider relationships from a dynamic perspective, they can reason variationally[1] when interpreting graphs (Cantoral & Farfán, 1998, 2003; Clement, 1989). Such reasoning is crucial for attending to the mathematics of change (Cantoral & Farfán, 2003). Even if students use a dynamic perspective, they can interpret the quantity represented on the horizontal axis as involving a "temporal change," meaning that variation in the quantity could be envisioned as if it were elapsing time (Janvier, 1998). Such interpretations can cause difficulties for students, particularly when variation in the quantity represented on the horizontal axis does not depend on elapsed time (e.g., a graph representing the diameter of a colony of bacteria as a function of temperature) (Janvier, 1998). If students were able to envision time as "conceptual," then students could envision variation in a quantity as happening in relationship to time, rather than because time has elapsed (Thompson, 2012).

The well-known bottle problem developed by the Nottingham University's Shell Centre (1985) requires students to envision bottles of varying width filling with liquid that is being dispensed at a constant rate. Researchers have used different versions of the bottle problem with university mathematics students (Carlson et al., 2002), prospective secondary mathematics teachers (Heid, Lunt, Portnoy, & Zembat, 2006), and prospective elementary teachers (Carlson, Larsen, & Lesh, 2003). Such versions of the bottle problem required students to sketch a graph such that the height of the liquid in the bottle is a function of the volume of liquid in the bottle. Analysis of prospective elementary teachers' work revealed that some treated variation in volume as if it were elapsing time, thereby reducing the complexity of the problem to variation in height, rather than covariation between height and volume (Carlson et al., 2003). Had the prospective elementary teachers in Carlson et al.'s study been able to envision time as conceptual, they might have been able to separate related changes in volume from elapsing time and therefore reason about the quantities involved in the bottle problem in more sophisticated ways (cf., Thompson, 2012). As suggested by the work of the prospective elementary teachers, distinctions between ways in which individuals can envision change as occurring might be useful for explaining how individuals make sense of Cartesian graphs in terms of variation and covariation (Castillo-Garsow et al., 2013).

3 Method

3.1 Research methodology

This paper reports a multiple case study (Stake, 2005; Yin, 2006) of three secondary students' reasoning about quantities involved in rate of change. In

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qualitative research, cross-case analyses generally take one of two forms: intrinsic case study and instrumental case study (Borman, Clarke, Cotner, & Lee, 2006). Researchers using intrinsic case study design attempt to understand "what is important about that case within its own world" (Stake, 2005, p. 450), while researchers using instrumental case study attempt to illustrate "how the concerns of researchers and theorists are manifest in the case" (Stake, 2005, p. 450). This multiple (or collective) case study extends an instrumental case study to multiple examples (Stake, 2005).

Basic steps in case study design include: defining the case, determining whether to conduct a single or multiple case study, and deciding how/if to use theory to inform case selection, data collection, and/or data analysis (Yin, 2006). For this study, a case was defined as a secondary student's reasoning about quantities involved in rate of change when working on tasks involving different contexts and representations. Assuming that students' rate-related reasoning might take many forms, I chose to conduct a multiple case study, with the goal of developing accounts of students' reasoning that could explain how students make sense of rate-related situations. To develop such accounts, I employed grounded theory methodology (Corbin & Strauss, 2008) for within-case and cross-case analysis, because of its utility for building explanations of complex phenomena.

When using grounded theory for data analysis, a researcher aims to abstract concepts from data to develop coherent explanations of phenomena being studied. Those explanations then could be used to build theory. The phenomenon being investigated for this study is secondary students' reasoning about quantities involved in rate of change. My decision to employ grounded theory methodology influenced my choice of research methods, including the use of the clinical interview (Clement, 2000) to elicit data from which accounts of students' reasoning could be developed. Consistent with the clinical interview, I made no attempt to teach the students.

Key aspects of grounded theory data analysis include conceptual saturation, open coding, and comparative analysis (Corbin & Strauss, 2008). To allow for conceptual saturation, I incorporated multiple interviews with each student. Anticipating that students' reasoning might be tied to their work on a particular task, I included multiple tasks incorporating different contextual situations and different representations of covarying quantities. My goal was to investigate students' current reasoning related to rate, not to foster their development of a new understanding of rate.[2] Drawing on Oresme's distinctions between three different types of change: uniform, uniformly difform, and difformly difform (Clagett, 1968), I designed a set of tasks[3] including representations of different types of change: constant, varying at a constant rate, and varying at a varying rate, respectively. Based on Stroup's (2002) recommendation, I incorporated constant and varying change across all tasks, rather than beginning with constant then moving to varying.

