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TASK DESIGN: FOSTERING SECONDARY STUDENTS' SHIFTS FROM VARIATIONAL TO COVARIATIONAL REASONING

Heather Lynn Johnson

University of Colorado Denver

Covariational reasoning is essential for secondary students, yet little is known about its development. Reporting on a study with five ninth grade students (~15 years old), this research documents a student's shift from variational to covariational reasoning. Recommendations for task design include: (1) Incorporate dynamic representations that can provide students' opportunities to attend to multiple changing quantities. (2) Include nontemporal quantities from the same measure spaces. (3) Provide students engaging in variational reasoning opportunities to interact with students engaging in covariational reasoning when making sense of task situations.

Despite the pervasiveness of the concept of change in the study of mathematics and science, secondary students may not form and interpret relationships between changing quantities—engage in covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)—when reasoning about rate of change (e.g., Lobato, Ellis, & Muñoz, 2003) or interpreting graphs (e.g., Leinhard, Zaslavsky, & Stein, 1990). If students consistently engaged in covariational reasoning, their conceptions of rate of change would be more robust (e.g., Carlson et al., 2002; Thompson, 1994). However, students may engage in variational reasoning—envisioning only one changing quantity—when interpreting situations involving multiple changing quantities (Johnson, 2013). Yet, little is known regarding how students might shift from variational to covariational reasoning.

Dynamic computer environments are useful for investigating students' reasoning about changing quantities (e.g., Kaput & Roschelle, 1999), and secondary students have demonstrated positive affect when interacting with a dynamic computer environment (SimCalc Mathworlds) that incorporated time as one of the changing quantities (Schorr & Goldin, 2008). However, few environments incorporate changing quantities such that neither is time (nontemporal quantities), for example volume and height of liquid in a filling bottle (Thompson, Byerly, & Hatfield, 2013), which can provide students opportunities to form and interpret relationships between changing quantities (Johnson, in press).

In Spring 2014, using a dynamic computer environment involving a turning Ferris wheel, I conducted a small-scale, exploratory study investigating five ninth grade students' reasoning. Employing design experiment method (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), building from tasks I designed and piloted (Johnson, 2013, in press), I investigated the following questions: How do secondary students shift from variational to covariational reasoning when interacting with dynamic

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computer environments that involve nontemporal changing quantities? What design aspects of mathematical tasks foster such a shift in reasoning?

THEORETICAL FRAMING: STUDYING SHIFTS IN REASONING

When studying students' shifts in reasoning, I investigate changes in the focus of students' attention. For example, when making sense of a situation involving a bottle filling with liquid, a student may shift from attending to only the changing height of the liquid (variational reasoning) to attending to both the changing height and volume of the liquid (covariational reasoning). I distinguish between a shift in a student's reasoning (a change in a student's focus of attention) and a student's learning of a new mathematical idea (a change in a student's understanding). In particular, I am not suggesting that a student who has shifted her reasoning has developed new conceptual structures, but I do argue that shifts in students' reasoning could play a role in students' learning of new mathematical ideas. For example, to come to understand rate of change as a single entity that represents a relationship between varying quantities, a student engaging in variational reasoning would need to shift to covariational reasoning.

To theoretically frame this inquiry, I coordinate constructivist and sociocultural perspectives (Cobb, 1994). Drawing on a constructivist perspective, my unit of analysis is individual students' reasoning, with reasoning referring to purposeful mental activity in which an individual could engage (Piaget, 1970). Drawing on a sociocultural perspective, I account for conditions (e.g., task design principles) that could foster shifts in students' reasoning (Cobb, 1994), explaining how tasks might be designed and small group instructional settings might be organised to provide students opportunities to shift their reasoning.

WHAT WOULD A SHIFT FROM VARIATIONAL TO COVARIATIONAL REASONING ENTAIL?

