TITLE

Quantitative Reasoning in Mathematics Education: Directions in Research and Practice[1][2]

AUTHOR

Heather Lynn Johnson

University of Colorado Denver

Heather.johnson@ucdenver.edu

Quantification and measurement are distinct processes at the heart of quantitative reasoning (QR) and quantitative literacy (QL), respectively. QR is useful for studying key mathematical concepts and plays an important role in students' engagement in algebraic reasoning. Research-based tasks fostering QR K-16 are being developed and implemented, and learning trajectories are beginning to address how students' QR might develop. Although QR is present in some educational policy and curricular documents, greater emphasis is needed. A focus on quantification and measurement could forge links between mathematics and science education, supporting the use of different, yet compatible lenses to investigate shared problems.

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Imagine that a bottle is being filled with water that is being dispensed at a constant rate[3]. How might one measure the amount of water in the bottle at any given time? How might the amount of water in the bottle relate to the height of the water in the bottle for differently shaped bottles? How might one relate the changing amount of water to the changing height of water? To respond to these questions, an individual would need to make sense of both the height of the water and the amount of the water as attributes that could be measured. Importantly, an individual would not necessarily need to measure (or know) particular numerical amounts to make sense of the situation. For example, dispensing water into a very wide bottle or a very narrow bottle would result in different relationships between the height and amount of water in the bottle. An individual's responses to the questions would depend on how that individual interprets the context of the problem situation (Johnson, 2014). For example, the sound of the water filling into the bottle, bottles with which an individual might be familiar, or dispensers an individual has used might play a role. These introductory questions are intended to provide examples of prompts that could foster individuals' quantitative reasoning.

This chapter stemmed from conversations that took place in the Quantitative Reasoning Working Group of the International STEM Research Symposium: Quantitative Reasoning in Mathematics and Science Education held in Savannah, GA in Spring 2012. Goals of the Working Group, facilitated by Johnson, were threefold: To identify high priority research problems and methods related to quantitative reasoning; To determine critical themes based on those problems and methods; and To identify future actions for the group. Broadly, conversations revolved around three themes: distinguishing between quantitative reasoning (QR) and quantitative literacy (QL), fleshing out interrelated constructs of quantity and quantification, and addressing

[3] This problem situation is derived from the well-known bottle problem developed by the University of Nottingham's Shell Centre (Shell Centre for Mathematical Education (University of Nottingham), 1985).

roles of context in quantitative reasoning. In the discussion that follows, each of these themes is situated within a larger discussion focusing on quantitative reasoning in mathematics education.

To organize this chapter, first relationships between the constructs of quantity and quantification are examined, a working definition of QR is provided, and distinctions between QR and QL are made. Second the role of QR in multiplicative, algebraic, and covariational reasoning is examined, arguing for a dynamic perspective of context in tasks designed to foster QR, and highlighting learning trajectories in mathematics education K-12 that foster QR. Next, the inclusion of QR in educational policy documents and practitioner resources is discussed, providing subsequent examples of research based tasks fostering students' QR in K-16 mathematics education. The chapter concludes by arguing how quantification and measurement could forge links between mathematics and science education, supporting the use of different, yet compatible lenses to investigate shared problems.

Background

Quantity and Quantification

Nearly three decades ago researchers defined a quantity as a pairing of a number and a unit (Nesher, 1988; Schwartz, 1988), meaning that quantities are numbers with additional characteristics (a unit). However, as suggested by the example of the bottle problem, a numerical amount could indicate a measure of a quantity, but a particular measure of a quantity does not need to be known to reason about the quantity (cf., Thompson, 1993). For example, an individual would not need to measure (or know) particular amounts of height and water to make relationships between them. Unlike previous definitions, Thompson (1994a) defined quantities as "conceptual entities," asserting that a quantity entails an individual's conception of some attribute of an object (e.g., height of water) as being possible to measure. Importantly, from this perspective, the entity of quantity is inextricably linked to an individual's conception of the measurability of some attribute. When referring to quantity in this chapter, it is an entity that an individual conceives as being possible to measure.

While a quantity is an object, quantification is a process. Thompson (2011) defined quantification as a three part process, involving "settling what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (p. 38). For example, quantification related to the bottle problem would involve deciding what it would mean to measure amounts of height and water, what one would measure to determine amounts of height and water, and what measures of amounts of height and water would mean in this situation. Building from Thompson's research, Mayes, Peterson, and Bonilla (2013) defined quantification as a mathematical process extending beyond assigning a numerical value to a quantity, involving "conceptualizing an object and an attribute of it so that the attribute has a unit measure" (p. 6). The unit of measure indicated by Mayes et al. may be a standard or non-standard unit of measure. Importantly, as illustrated by both definitions, the process of quantification is distinct from the process of measurement, with quantification setting the stage for subsequent measurement (Thompson, 2011). When referring to quantification in this chapter, it is a process of conceiving of an attribute that could be measured, deciding how one might measure that attribute (which could include selecting a unit of measure), and making meaning from measures that would result.