An aim of this research was to develop explanations accounting for students' quantitative and covariational reasoning related to rate. When engaging in open coding, I drew on the corpus of rate related research articulating the multi-dimensional nature of rate-related reasoning, including how students envisioned change as occurring, formed and interpreted relationships between quantities, expressed relationships with numerical amounts or verbal description, and represented relationships with graphs, diagrams, or other means. When engaging

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in comparative analysis, I attempted to adopt the perspective of each student, engaging in thought experiments regarding how each student might reason quantitatively and covariationally related to rate on and across different tasks. Theoretical saturation (Corbin & Strauss, 2008) was reached when explanations of students' reasoning meaningfully integrated the multi-dimensional nature of rate-related reasoning. When I was able to articulate how a student's forming and interpreting of relationships between quantities and envisioning of change as occurring could explicate how he used numerical amounts, verbal description, and representations on rate-related tasks, I assumed an explanation of a student's reasoning was viable.

3.2 Setting

Results reported in this paper come from individual, task-based clinical interviews (Clement, 2000; Goldin, 2000) that I conducted with six secondary students from a small rural high school during fall 2009. Each student had completed at least one year of algebra, but had not taken a calculus course. Students participated in a series of five interviews during which they completed seven mathematical tasks. The tasks incorporated different contextual situations and different representations of covarying quantities. This paper reports three students' work on the final task, which students completed during the fifth interview. For a complete description of all of the tasks used in this study, see Johnson (2010). I chose to present data from the final task because it adapted a well-known problem for which researchers have documented students' difficulty (see section 4).

This paper reports results based on three participants—Austin, Mason, and Jacob (pseudonyms). During the time of the interviews, Austin, Mason, and Jacob were enrolled in a pre-calculus course. Johnson (2012) reports on the reasoning of one of the other three participants. The remaining two participants provided limited evidence of covariational reasoning related to rate of change. Johnson (2010) reports on the reasoning of one of the remaining two participants.

3.3 Data collection and analysis

Students were interviewed during the regular school day once per week for a period of 5 weeks.[4] Interviews occurred in a quiet room during a time when students did not have an academic class. I served as the interviewer, and another researcher was in the room operating the video camera. Generally, the second researcher played the role of camera operator and observer. However, while the interviews were in progress, if the second researcher had suggestions for additional probing questions that I might ask, the researcher passed me a note card with the question written on it. I would then choose to ask or not ask that question depending on the flow of the interview.

Audio-recordings, video-recordings, annotated transcripts, and students' written materials served as sources of data to be analyzed. Data included students' explanations, written work, and gestures in which students provided evidence of attending to quantities, amounts of change in quantities, and quantities changing together. Evidence included a student's specifying particular amounts of change in quantities, explaining or gesturing to indicate the intensity and/or direction of change in quantities, comparing and/or ordering changing quantities, or the interpreting and/or creating of representation of quantities changing together.

Data analysis incorporated ongoing and retrospective analysis. Ongoing analysis included reflective notes, written after conducting each interview, to inform decisions made regarding future interviews, including subsequent interviews with the same student as well as subsequent iterations of the same interview with different students. Retrospective data analysis included multiple passes through the data. The first pass incorporated the use of open coding (Corbin & Strauss, 2008) to identify chunks of data when students were relating covarying quantities. I defined 'relating covarying quantities' broadly to include instances when students related amounts of change, indicated direction of change, described how change was occurring, or compared/ordered changing quantities. The second pass incorporated comparative analysis (Corbin & Strauss, 2008) to elaborate on concepts developed through coding. I completed both passes for each student, working task by task. Subsequent passes incorporated asking questions of the data and making further comparisons between chunks of data for individual students. A goal of subsequent passes was to search for confirming and disconfirming evidence for emerging accounts of students' reasoning. As concepts emerged from the data and accounts of students' reasoning began to develop, I vetted their viability with two other researchers. Saturation was reached when an account of student reasoning held across a student's work on all seven tasks. After developing accounts of individual students' reasoning, I made comparisons across those accounts to develop overarching explanations that could include multiple accounts of students' reasoning.

4 The Filling Bottle Task

Accounts of students' reasoning held across students' work on all seven tasks.[5] This paper reports students' work from the final task, the filling bottle task. I developed the filling bottle task by adapting The Nottingham Shell Centre's well-known bottle problem. Anticipating that this task would be difficult for secondary students, I made a few key adaptations: (1) Provided actual soda bottles for students to examine, (2) Provided students with graphs that they could match to actual bottles, (3) Included a graph for an unknown bottle that represented the height of the liquid in the bottle on the horizontal axis and the volume of the liquid in the bottle on the vertical axis, and (4) Prompted students to sketch a bottle that could be represented by a graph of volume as a function of height. I provided actual bottles (shown in Fig. 1), to familiarize students with the filling bottle situation. I provided graphs for students to match to the given bottles so that they could connect the next part of the task (sketching a bottle) to something they had just done. I chose to represent volume as a function of height, because the rate of change of height with respect to time was not constant for bottles of varying width. Choosing the quantity represented on the horizontal axis to be a quantity not changing at a constant rate with respect to time might provide opportunities to distinguish between students envisioning time as elapsing vs. time as conceptual. Given the difficulty that university students with extensive mathematics background have had when sketching a graph given a diagram of a bottle (Carlson et al., 2002; Heid et al., 2006), I anticipated that asking secondary students to interpret a graph then sketch a bottle might be more likely to elicit useful data.