When students shift from variational to covariational reasoning, tasks or task situations that, from a student's perspective, once involved only variation (one changing quantity) now involve covariation (quantities changing together). By quantity I mean an individual's conception of a measurable attribute of an object (Thompson 1994), which is not synonymous with determining a particular amount of measure. For example, one can envision measuring the height from the ground of a Ferris wheel car without actually determining particular amounts of height.

Shifts from variational to covariational reasoning can occur within tasks, across tasks, or across task situations. By tasks I mean problems that are purposefully designed for a particular audience (Sierpinska, 2004). By task situations I mean common experiences in which students have might have engaged or which students could envision occurring (e.g., riding a Ferris wheel or "filling" shapes with area), used to unite multiple tasks. Task situations I have used include filling bottles (Johnson, in press), shapes "filling" with area (Johnson, 2013), and a turning Ferris wheel.

DESIGNING THE FERRIS WHEEL ENVIRONMENT

To provide students opportunities to form and interpret relationships between changing quantities, using Geometer's Sketchpad Software (Jackiw, 2009), I designed a dynamic computer environment that incorporated dynamically linked representations of nontemporal quantities (Figures 1 and 2). The Ferris wheel environment links an animation of a Ferris wheel and a dynamic Cartesian graph. To depict a Ferris wheel, I used a circle containing an active point, representing a car on the Ferris wheel. Represented quantities on the Ferris wheel animation (Figures 1 and 2, left) included *distance* the car travelled around the Ferris wheel (arc length, shown in Figures 1 and 2), *height* from the ground (vertical distance shown at left in Figure 1), and *width* from the center (horizontal distance shown at left in Figure 2).