Quantitative Reasoning (QR)

Quantitative reasoning (QR) entails both the objects of the reasoning (quantities) and the operations (Piaget, 1970) that an individual could enact as thought processes or physical acts

(Thompson, 1994a). For example, an individual could envision what attribute the height of water in a bottle would represent or physically investigate heights of water in different bottles. In contrast, Mayes et al.'s (2013) definition of QR focuses on the how an individual might apply her knowledge: "mathematics and statistics applied in real-life, authentic situations that impact an individual's life as a constructive, concerned, and reflective citizen" (p. 6). For example, an individual could apply her knowledge to determine an ideal bottle shape given particular constraints. A key distinction between these definitions involves what is claimed to be central to QR: application (Mayes et al.) vs. quantification (Thompson). When referring to QR in this chapter, it is a process composed of operations on objects (quantities), for which quantification is central. This is not to say that application of knowledge could not be involved in QR, but that quantification should be central to QR.

Quantitative Literacy (QL)

Steen (2001) defined quantitative literacy (QL) as a "habit of mind", involving a variety of elements, including solving problems involving different contexts, making meaning from numbers, and using numbers to measure different items. More recently, the American Association of Colleges and Universities,AAC&U; (2010) defined QL as follows: "Quantitative Literacy (QL) – also known as Numeracy or Quantitative Reasoning (QR) – is a "habit of mind," competency, and comfort in working with numerical data." Notably, working with numbers plays a key role in both Steen's and the AAC&U's definitions of QL.

Some use QR and QL synonymously (e.g., American Association of Colleges and Universities, 2010; Dingman & Madison, 2010), while others identify QL as a component of QR (e.g., Mayes et al., 2013). Mayes et al. (2013) defined QL as the "use of fundamental mathematical concepts in sophisticated ways for the purpose of describing, comparing, manipulating, and drawing conclusions from variables developed in the quantification act" (p. 6). Importantly, Mayes et al.'s definition indicates that QL results from quantification, which may or may not include working with numbers. Here, it is argued that work with numbers does not capture the scope of QL, which results from quantification and does not necessitate work with numbers.

Quantitative Literacy (QL) and Quantitative Reasoning (QR)

Quantification and measurement are distinct processes at the heart of QR and QL, respectively (See Figure 1). Simply put, measurement involves actual measuring (or knowing actual amounts of measure), whereas quantification involves conceiving of how an attribute of an object might be measured and what those measures might mean. For example, an individual might be able to use numerical amounts of city and highway miles per gallon calculations to make comparisons between the fuel efficiency of different cars (measurement). Notably, fuel efficiency comparisons would not necessarily require one to envision how one might measure miles per gallon (QR), rather they would involve being able to use determined amounts of miles per gallon to make informed decisions (QL). In contrast, an individual determining what it might mean to measure miles per gallon (quantification) could form and interpret relationships between the quantities of miles and gallons. Similarly, an individual determining what it might mean to measure miles per gallon (QR) may or may not use those relationships to make informed decisions about automobile travel (QL). Figure 1 illustrates a reflexive relationship between QR and QL, such that QL results from QR, and subsequent QL could afford an individual's further engagement in QR.

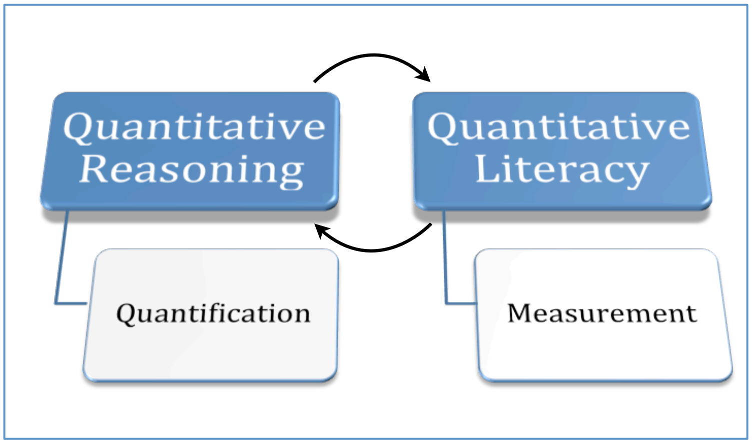

Figure 1. A reflexive relationship between QR and QL

QR and Multiplicative, Algebraic, and Covariational Reasoning

QR can form a foundation for multiplicative reasoning (Thompson & Saldanha, 2003), algebraic reasoning (Carraher & Schliemann, 2007; Mason, 2008; Smith & Thompson, 2008), and covariational reasoning (Saldanha & Thompson, 1998). Here the focus is on relationships between QR and these three forms of reasoning because of the key role multiplicative, algebraic, and covariational reasoning play in K-16 mathematics. Multiplicative reasoning is essential for students' development of algebraic reasoning (Greer, 1994). Algebraic reasoning encompasses more than reasoning about generalized arithmetic (Carraher & Schliemann, 2007), and QR can support students development of algebraic reasoning that extends beyond generalized arithmetic (Smith & Thompson, 2008). Covariational reasoning is essential for students' developing understanding of the mathematics of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002).