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When working on the filling bottle task, students responded to the following prompts in the order given below. As the interviewer, I posed the prompts verbally to each student during an individual clinical interview. Students received a copy of the graph shown in Fig. 2.

1. Given four soda bottles of varying width (Fig. 1), describe how the volume of liquid being poured into the bottle would change as the height of the liquid in the bottle increased.
2. Match (or sketch) four graphs that could relate the height of the liquid in the bottle and the volume of liquid being poured into the bottle as each of the bottles A, B, C, D (Fig. 1) is being filled.
3. Sketch a graph that could relate the height of the liquid in the bottle and the volume of liquid being poured into the bottle for a narrow bottle that is being filled.
4. Sketch a bottle of varying width that a given graph (Fig. 2) could represent.

Fig. 1 Four soda bottles that students worked with on the task

Fig. 2 Graph relating volume and height for a bottle not given

5 Results

This section includes Austin, Mason, and Jacob's response to one of the prompts in the filling bottle task. The selected responses are typical of how each of the individual students reasoned on other prompts in the filling bottle and other tasks. This section begins with Austin's response to prompt 3, then Mason's response to prompt 4, then Jacob's response to prompt 2. By presenting students' work in this order, I intend to demonstrate how students' association of amounts of change in covarying quantities can result in increasingly complex representations of relationships between quantities.

5.1 Sketching a graph that related volume and height for a narrow bottle: Austin

Austin estimated numerical amounts of volume and height, comparing those associated amounts for the purpose of making claims about the filling bottle. When asked to sketch a graph for a narrow bottle, Austin drew a line with a small positive slope (Fig. 3).

Interviewer: So if you were to sketch me a graph that related volume and height for a bottle that was pretty narrow.
Austin: That would, it would just go more height than volume. *[While talking, sketches graph shown in Fig. 3.]* So it would just be. It'd travel, it uh, cover more inches than it did in volume.

Fig. 3 Austin's sketch of a graph for a narrow bottle

When describing the linear graph he sketched (See Fig. 3), Austin associated amounts of height and volume that would accrue during the filling of a narrow bottle: "go more height than volume" (line 3), "cover more inches than it did in

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volume" (lines 5-6). Earlier in the interview, Austin had matched graphs containing curved portions (See Fig. 4) to physical bottles A, B, C, and D (See Fig. 1). Regardless of whether a graph was linear or nonlinear, Austin made comparisons between the amounts of change in quantities in given intervals, using verbal descriptors (more/less) or estimated numerical amounts.

Interviewer: And say, if you wanted to compare the rate at which the volume was changing say for this graph that you drew *[Fig. 3]* and to this graph here for the beginning part of B *[Fig. 4]*, how would you compare those?

Austin: You could just have, just compare the slopes as a ratio of ounces to inches and then just compare the two.

Interviewer: Would you show me how you might do that?

Austin: Yeah I'll just pick something here. So that *[Graph shown in Fig. 4]* goes up close to like a slope of 2 almost or maybe steeper, and this one, *[Puts tick marks on axes of graph shown in Fig. 3].* This one, it kind of goes 1 in height as it does 2 in length so I would say it's 1 to 2. *[Writes 1/2 next to the graph shown in Fig. 3.]* And this one, *[Puts tick marks on axes of graph shown in Fig. 4.]* we'll say maybe goes like 2 to 1. *[Writes 2/1 next to graph shown in Fig. 4.]*

Interviewer: And can you tell me what these numbers mean?

Austin: The 2 to 1 means it goes 2 ounces to 1 inch. *[Adds units to 2/1 written next to graph shown in Fig. 4.]* And then, 1 ounce to 2 inches. *[Adds ounces and inches to 1/2 written next to graph shown in Fig. 3.]* We need to put them in the same form, so 1 on the bottom or we'll just put this 1 to 2 so it would be 4 to 2. *[Writes 4oz/2in next to graph shown in Fig. 4.]* That way they have the same 2 inches on the bottom.