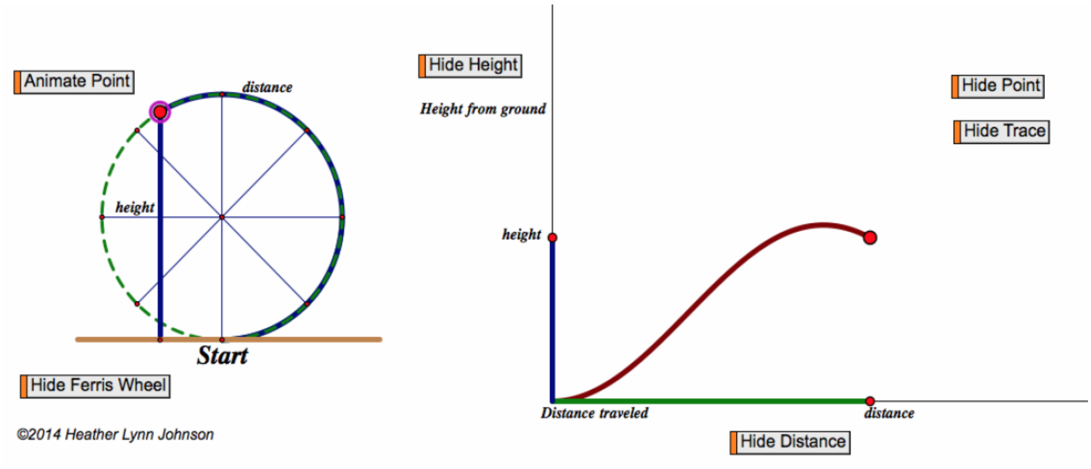

Figure 1. The Ferris wheel: Distance and height

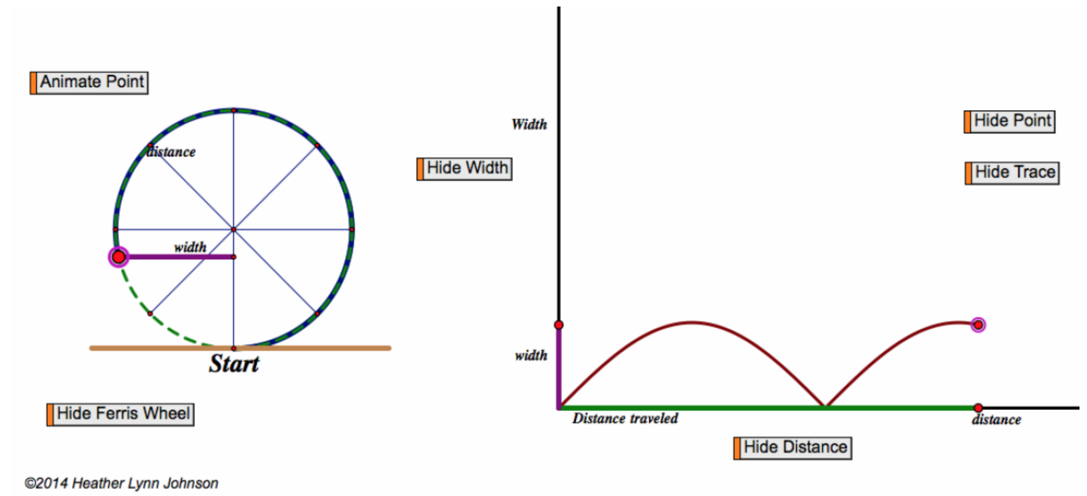

Figure 2. The Ferris wheel: Distance and width

To interact with the Ferris wheel environment, students could press Animate Point to move the car (active point) around the Ferris wheel or they could click and drag the car to control the motion. As a student moves the car around the Ferris wheel, the lengths representing distance and height (Figure 1, left) or distance and width (Figure 2, left) on the Ferris wheel animation dynamically change.

Linked to the Ferris wheel animation is a dynamic Cartesian graph containing added features not seen on typical Cartesian graphs. On each axis, a dynamic segment represents changing *distance* (green segment on horizontal axis; Figures 1 and 2), *height* (blue segment on vertical axis; Figure 1), or *width* (purple segment on vertical axis; Figure 2). Although the Cartesian graphs in Figures 1 and 2 show both the trace and the moving point, either can be shown separately. Notably, the Ferris wheel and graph can be hidden or shown to allow students to *make predictions without seeing the actual motion*—a key design feature of tasks fostering students' reasoning about changing quantities (Johnson, 2013).

RESEARCH METHODS

Researchers have described design experiments as "test-beds for innovation" (Cobb et al., 2003, p. 10). A goal of design experiment research is to develop theory that is closely tied to practice. Through this exploratory study I intended to (1) develop empirically based explanations regarding how students might shift from variational to covariational reasoning and (2) hypotheses regarding the design of tasks that might foster such a shift in reasoning.

Setting

Gutiérrez (2008) called for research that avoids focusing on gaps between groups of students from different races or socioeconomic statuses, but rather focuses on complexities within a group of students. I conducted this exploratory study at a 6-12 neighborhood school, serving primarily Mexican-American students, in a working class community in an industrial area of a large midwestern city in the United States. In 2013-14, 97.8% of students were eligible for free and reduced lunch and 96.4% of students were nonwhite. I have partnered with this neighbourhood school since 2012, having developed relationships with administrators, teachers, administrative staff, and students in the school. Although I am not from the community that the school serves, my longstanding relationship with stakeholders at the school has demonstrated my intent to collaborate in mutually beneficial ways that can support students' development of robust mathematical reasoning—a critical resource that students can carry with them beyond the bounds of a mathematics classroom or research study.

Task design and sequencing

I drew on variation theory (Marton & Booth, 1997) when designing and sequencing tasks and task situations through which students could experience differences that could provide them opportunities to change the focus of their attention. I incorporated two different task situations, the Filling Bottle and the Ferris wheel, with the Filling Bottle situation involving quantities from different measure spaces (height and volume) and the Ferris wheel situation involving quantities from the same measure space (distance and height or width). Within the Ferris wheel task situation, I incorporated different quantities (distance and height, distance and width), represented quantities on different axes of a Cartesian graph (e.g., distance represented on horizontal and vertical axes), and changed the orientation of the axes on the Cartesian graph (axes opening left rather than right).