QR and Multiplicative Reasoning

Researchers investigating students' reasoning in the early grades have made distinctions between the types of units of which a quantity could be composed (Behr, Harel, Post, & Lesh, 1994; Steffe, 1992, 1994). Two such types of units are single units and composite units. Composite units are entities composed of single units (Steffe, 1994). For example, a tower composed of four unit cubes represents a composite unit because it is an entity composed of single units (unit cubes). To reason multiplicatively, an individual needs to be able to operate on composite units (Behr et al., 1994). For example, consider a situation involving seven towers containing four cubes in each tower. An individual reasoning multiplicatively could operate on composite units (towers) to determine the total number of cubes needed to create seven towers with four cubes in each (cf., Tzur et al., 2013). Individuals operating on composite units in this way would be using QR because they would be envisioning towers as “things” that could be measured by the number of unit cubes contained in each tower. As illustrated by this example, a strong foundation of QR could lay groundwork for operations on composite units, which are at the heart of multiplicative reasoning.

QR and Algebraic Reasoning

QR can support algebraic reasoning that may not depend on symbolic expressions and help students from the early grades through post secondary to make sense of symbolic expressions in terms of related quantities (Madison & Dingman, 2010; Smith & Thompson, 2008). For example, an individual making sense of the bottle problem might realize that for a wide bottle, the height of water would change more slowly in relation to the increasing amount of water than would the height for a narrow bottle, thereby allowing that individual to make sense of relationships represented by a symbolic expressions. To be clear, symbol manipulation is not mutually exclusive from QR. Rather, QR can provide a meaningful basis from which generalizations could emerge, and then need to be represented with abstract symbols (Smith & Thompson, 2008).

QR and Covariational Reasoning

Covariational reasoning involves reasoning about quantities that change together (Carlson et al., 2002), thereby extending QR to include forming and interpreting relationships between quantities (e.g., the height and amount of water in the bottle filling situation). Covariational reasoning can take different forms, depending on how an individual envisions the ways in which the quantities change together. Two different forms of covariational reasoning are chunky and smooth (Castillo-Garsow, Johnson, & Moore, 2013). Chunky covariational reasoning involves envisioning change as having happened (e.g., envisioning a bottle being filled in two-ounce increments, then comparing the change in the height of the liquid in each of those increments). In contrast, smooth covariational reasoning involves envisioning change as continuing (e.g., envisioning a bottle being filled from a soda dispenser and coordinating change in the height of the liquid with continuing change in the volume of the liquid). Envisioning change as continuing is more powerful than envisioning change as having occurred (Castillo-Garsow et al., 2013), in part because it affords the consideration of variation in the intensity of a change (Johnson, 2012), and is involved in robust quantification of rate of change (Johnson, 2015a). Thus, QR could provide a meaningful foundation on which the mathematics of change could be developed.

A Role of Context in Tasks Fostering Students' QR

Tasks designed to foster students' QR have incorporated situations that have realistic contexts (e.g., Johnson, 2012; Lobato & Siebert, 2002; Moore & Carlson, 2012; Smith & Thompson, 2008). As such, asking whether or not a student is familiar with a context seems a natural question. However, students' familiarity with the situation described by a task is only one aspect of the multifaceted interactions that can occur as students work on a task. When referring to context in this chapter, a dynamic stance is taken, meaning that context is not something external to an individual, but rather that context "constitutes an individual's conception of a given situation" (Johnson, 2014, p. 339). For example, two students encountering the filling bottle problem could have different conceptions of the given situation, depending on how they conceived of quantities involved.

A dynamic stance on context proved useful for the analysis of secondary students' work on an adapted version of the bottle problem (Johnson, 2012, 2015a, 2015b). The adapted bottle problem incorporated a graph that represented the amount of liquid in the bottle as a function of the height of the liquid in the bottle. Given the graph, students were asked to discuss how the volume of liquid in the bottle was changing as the height of the liquid in the bottle was increasing. Students had different conceptions about what the filling bottle problem entailed. Some students made and interpreted relationships between the varying quantities of volume and height. Two distinct types of relationships include (1) making comparisons between associated amounts of volume and height (Johnson, 2015b) and (2) coordinating changes in volume with continuing change in height (Johnson, 2012).

A key distinction between the relationships lay in how students formed and interpreted relationships between the changing quantities—comparison or coordination (Johnson, 2015a). Students making Type 1 relationships compared amounts of change in volume and height that *occurred* on particular intervals (chunky covariational reasoning). In contrast, the student making Type 2 relationships coordinated change in volume *continuing* change in height (smooth covariational reasoning). Despite the bottle problem's intentional focus on the quantities of height and volume of liquid in the bottle, students might consider other quantities or relationships. Consider this student's response: "I'm thinking as the water bottle, you are filling it up and as the rim comes, you can hear it filling up faster" (Johnson, 2010). Although this student was working on the same printed task as students forming and interpreting relationships between height and amount of water, she drew on her auditory experiences related to filling a bottle as she wrestled with the problem of actually filling a bottle. Taken together, these student responses suggest that a student's conception of a situation represented by a task might afford or constrain his or her QR.

Students' perspectives of quantities involved in a task may explain in part why students may not seem to transfer knowledge to a novel task situation (Lobato & Siebert, 2002). Using a task requiring students to investigate the steepness of a ramp, Lobato and Siebert (2002) identified multiple quantities that a student could use to quantify the steepness of a ramp. For a student to interpret that constructing a ratio between height and length of a ramp would be an appropriate activity in this situation is neither transparent nor trivial. Because students working on the same printed task can interpret the task differently and potentially reason about different quantities, conceiving of the task context as a "textual cover story" does not account for how students might draw on QR in different task situations (Lobato & Siebert, 2002).