Fig. 4 Austin's marking of a graph matched with bottle B

Austin identified a single numerical value, 2, for the slope of the leftmost part of the graph shown in Fig. 4. However, Austin's interpretation suggested that he envisioned a slope of 2 as an association between height and volume (e.g., "The 2 to 1 means it goes 2 ounces to 1 inch," line 23). Importantly, although a student may use a single numerical value to represent a relationship between quantities, that student may interpret that value as an association of quantities, rather than as a single quantity.

Austin put each slope into what he referred to as the "same form" (line 27), meaning that one of the associated quantities in each slope would be the same amount (e.g., 2 in). Although it might seem as if Austin needed to have the same amount of one quantity to make comparisons between slopes, Austin's prior comparisons suggest otherwise (e.g., his description of how a graph "goes" in terms of "height" and "length," lines 17-18). Given his use of the pronoun "we" rather than "I" (line 26), it might be that putting things in the "same form" was a procedure that Austin had been taught and was accustomed to performing.

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Interviewer: So what does that mean in terms of these bottles?
Austin: This one, I mean, in two inches we'll say is like right here. *[Points to the portion near the base of bottle B, shown in Fig. 1.]* It's going to have four ounces compared to, and two inches than maybe say a thinner bottle at the same height it's going to have only one ounce so it has three less ounces of liquid.
Interviewer: So how would the rate at which the volume is changing in relationship to the height, how would these *[Graphs shown in Figs. 3 and 4.]* give you information about that?
Austin: That would just tell you I mean the more volume it changes compared to height the wider or the fatter the bottle is going to be. The more height than volume, the thinner the bottle is going to be.

Austin's purposeful association of amounts of increase in height to amounts of increase in volume was useful for making predictions about the width of a bottle (e.g., "the more volume it changes compared to height the wider or the fatter the bottle is going to be," lines 42-45). To make comparisons between quantities from different measure spaces, he used numerical amounts, such that an increase of 2 ounces of volume would be more than an increase of 1 inch of height. Although it might seem as if numerical values were necessary for Austin to make comparisons, in other tasks he compared amounts of horizontal and vertical distance on a graph representing amounts of change in each quantity without attaching numerical values to those amounts of distance.[6]

A hallmark of Austin's reasoning is his envisioning of completed amounts of change in making predictions about variation in the width of the bottle. In doing so, he determined a section of the bottle to which he would attend, estimated amounts of increase in height and volume in that section, and then compared those amounts of increase to make predictions about the width of a bottle. Comparing amounts of increase in linear units to amounts of increase in cubic units did not present a conflict for Austin, because he envisioned an amount of change in each quantity to be represented by an amount of horizontal or vertical distance on a graph. Although Austin could create sections having the same or different heights, the height of the section was secondary to the utility of a section for comparing associated amounts of change in height and volume.

5.2 Sketching a bottle that a given graph could represent: Mason

By making comparisons between associated amounts of change in volume and height of liquid, Mason made predictions about variation in the width of a bottle. When sketching a bottle that the graph shown in Fig. 2 could represent, Mason drew one bottle, scratched it out and then drew a second bottle (See Fig. 5).

Interviewer: And for the graph that's drawn here. Could you draw a picture of a bottle that would have this graph that's given here?

Mason: *[Draws bottle shown at left in Fig. 5.]* I guess something weird like that.

Interviewer: And could you use the graph to explain why the bottle would look that way?

Mason: Because it is taking up more volume than the height here. I got this backwards. It's taking up more of volume. Yeah I have it backwards *[Scratches out bottle shape shown at left in Fig. 5.]*. It's going to be wider down at the bottom. It gets smaller up at the top because it's taking up more inches and its less amount of volume getting into it. So I think it is going to start out wide. *[While talking, draws bottle shown at right in Fig. 5.]* And then once it reaches that certain point it's just going to taper off kind of like D *[See bottle D shown in Fig. 1.]* I guess. Going to start off wide and then—

Interviewer: And how can you tell that the bottle would taper off based on that graph?

Mason: Because you can see that it's filling up the height more than it's putting in ounces. I am guessing it is just going to get thinner there.

Interviewer: And so for the new bottle that you drew could you show me where the height would be filling up faster than you're putting in ounces?

Mason: Right here. This area. *[Shades top portion of bottle drawn at right in Fig. 5.]*

Interviewer: And that area that you just shaded. Where would that be on the graph?

Mason: This area. *[Runs marker along the concave down portion of the graph shown in Fig. 2.]*

Interviewer: And why?

Mason: Because it's arcing and it's putting less ounces in but it's taking up a lot of height. *[Runs marker back and forth along the height axis of the graph shown in Fig. 2.]* So it's filling up the height faster but it's not putting in as much volume as it did down here. *[Points to base of bottle drawn at right in Fig. 5.]*

Fig. 5 Mason's bottle sketches.