Data Collection

I conducted a series of six clinical interviews with individuals, pairs, or trios of students. Table 1 shows the task situations, *tasks*, and represented quantities.

Interview	Task Situations/*Tasks*
1	Filling Bottle: *Volume and Height;* Cartesian Graph: vertical axis (volume), horizontal axis (height)
2	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (height), horizontal axis (distance)
3	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (height), horizontal axis (distance)
4	Ferris wheel: *Width and Height*; Cartesian Graph: vertical axis (width), horizontal axis (distance)
5	Ferris wheel: *Distance and Height*; *Width and Height*; Cartesian Graph: vertical axis (distance), horizontal axis (height and width, respectively)
6	Ferris wheel: *Distance and Height;* Cartesian Graph: vertical axis (distance), horizontal axis (height), with axes opening left Filling Bottle: *Volume and Height;* Cartesian Graph: vertical axis (volume), horizontal axis (height)

Table 1. Task Situations, by Interview

Each of the five students participating in the study was a ninth grade student (~15 years old) enrolled in an Algebra course, which was typical for ninth grade students at the school where I conducted this research. For each task in the Ferris wheel task situation, I implemented a five-part task sequence, shown in Table 2.

Part	Task Description: Ferris Wheel Task Situation
1	Predict then view how quantities change in the Ferris wheel animation.
2	Without viewing dynamic Cartesian graph, sketch a graph that represents a relationship between quantities in the Ferris wheel animation (distance and height or distance and width).
3	Predict then view how vertical and horizontal segments shown on the dynamic Cartesian graph relate to quantities in the Ferris wheel animation.
4	With the Ferris wheel hidden and only the moving horizontal and vertical segments showing on the Cartesian Graph, predict the car's location on the Ferris wheel.

5	Compare graphs sketched in Part 2 with the trace shown on the dynamic Cartesian graph.

Table 2. Description of the five parts of each task in the Ferris wheel task situation

Data analysis

Data analysis encompassed both ongoing and reflective analysis. Ongoing analysis, including reflective notes compiled after each interview, informed future interviews. I conducted multiple passes of analysis. In the first pass, I used open coding (Corbin & Strauss, 2008) to identify and describe data when students were focusing on change in one quantity (variation) or coordinating change in quantities (covariation), attending to the types of quantities, the prompts I used, and the interaction between students. In subsequent passes, I used comparative analysis, examining data when shifts in reasoning seemed likely to occur (e.g., parts of tasks that have potential to problematise the use of only one quantity to make predictions), then looking across all tasks for each student to trace shifts in students' reasoning within and across tasks.

RESULTS: A PROMISING EMPIRICAL FINDING

Prior to implementing the Ferris wheel task situation, I had not documented a student shift from variational to covariational reasoning in an empirical research study. To illustrate, I share data from Lucia and Sofia's work in Part 4 of Interview 4. I prompted Sofia to hide the Ferris wheel, choose when to stop the moving segments, then ask Lucia to predict the car's location. Lucia (Figure 3, right) predicted the car would be on the right side of the Ferris wheel, just before the width would have reached its longest amount. When prompted to explain, Lucia responded: "Cause, in the graph it's (purple segment representing width, Figure 2) like going up." Sofia (Figure 3, left) predicted the car would be on the left side of the Ferris wheel, just before the width would have reached its longest amount.

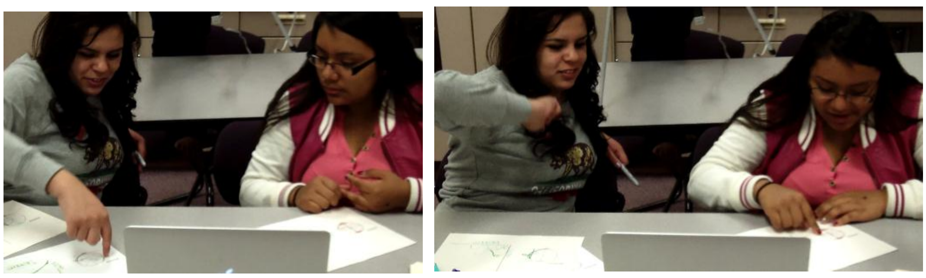

Figure 3. Sofia's prediction (left); Lucia's prediction (right)