Working with undergraduate precalculus students, Moore and Carlson (2012) found that students' images of a problem's context were not static. When solving a problem involving constructing an open box from a sheet of paper, students' interpretations of the work to be carried out did not necessarily align with the researchers' interpretations of the work to be carried out. During interactions with a researcher, students adjusted their image of the problem's context, and their emerging images influenced their QR (Moore & Carlson, 2012). Taken together, the findings of Johnson, Lobato and Siebert, and Moore and Carlson suggest that students' perspective of the task context is nontrivial and may influence QR.

Learning Trajectories Involving QR

Beginning with Simon's (1995) articulation of a hypothetical learning trajectory, learning trajectories (LTs) have become more pervasive in mathematics education and can be interpreted in different ways (Daro, Mosher, & Corcoran, 2011). Here, when referring to LTs, Clements' and Sarama's (2004) definition is used:

> descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (p. 83).

In this section, aspects of LTs (or groundwork supporting the development of a LT) related to geometric measurement in the early grades, multiplicative reasoning in the intermediate grades, and exponential and quadratic function in the middle-early secondary grades are addressed. In doing so, the goal is to illustrate how QR can play a role in students' understanding of K-12 mathematics.

Geometric Measurement: LTs

Sarama and Clements (2009) developed LTs on how early elementary students might develop understanding of length, area, and volume measurement. QR is rooted in the LTs, each LT begins with students' identification of length, area, or volume as an attribute of an object that could be measured and then indicates how students could develop the ability to make comparisons between amounts of length, area or volume. Importantly, it is only after students engage in quantification that the LTs address how students might begin to determine particular amounts of length, area, or volume (measurement). Notably, to determine particular amounts of length, area, or volume, first students would need to conceive of length, area, or volume as composite units. Figure 2 illustrates a general sequence of the operations for LTs related to length, area, and volume measurement.

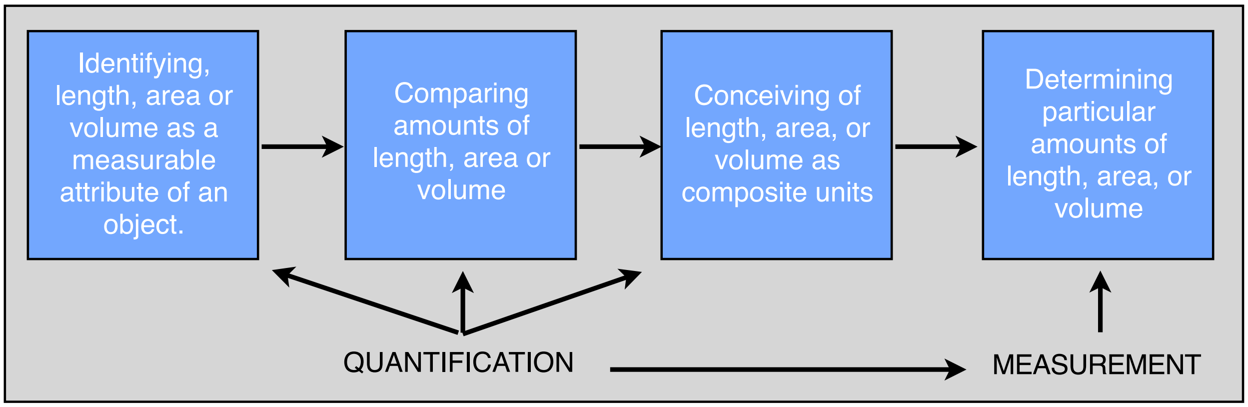

Figure 2. Sequence of operations for learning trajectories related to length, area, and volume measurement (Sarama & Clements, 2009)

Figure 2 illustrates how the LTs are rooted in students' quantification of length, area, and volume. The identifying portion of each LT involves determining what it means to measure length, area, or volume. By comparing amounts of length, area or volume a student could begin to determine what one might measure to determine an amount of length, area or volume. Conceiving of length, area or volume as a composite unit is necessary for determining what a measured amount of length, area, or volume might mean. Notably, measurement—involving the determination of particular amounts of length, area, or volume—occurs after quantification.

Multiplicative Reasoning: Laying Groundwork for LTs

Tzur et al. (2013) proposed a set of six schemes (von Glasersfeld, 1995) and operations (Piaget, 1970) related to students' development of multiplicative reasoning with whole numbers.

Schemes are goal directed mental activities that an individual could enact in anticipation of a particular result (von Glasersfeld, 1995). The schemes and operations build on students' envisioning of whole numbers as composite units. As students progress through the schemes, they develop an ability to operate on composite units. The set of schemes indicates students' operations on composite units would develop as follows: (1) coordinating equally sized composite units, (2) additively operating on composite units, (3) comparing collections of differently sized composite units, (4) coordinating additive operations on composite units with a segmenting operation on a group of single units (5) segmenting a set of objects into groups of equally sized composite units (determining the number of groups), then (6) partitioning a set of objects into groups of equally sized composite units (determining the amount in each group) (Tzur et al., 2013).