Using comparisons between associated amounts of increase in height and volume, Mason sketched each bottle shape in sections. When the quantity that had been increasing more (or less) than the other now began to increase less (or more) than the other, Mason created a new section. Mason needed two sections for the bottle represented by the graph in Fig. 2. In the first section (from the base of the bottle to when the height is ~3 inches), Mason indicated that volume was increasing faster than the height (e.g., "it is taking up more volume than the height here," line 8). In the second section (from the point when the height is ~3 inches to the top of the bottle), Mason indicated that height was increasing faster than the

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volume (e.g., "it's filling up the height more than it's putting in ounces," lines 20-21).

By comparing associated amounts of change in height and volume, Mason could determine whether height or volume would be increasing "more" than the other (e.g., "filling up the height more than it's putting in ounces," lines 21-22). Mason envisioned the bottle as filling in a particular way throughout sections that he determined based on his comparisons of increases in volume and height: If an amount of increase in height were greater (or lesser) than an amount of increase in volume in a section, then height would be increasing faster (or slower) than volume in that section. Likewise, if an amount of increase in volume were greater (or lesser) than an amount of increase in height in a section, then volume would be increasing faster (or slower) than height in that section. Although Mason noticed curvature, as indicated by his comment regarding a change in "arcing" of the graph (line 34), he did not create a third section to distinguish between the linear and concave down portions of the graph. Because the increase in the amount of height as time elapsed exceeded the increase in the amount of volume as time elapsed for both the linear and concave down portions of the graph, Mason treated the parts of the bottle represented by those portions as a single section.

Interviewer: And originally you drew a bottle that looked pretty different.
Mason: Yeah.
Interviewer: What, what made you change?
Mason: Because I wasn't looking at the numbers. I was just looking at the way the graph went. And once I looked at the numbers *[Runs marker along the height, then volume axis of the graph shown in Fig. 2.]* and the way the graph was going it made me change my mind on how the bottle would look.
Interviewer: So when you said you weren't looking at the numbers, could you tell me what you looked at with the numbers that helped you to change your, to change what you were working with?
Mason: I was looking at how much height *[Runs finger along the height axis of the graph shown in Fig. 2.]* it took and it only went to three but it put a lot of volume *[Points to the volume axis of the graph shown in Fig. 2.]* in just those three inches of height. That means the bottle is wider because it's putting in more volume and it's not taking up as much space so it must be a wider bottle at the bottom. And then the top part is putting in less volume and more inches which I take to mean that it's filling up the height faster than it's, and it doesn't have to put in as much volume.

Initially, Mason had focused on the shape of the graph show in Fig. 2, rather than on relationships between the quantities of height and volume. Mason's decision to draw a new bottle resulted from associating amounts of change in height and volume: "I wasn't looking at the numbers. I was just looking at the way

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the graph went." (lines 44-45). When Mason elaborated on what he meant by "looking at the numbers," he gestured to both the height and volume axes, moving his hand from right to left, then up and down (lines 46-47). Mason's hand motions provide evidence that "looking at the numbers" entailed attending to the quantities of height and volume.

The first bottle Mason drew was shaped similarly to the graph shown in Fig. 2, with the left contour of the first bottle that Mason drew (Fig. 5, left) resembling the shape of the curved portion of the graph shown in Fig. 2. When explaining the reason for sketching a new bottle shape, Mason's gestures suggest that he was attending to the changing quantities of volume and height being related by the given graph (e.g., gesturing to the height and volume axes when making comparisons between amounts of change in height and volume in a section, lines 54-57). By comparing amounts of change in associated quantities, Mason interpreted the given graph as a relationship between quantities rather than as a picture of a bottle.

Like Austin, creating sections of the bottle in which he could compare amounts of change in volume with amounts of change in height was essential to Mason's work. Unlike Austin, who envisioned change *as having occurred* within a section, Mason envisioned change *as occurring* throughout a section (e.g., "putting in less volume and more inches," line 61). Despite Mason's envisioning of change as occurring throughout a section, the essential characteristic of the section was the amount of change that eventually occurred. Further, as did Austin, when Mason created sections, the height of the section was secondary to the comparison between amounts of change in height and volume that could be made.

5.3 Sketching a graph that related volume and height for bottles of varying width: Jacob

When prompted to draw something to represent what he meant by his statement: "slope, almost, that's steady," Jacob drew a graph that related volume and height for a filling bottle of varying width. In prompting Jacob "Could you draw that for me?" my intent was for him to draw what he meant by a "steady" slope. Instead he responded to my prompt by sketching a graph that represented a relationship between the volume and height of the liquid in the bottle. Because Jacob decided to sketch a graph for bottle A, I adapted the interview in two ways: (1) by not providing him the pre-drawn sketches (e.g., sketch shown in Fig. 4) that other participants received and (2) by asking him to sketch graphs for bottles B, C, D as well as A (Bottles shown in Fig. 1). Transcript included in this section involves Jacob's sketching of graphs representing the height and volume of liquid filling into bottles A and C, respectively.