When prompted to explain, Sofia responded: "Because the distance is really great here, and this distance (points to location Lucia predicted) is shorter." Next, I suggested we show the Ferris wheel, and after seeing the car's location, with a smile Sofia said: "See, I told you." Lucia grinned in response, moving her index finger up and down (Figure 4) and saying: "I basically focused on that (purple segment representing width, Figure 2)."

Figure 4. "I basically focused on that."

Once Lucia conceived of the task as involving multiple changing quantities, she engaged in covariational reasoning on other tasks involving the turning Ferris wheel. Specifically, she focused on how *both* the horizontal and vertical segments were changing (covariation), rather than focusing on how only one segment was changing (variation). Importantly, Lucia's shift provides empirical evidence of a student's shift from variational to covariational reasoning in a small group interview setting.

TASK DESIGN PRINCIPLES

I argue that three key design principles contributed to a student's shift in reasoning. First, incorporating *graphs with dynamic segments* (e.g., vertical and horizontal segments on graphs in Figures 1 and 2), drew students' attention to two changing quantities rather than just one. Second, incorporating *changing quantities from the same measure spaces* (e.g., height and distance) provided richer opportunities for students to attend to multiple changing quantities than did tasks incorporating changing quantities measured with different kinds of units (e.g., height and volume in the filling bottle task situation). Third, *pairing a student engaging in variational reasoning* (e.g., Lucia) *with a student engaging in covariational reasoning* (e.g., Sofia) provided students opportunities to discuss different ways in which they were making sense of the situation, thereby fostering a shift from variational to covariational reasoning (cf., Vygotsky, 1978).

IMPLICATIONS

When students engage in covariational reasoning, it expands not only their mathematical horizons, but also their ability to make sense of change in science and social science (e.g., the unemployment rate is decreasing more rapidly in 2015 than in 2014). Promoting students' covariational reasoning can support their success in algebra and open doors of opportunity that might otherwise have been closed. In fact, during the Spring 2014 study, Sofia said that working on the Ferris wheel tasks helped her to make sense of algebra problems in new, useful ways. Important, such tasks have potential to foster students' study of mathematics as an investigation of relationships between quantities rather than a pursuit of answers. The study of relationships, not the finding of answers, imbues students with mathematical power.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*(7), 13-20.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). London: Sage Publications.

Gutiérrez, R. (2008). A "gap-gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4),357-364.

Jackiw, N. (2009). *The Geometer's Sketchpad (Version 5.0) [Computer Software]*. Emeryville, CA: Key Curriculum Technologies.

Johnson, H. L. (2013). Designing covariation tasks to support students reasoning about quantities involved in rate of change. In C. Margolinas (Ed.), *Task design in Mathematics Education*. *Proceedings of ICMI Study 22* (Vol. 1, pp. 213-222). Oxford

Johnson, H. L. (in press) Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1).

Kaput, J. J., & Roschelle, J. (1999). The mathematics of change and variation from a millenial perspective: New content, new context. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 155-170). London: Falmer Press

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning and teaching. *Review of Educational Research, 60*(1), 1-64.

Lobato, J., Ellis, A. B., & Muñoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning, 5*(1), 1-36.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum Associates.

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press.

Schorr, R. Y., & Goldin, G. A. (2008). Students' expression of affect in an inner-city simcalc classroom. *Educational Studies in Mathematics, 68*(2), 131-148.

Sierpinska, A. (2004). Research in mathematics education through a keyhole: Task problematization. *For the Learning of Mathematics, 24*(2), 7-15.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: New York Press

Thompson, P. W., Byerly, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in Schools, 30*, 124-147.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.