Tzur et al.'s (2013) set of schemes is rooted in students' quantification of composite unit. By developing the operations involved in this set of schemes, students could interpret whole number multiplication problems in terms of quantity. For example, the problem 7×6 could be envisioned as a unit composed of 7 single units replicated 6 times. This is not to be confused with an image of multiplication as repeated addition. Conceiving of 7 single units replicated 6 times would require an individual to conceive of 3 different units: the number of single units in a composite unit (7), the number of composite units (6), and the total number of single units (42). In contrast, repeatedly adding 7 would not necessarily require a student to consider anything other than single units. A purpose of this example is to communicate the central role of quantity in envisioning operations on composite unit. In addition, the example highlights a critical distinction between multiplicative and additive reasoning: multiplicative reasoning requires a transformation of units, while additive reasoning involves operation with the same kind of units.

Quadratic and Exponential Function: LTs

QR is an important aspect of students' reasoning about functions as relationships (Ellis, 2011a; Lobato, Hohensee, Rhodehamel, & Diamond, 2012). Recently, groundwork for a LT rooted in QR has been accomplished for quadratic function (Lobato et al., 2012), and a LT rooted in QR has been proposed for exponential function (Ellis et al., 2013).

Lobato et al. (2012) investigated 24 eighth grade students' quantitative reasoning related to quadratic function, with the goal of supporting students' understanding that the rate of change of the rate of change in a quadratic function is constant. Students participated in 15 hours of classroom instruction in a small group setting of seven to nine students and in an individual follow up task-based interview. Results of the study include five pivotal intermediate conceptions (PICs) related to quadratic function. PICs build on students' conception of quantities. The first PIC is: "Students comprehend a quadratic motion situation as containing a set of elapsed distances as quantities— entities that can be mentally operated upon (e.g. compared), interpreted in terms of their meaning in context, and assigned correct units of measure" (p. 93). Lobato et al. argued that conceiving of a quadratic situation in terms of quantities was critical for students' progress in the conceptual domain of quadratic function.

Findings reported in Ellis (2011b) provide evidence to support Lobato et al.'s argument. Using the teaching experiment methodology (Steffe & Thompson, 2000), Ellis conducted a study with six middle school students. Given a task involving a table of values of side length and area for a growing square with dimensions 1×1, 2×2, 3×3, 4×4, 5×5, and 6×6, students determined first and second differences. During a class discussion, a student used another student's diagram of a growing square to make sense of the second difference as the increase in the number of new squares being added to the growing square as the side length increased by one unit. Particularly powerful was the student's ability to relate the amount of growth in new unit squares being added to the growing square to the amount of growth in the side length. Interestingly, the student was not able to determine this relationship prior to using the diagram. By interpreting the diagram in terms of the quantities involved in the growing square (side length and unit squares), the student could determine a relationship between amounts of change in quantities.

Recently, Ellis et al. (2013) proposed a LT related to exponential function, resulting from a teaching experiment study with three middle school students. The LT indicates three types of reasoning involved in developing understanding of exponential function: pre-function, covariation, and correspondence, with the development of the different types of reasoning being not entirely sequential. Although Ellis et al. indicate that pre-functional reasoning develops prior to covariation and correspondence reasoning, they found that the covariation and correspondence reasoning developed in a non-linear fashion, and students could shift between covariation and correspondence forms of reasoning to make sense of situations involving exponential function. Central to the study were task situations involving a growing plant. Like the students interpreting the growing square reported in Ellis (2011b), these students interpreted numerical amounts in terms of measurable attributes of the growing plant. Together, this collection of LTs provides some indication of how QR might foster K-12 students' development of important mathematical concepts.

Educational Policy Documents and QR

Principles and Standards for School Mathematics

Over a decade ago, the National Council of Teachers of Mathematics (NCTM) published *Principles and Standards for School Mathematics* (PSSM) (National Council of Teachers of

Mathematics, 2000). The PSSM organizes the standards into standards for mathematical content (Number & Operations, Algebra, Geometry, Measurement, and Data Analysis & Probability) and standards for mathematical process (Problem Solving, Reasoning & Proof, Communication, Connections, and Representations). Quantitative relationships are included as one of the Algebra Standards to be addressed in K-12: “Use mathematical models to represent and understand quantitative relationships.” (p. 90). Expectations related to this standard included modeling problems using multiple representations (K-8) and identifying quantitative relationships in function situations (9-12). Quantity appears in expectations at the 6-8 level in the Number and Operation Standard: “understand and use ratios and proportions to represent quantitative relationships” (p. 214) and the Algebra Standard: “use graphs to analyze the nature of changes in quantities in linear relationships” (p. 222). Although quantity and quantitative relationships were mentioned, they were not central to the standards. Furthermore, QR is not included in a later publication that addressed high school students’ mathematical reasoning: *Focus in high school mathematics: Reasoning and sense making* (National Council of Teachers of Mathematics, 2009).