Interviewer: So if you were to take a look at bottle A, and to talk to me about how the volume of soda in that bottle is changing as the height increases, same scenario where it's being filled from the dispenser and soda's being poured in at a constant rate.
Jacob: Um, if this is all the same shape going up, so it'd be like— like a slope almost, that's steady.
Interviewer: Could you draw that for me?

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Jacob: Mhmm.
Interviewer: Here's a little piece of paper.
Jacob: Okay. Do I have something to write with?
Interviewer: Yes.
Jacob: Okay. And, I'll just say it's going up this much over here. *[Draws the leftmost part of graph A, shown in Fig 6. Has not yet labeled axes.]* What goes on the side is it volume, or?
Interviewer: What would you put on the side?
Jacob: I don't know. Okay, volume it is. And then down here it'll be height. I think it's supposed to be the other way. Either way it's getting, the volume's getting just as high as the height is as you're down here near at base of it. But once you get up here to the neck *[Points to bottle A, bottle shown in Fig. 1.]* then the height starts getting, getting bigger. *[While talking, draws the rightmost part of graph A, shown in Fig. 6.]* The height's still getting bigger but the volume is not getting as big. And then finally the height and volume just close off cause there's no more.
Interviewer: And would you just write letter A next to that?
Jacob: Okay.

Fig. 6 Jacob's sketch of a graph that could represent the filling of bottles A (left) and C (right)

When Jacob began sketching a graph that could represent a relationship between the changing quantities of height and volume for bottle A, he sketched the leftmost linear portion of the graph (Fig. 6, left) before labeling the axes. Noteworthy is his question regarding which quantity—height or volume—should be placed on the vertical axis (lines 15-16). As the interviewer, I purposefully asked him which he would choose (line 17), and he chose volume. Although Jacob questioned his choice of axes (line 19), he went ahead with the labeling. Importantly, his graph would have looked the same at this point regardless of the axis on which he chose to label volume ("Either way it's getting…" lines 19-20).

To sketch a graph for bottle A, Jacob made purposeful associations between amounts of change in height and volume in different sections of the bottle. Jacob sketched a piecewise linear graph with two sections (Fig. 6, left), the cylindrical portion, where "the volume's getting just as high as the height is as you're down here near at base of it" (lines 20-21), and the "neck" of the bottle, where "the height's still getting bigger but the volume is not getting as big" (lines 25-26). This is not to say that Jacob considered any cylindrical bottle to have amounts of change in height to be equal to amounts of change in volume. When working on prompt 3, Jacob drew a graph with a small positive slope to represent a narrow bottle. It may have been that the shape of the bottles shown in Fig. 1 made it seem as if height and volume were increasing by equal amounts. Alternatively, Jacob may have needed the cylindrical sections to have amounts of increase in height be equal to amounts of increase in volume to make distinctions between cylindrical and curved sections of the bottles.

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Interviewer: Cool. And here's bottle C.

Jacob: Okay. We'll just leave them the same on the sides so they match. *[Draws a new pair of axes and labels volume on the vertical axis and height on the horizontal axis of graph C shown in Fig. 5.]* Okay. C, C's like it starts out skinny and then gets fatter here. *[Moves fingers along the height of bottle C, bottle shown in Fig. 1.]* So we'll say the height's getting bigger but the volume's not getting too—Could that mean the volume's getting? Yeah. So the volume's getting bigger than the height. I think. Yeah. And then it goes the other way I think. Wow. This is the most confusing one we've had yet for me at least. Cause it, the height's getting bigger, but not as much is going in it, then it gets fatter again so then it goes up a little bit, then it goes over again. *[While talking, draws graph C, shown in Fig. 6.]* That is a really confusing graph. It's like this. *[Draws the line part of the sketch to the upper right of graph C, shown in Fig. 6.]*

Interviewer: And when you say, "it's like this" could you tell me what that means again in terms of the bottle?

Jacob: Okay this counts for the part where it goes like this at the very bottom *[Draws a picture of part of the bottle next to the first section of the sketch to the upper right of graph C shown in Fig. 6.]* where it's getting, right here. *[Points to bottle C.]* From here to here, where it, there it's getting more volume in it but not that high. This part describes where it gets skinny going up but it's still getting filled so it's like this. *[Draws a picture of part of the bottle next to the second section of the sketch to the upper right of graph C shown in Fig. 6.]* This little part here describes just this tiny portion of where it gets a little wider. *[Draws a picture of part of the bottle next to the third section of the sketch to the upper right of graph C shown in Fig. 6.]* And then this part shows that it gets skinnier again. *[Draws a picture of part of the bottle next to the fourth section of the sketch to the upper right of graph C shown in Fig. 6.]*

Interviewer: And can you tell me why that shows that on the graph?

Jacob: Yeah cause, cause the height's, the height's going the whole time. *[Moves a marker along the height axis.]* But the volume, *[Points to the volume axis with marker.]* you're getting more volume in the fatter parts than you are in the skinny parts. So whenever you're, when you're putting soda in the fatter parts then there's more volume as compared to height. Then for this part *[Points to the second section of the sketch to the upper right of graph C shown in Fig. 6.]* like the skinny parts you're not getting as much soda in there but it's still getting higher so you get less volume more height. Here *[Points to the third section of the sketch to the upper right of graph C shown in Fig. 6.]* you're getting more soda filled in there so you're getting more

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volume, more height still. This one *[Points to the fourth section of the sketch to the upper right of graph C shown in Fig. 6.]* you're getting less volume, more height.

Jacob sketched a piecewise linear graph with four sections (Fig. 6, right), making distinctions between the sections depending on whether height or volume would be increasing more than the other. When a bottle had curved or slanting portions, Jacob used the width of the bottle to make comparisons between amounts of increase in height and volume. When the bottle had a narrowing width, he interpreted the amounts of increase in volume to be less than the amounts of increase in height (e.g., "when you're putting soda in the fatter parts then there's more volume as compared to height," lines 70-72). When the bottle had a widening width, he interpreted the amounts of increase in volume to be greater than the amounts of increase in height (e.g., "you're getting less volume, more height," line 81). Although a literal interpretation of "getting less volume" could mean that volume is diminishing, Jacob's comparison (lines 70-72) suggests that he was focusing on amounts of increase in both volume and height.

When representing a new section of the bottle, Jacob altered the slant of each linear segment. In the sections when he determined the amounts of increase in volume to be the same as the amounts of increase in height, the slope of each linear segment appeared to be close to one. In the sections when he determined the amounts of increase in volume to be greater than the amounts of increase in height, the slope of each linear segment appeared to be greater than one. In the sections when he determined the amounts of increase in volume to be less than the amounts of increase in height, the slope of each linear segment appeared to be less than one.

Despite having worked with nonlinear graphs on other tasks, Jacob did not sketch graphs with nonlinear portions for any of bottles A, B, C, or D. Further, when presented with the nonlinear graph shown in Fig. 2, Jacob did not indicate that the graphs he had drawn for bottle A, B, C, and D could have been nonlinear. Whether interpreting nonlinear graphs or sketching piecewise linear graphs, Jacob associated amounts of change in height and volume, then made comparisons between those amounts. The piecewise linear graphs Jacob created illustrate results of such comparison. It is as if Jacob were imagining a graph being "pulled" toward the axis that represents the quantity with a greater amount of increase. For Jacob, linear segments were sufficient to make distinctions between three different ways in which amounts of change in volume could relate to amounts of change in height in a section (same, greater, or less).

Jacob labeled the axes for graph A, representing height and volume on the horizontal and vertical axes, respectively. However, when sketching a graph for bottle C he did not immediately label the axes in the same way. His statement, "We'll just leave them the same on the sides so they match" (lines 31-32) suggests that his labeling of graph C was more to preserve a convention he had established than to indicate a particular kind of relationship between quantities. Importantly, were Jacob to have changed the labeling of the axes, he still could have compared amounts of change in height and volume of liquid, but he would have needed to interpret the linear segments differently. If height and volume were represented on the vertical and horizontal axes, respectively, a steeper linear segment would have indicated a section in which an amount of change in height was greater than an amount of change in quantity.

Like Austin and Mason, at the heart of Jacob's reasoning is the comparison of amounts of increase in the quantities changing together. Like Mason, Jacob seemed to envision change *as occurring* throughout a section (e.g., "like the skinny parts you're not getting as much soda in there but it's still getting higher," lines 74-75). Like Mason, despite Jacob's envisioning of change as occurring throughout a section, the essential characteristic of the section was the amount of change that eventually occurred. Further, as did Austin and Mason, when Jacob created sections, the size of the section was secondary to the comparison between amounts of change in height and volume that could be made.