Common Core State Standards for Mathematics

The Common Core State Standards for Mathematics (CCSSM) include standards related to quantity and QR (National Governors Association Center for Best Practices Council of Chief State School Officers, 2010). The CCSSM is organized into standards for mathematical practice and standards for mathematical content, with “Reason abstractly and quantitatively” being one of the eight standards for mathematical practice. The standards for mathematical content are organized by grade level for K-8, then by conceptual categories for grades 9-12. Number and quantity is one of the conceptual categories for grades 9-12. In addition, quantities are related to number and operation in grades K-5, ratio and proportion in grades 6-8, and algebra and function in grades 8-12. Despite a greater inclusion of quantity and QR than PSSM, the CCSSM define quantities as “numbers with units,” and do not distinguish between quantification and measurement. Below are the CCSSM’s definitions for quantity, QR, and quantification:

- *Quantity.* “In real world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement.” (p. 58)
- *QR.* “Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.” (p. 6)
- *Quantification. “*In high school, students encounter a wider variety of units in modeling… They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification.” (p. 58)

As argued previously, defining quantity as a pairing of a number and unit conflates number and quantity. With such a definition, quantities do not arise without numbers. However, quantities such as length or area can be envisioned without determining particular numerical amounts. Although the CCSSM’s definition for QR distinguishes the meaning of quantities from the computation of quantities, the standards do not adequately explicate how such distinctions maybe developed. Despite the CCSSM’s example of finding an appropriate measure as part of the conceptual process of quantification, more elaboration is needed to address the scope of quantification, including attributes one would measure to arrive at those given measures, and what the measures would mean in terms of the situation. Although the CCSSM include QR, they do not emphasize the forming and interpreting relationships between quantities that change (covariational reasoning), which limits the scope of QR as a coherent thread supporting students' mathematical meaning making (Thompson & Carlson, in press).

Educational Resources Connecting Research to Practice

Connecting to the CCSSM's Standards for Mathematical Practice

In 2013, NCTM published *Connecting the NCTM Process Standards & The CCSSM Practices*, which is intended to provide a "roadmap to help teachers navigate these practices" (Koestler, Felton, Bieda, & Otten, 2013, p. ix). In unpacking the mathematical practice of reasoning quantitatively and abstractly, Koestler et al. explicitly indicate that students can reason quantitatively without assigning specific measurements to those quantities. This is a useful step for supporting teachers' making sense of quantities as more than a pairing numbers with units. Drawing on research reported in Johnson (2012) and tasks described by van Dyke and Tomback (2005), Koestler et al. argue that a common theme is "the clear emphasis on the relationships among the various quantities *before* turning attention to the specific numerical values" (p. 26). Importantly, this publication begins to elaborate teaching practices that could provide students opportunities to reason about quantities as more than just a pairing of a number and a unit.

NCTM'S Essential Understanding Series

NCTM's *Essential Understanding* (EU) series, edited by Zbiek (2010-2014), has a twofold purpose: to help teachers (1) develop their mathematical knowledge and (2) support their students' development of big ideas of mathematics. The EU series addresses mathematical topics

PreK-12 that are difficult to learn and challenging to teach, and identifies big ideas and essential understandings for the different topics. The big ideas relate to the areas of mathematics, and the essential understandings provide specifics to support the big ideas. The discussion that follows addresses a role of quantity and QR in the EU series across PreK-12.

PreK-2: number & numeration. One big idea defines number as an extension of quantity: "*Number is an extension of more basic ideas about relationships between quantities.*" (Dougherty, Flores, Louis, & Sophian, 2010, p. 10) One essential understanding related to this big idea involves comparing quantities without assigning particular numerical amounts: *"Quantities can be compared without assigning particular numerical values to them."* (p. 10) Another essential understanding related to this big idea distinguishes quantity from physical objects: *"Physical objects are not in themselves quantities. All quantitative comparisons involve selecting particular attributes of objects or materials to compare."* (p. 11) By focusing on quantities as attributes of objects (not objects themselves) and distinguishing quantities from particular numerical amounts, Dougherty et al. highlight a central role for quantity in PreK-2.

3-5: algebraic thinking. One big idea explicitly involves QR: "*Quantitative reasoning extends relationships between and among quantities to describe and generalize relationships among these quantities.*" (Blanton, Levi, Crites, & Dougherty, 2011, p. 39). One essential understanding related to QR addresses the use of comparison to make relationships between quantities. Examples indicate how comparisons can be done without assigning numerical values to quantities or determining particular amounts of quantities. Another big idea relates quantity and functional thinking: "Functional thinking includes generalizing relationships between covarying quantities..." (p. 47) An essential understanding related to functional thinking explicitly describes functions as tools for expressing covariation: *"Functions can be viewed as* *tools for expressing covariation between two quantities."* (p. 48) Researchers have argued that a covariation perspective on function is a more intuitive approach to function than a correspondence approach (Confrey & Smith, 1994, 1995; Thompson, 1994b). Blanton et al. indicate how such a view of function can be developed in the early grades.