6 Discussion

6.1 Interpreting a graph as a relationship between quantities

Researchers (Carlson et al., 2002; Saldanha & Thompson, 1998) have identified that students can associate quantities to make sense of situations involving quantities that change together. This research extends previous work by developing accounts of how students are engaging in the association. Such accounts can provide insight into how students interpret and create Cartesian graphs. It is known that students interpreting Cartesian graphs may make sense of the graphs as a picture rather than as a relationship between quantities (Leinhardt et al., 1990). The account of Mason's reasoning suggests that interpreting a graph as an association of amounts of change in quantities might foster a student's interpretation of an unfamiliar graph as relating changing quantities rather than as a picture of a situation. Further, it is known that students working on the bottle problem might sketch piecewise linear graphs when nonlinear graphs would be appropriate (e.g., Carlson et al., 2002). The account of Jacob's reasoning suggests that when a student draws a piecewise linear graph to represent a situation modeled by a nonlinear graph, associating amounts of change in covarying quantities could be at the root of the activity.

6.2 Together, yet separate

Students' association of quantities involved in rate of change suggests that an identical groups conception of ratio could also extend to a conception of rate. Central to an identical groups conception of ratio is envisioning ratio as a purposeful association of quantities, rather than a single entity in and of itself (Heinz, 2000). The account of Austin's reasoning illustrates how a student can use a single number to indicate a constant rate of change represented by a graph of a linear function, yet still be considering the number as representing an association of quantities rather than a single entity. This is to say that when association is at the root of a relationship between changing quantities, it seems unlikely that students would consider rate of change as a single entity.

6.3 Envisioning change in covarying quantities

Envisioning change as occurring in completed chunks is less powerful than envisioning change as a continuing process (Castillo-Garsow et al., 2013). Austin,

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Mason, and Jacob attended to completed amounts of change as they associated amounts of change in height and volume for different sections of a bottle. Even though Mason and Jacob envisioned change in height and volume as occurring in a continuing process throughout a section, they focused on amounts of change that resulted at the end of each section. Further, the type of comparison between the amounts of change in the quantities (more, less, or the same) impacted students' choice of sections.

Students' reasons for choosing sections seem likely to impact how students might come to understand instantaneous rate of change. To support students' move from conceiving of average rate of change to conceiving of instantaneous rate of change, instruction that involves graphs can include opportunities for students to shrink the length of a given interval. However, even when a student has shrunk the length of an interval, he could still make comparisons between amounts of change in quantities. Because a student continuing to make comparisons could remain focused on average, rather than instantaneous rate of change, it may be that a root for reasoning about instantaneous rate of change is something other than envisioning change as having occurred in completed chunks.

7 Concluding Remarks

If a student's conception of rate of change involves a reflective abstraction of her conception of ratio (Thompson, 1994), then it would seem that a student reflectively abstracting a ratio as identical groups conception would be constructing a fundamentally different conception of rate than would a student reflectively abstracting a ratio as measure conception. This is not to say that it would be impossible for a student conceiving of ratio as a purposeful association of quantities (identical groups conception) to conceive of rate of change as a single quantity. However, it seems reasonable that a student with a ratio as identical groups conception would also conceive of rate of change as an association of quantities. Although researchers have made distinctions between essential characteristics of ratio and rate (e.g., Confrey & Smith, 1995; Lesh, Post, & Behr, 1988), less is known about how students' conceptions of ratio might be related to students' conceptions of rate. Future research investigating connections between students' conceptions of rate and ratio seems worthwhile for investigating how students might come to conceive of rate of change as a single quantity.

When students associated amounts of change in quantities involved in rate of change they were able to compare amounts of change in height and volume within various sections and to differentiate between sections based on those comparisons. Although such comparison is useful, it has limitations. If students compare amounts of change within a section, then for any given section, one quantity will always be changing more, less, or the same as another quantity. Further, even if students were to create new sections of lesser height they would still engage in comparison between quantities in each of those new sections. When students make comparisons between amounts of change in covarying quantities, there does not seem to be a need to account for variation in the intensity of change in a quantity (e.g., a rate that is increasing at a decreasing rate). Therefore, attending to variation in the intensity of a change seems to require reasoning rooted in something other than comparison of amounts of change in quantities.

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One such way of reasoning is the consideration of covarying quantities such that one quantity varies with respect to a related covarying quantity (Johnson, 2012). The ways in which students relate covarying quantities involved in rate of change point to differing conceptions of rate of change that extend beyond the ECA of rate identified by Herbert and Pierce (2012). The further investigation of students' reasoning about quantities involved in rate of change could inform learning trajectories focused on students' development of understanding of rate of change.

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[1] In using variationally, I intend to communicate an envisioning of variables as varying, not to distinguish between variation and covariation.

[2] This choice, however, does not preclude the possibility of a learning effect.

[3] By task, I mean a problem that has been purposefully designed for a particular audience (Sierpinska, 2004).

[4] Due to scheduling constraints, Jacob was interviewed twice during one week.

[5] For more detail regarding students' work on other tasks, see Johnson (2010).

[6] See Johnson (2010) for a more comprehensive discussion.