6-8: ratio, proportion, & proportional reasoning. Two essential understandings underlying the big idea of proportionality articulate how understanding ratio involves coordinating covarying quantities and making multiplicative comparisons between covarying quantities (Lobato & Ellis, 2010). Lobato and Ellis provide examples of tasks and questions that focus on these essential understandings. For instance, students could additively or multiplicatively compare amounts of orange juice and water in different mixtures. To illustrate, consider the sample question provided by Lobato and Ellis:

> Does a batch of orange juice made with 2 cans of orange concentrate and 3 cans of water taste equally orangey, more orangey, or less orangey than a batch made with 4 cans of orange concentrate and 6 cans of water? (p. 22)

A student reasoning additively might determine that the first batch has 1 more can of water than orange concentrate, while the second batch has 2 more cans of water than orange concentrate. The student could use this to conclude (incorrectly) that the first batch is more orangey, because there is less difference between the amounts of water and orange concentrate. In contrast, a student reasoning multiplicatively might determine that doubling the first batch would result in the second batch or that both batches have 1 ½ times more water than orange concentrate, meaning that both batches would taste the same. These examples illustrate the importance of how students make relationships between amounts of quantities—to reason meaningfully about ratios, students need to make multiplicative rather than additive comparisons.

9-12: functions. One big idea characterizes functions in terms of relationships between quantities: *"Functions provide a means to describe how related quantities vary together. We can classify, predict, and characterize various kinds of relationships by attending to the rate at which one quantity varies with respect to another."* (Cooney, Beckmann, & Lloyd, 2010, p. 23). Cooney et al. 's big idea indicates a covariation perspective on function. They distinguish this perspective from a correspondence perspective, arguing for the usefulness of a covariation perspective for making sense of the mathematics of change. Cooney et al. identify a rate of change of a function as providing a "mechanism to describe and quantify the covariation between two variables" (p. 25). Consequently, the covariation of variables is an attribute quantified by a rate. Envisioning rate as quantification of covariation of variables is far more powerful than envisioning rate as a procedure to be executed based on a formula. By forming and interpreting relationships between quantities, students could develop a need for different types of functions, such that different types would indicate different kinds of relationships between quantities.

Research-based Tasks Focusing on QR

Researchers working at the elementary (Tzur et al., 2013), secondary (Johnson, 2013), and post-secondary (Dingman & Madison, 2010) levels have designed and implemented mathematical tasks having the potential to engender students' QR. The intent of this section is not to provide an exhaustive list of tasks focusing on QR, but rather to highlight examples of tasks fostering students' QR in K-16 mathematics education.

Elementary: Please Go Bring for Me (PGBM)

Tzur designed a platform game, Please Go Bring for Me (PGBM), to be used for the purpose of developing elementary students' multiplicative reasoning (Tzur et al., 2013). PGBM requires students to engage in coordinated counting activities, beginning with physical manipulatives, then drawn representations of those manipulatives, followed by symbols representing those manipulatives. The game is played in pairs with one student playing the role of "sender" and the other playing the role of "bringer". The game begins with the sender asking the bringer to produce an amount of equally sized towers composed of cubes (e.g., 4 towers containing 3 cubes each). The sender then asks the bringer four key questions: "(1) How many towers did you bring? (2) How many cubes are in each tower? (3) How many cubes are there in all? (4) How did you figure it out?" (Tzur et al., 2013, p. 88). The questions are central to the task because they focus on the different types of units involved in the PGBM game: the amount of towers, the amount of cubes per tower, and the total amount of cubes. When students can play the PGBM game proficiently with manipulatives, a teacher would introduce sequential constraints including: covering physical manipulatives, then drawing pictorial representations of physical manipulatives, then using symbols rather than pictorial representations. Results from research conducted by Tzur et al. (2012) suggest that the strategic use of representations involved in PGBM supported students' ability to operate on composite units, and hence provided students opportunities to develop their multiplicative reasoning. Such use of representations is grounded in QR, because it fosters students' envisioning of towers as "things" that could be measured by the number of unit cubes contained in each tower.

Secondary: A Filling Triangle

Using Geometer's Sketchpad Software (Jackiw, 2009) Johnson (2013) developed dynamic, interactive computer environments suitable for middle school students. One environment linked a pictorial representation of a right triangle "filling with area" with a dynamic graph relating the changing quantities of area and height (see Figure 3). To vary the area and height of the triangle, students could click and drag on point D (see Figure 3) or press the action button Animate Point D. If a student pressed animate point when AD was greater than zero, the area would "fill" until point D reached the maximum height of the triangle, then reset AD to zero and begin filling area again until the initial length of AD was reached.

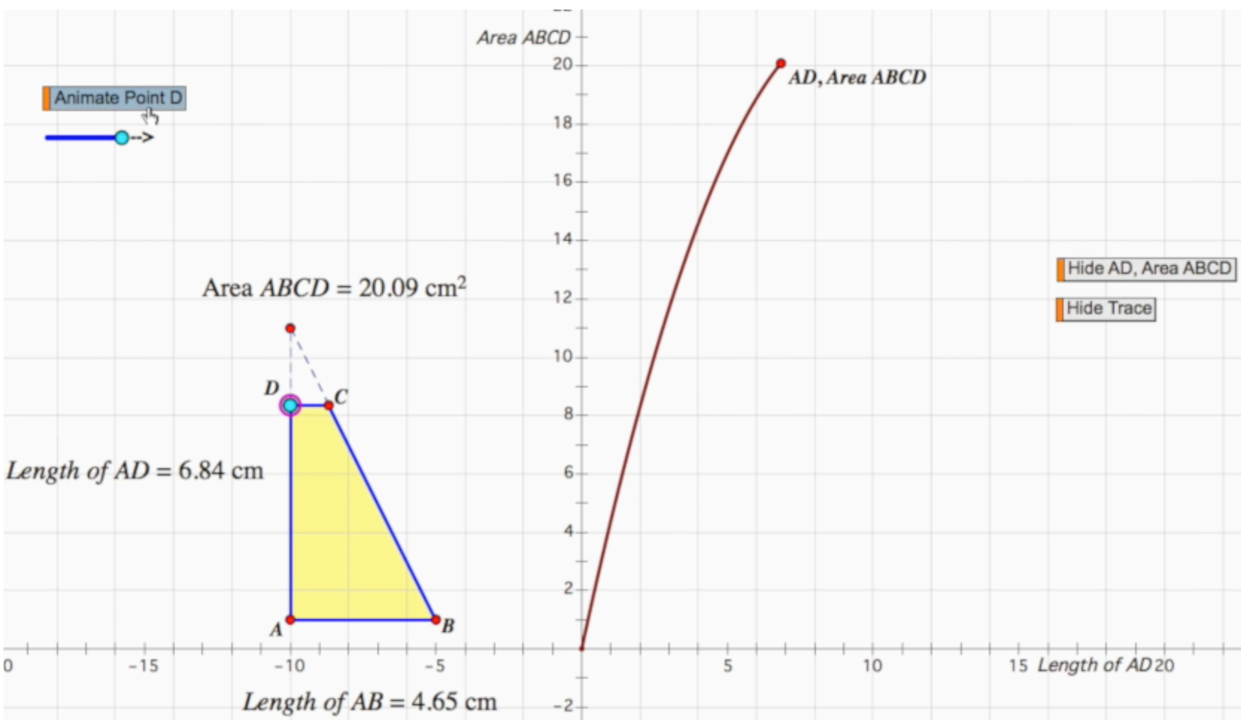

Figure 3. Filling triangle environment

By linking dynamic representations of geometric objects with dynamic graphical representations, the goal was to provide students working on the task with the opportunity to interpret graphical representations in terms of related quantities. Animations were designed to continually "fill" the objects with area. It is known that continuous covariational reasoning is important for students to develop (Thompson, 2008, 2011), and even successful mathematics students working at advanced levels may not necessarily engage in continuous covariational reasoning (Carlson et al., 2002). However, research has shown that secondary students can engage in continuous covariational reasoning (Castillo-Garsow, 2012; Johnson, 2012). Tasks requiring students to make predictions about relationships between covarying quantities prior to viewing an animation may provide opportunities for middle grades students to engage in continuous covariational reasoning (Johnson, 2013). Yet, more needs to be known regarding how students might develop such reasoning.

Post-Secondary

Researchers at the University of Arkansas (Dingman & Madison, 2010) designed and implemented a QR-focused mathematics course titled Quantitative Reasoning in the Contemporary World (QRCW). QRCW was implemented on a pilot basis in 2004, and by 2009 QRCW was being implemented to nearly 500 students. QRCW incorporated contextually rich tasks requiring students to reason about mathematical ideas to make sense of the situations. One such task required students to compare the fuel efficiency of a gasoline and hybrid car. After engaging in comparison, students were then required to respond to a newspaper article arguing that the cost of creating more fuel-efficient cars could outweigh savings in fuel efficiency. Based on their interactions with students in QRCW, Dingman and Madison argued that students' experiences in secondary mathematics did not seem to engender the kind of reasoning required for tasks used in QRCW. Were elementary and secondary students to have more opportunities to engage in QR through the kinds of tasks described in this chapter, they could be better prepared to engage in the QR necessary for tasks used in QRCW.

Discussion: Connections to White Paper

In the Waterbury Summit White Paper, Duschl et al. (this volume) argued that QR could serve as a link between mathematics and science education: "Quantitative reasoning is represented as a component of model-based reasoning that bridges the divide between mathematics and science." Although Duschl et al.'s argument addressed only QR, it could be extended to include both QR and QL. Mayes et al.'s (2013) definition of QL, which stated that QL results from quantification, provides an example of how QR and QL could be integrated to build bridges between mathematics and science.

During the QR Working Group of the International STEM Research Symposium, Duschl proposed that a focus on measurement could forge relationships between mathematics and science. While Duschl's proposition included only measurement, it could be extended to include the reflexively related constructs of measurement and quantification. This is to say that quantification precedes measurement, and subsequent measurement could support further quantification. Importantly, working from results of measurement may or may not incorporate the process of quantification. Thompson (2011) argued that expanding a definition of quantification could bridge mathematics and science education. Mayes et al.'s (2013) inclusion of quantification as part of a framework for QR in science education provides a beginning example that could support such bridging.

To conclude, distinctions between QR and QL could have implications for the design and implementation of tasks focused on QR. Moving too quickly into computation of amounts of quantities (or measuring particular amounts of quantity) can result in too little attention to the process of quantification and too much attention to the interpretation of the results of measurement. During the QR Working Group of the International STEM Research Symposium, the group did not come to consensus regarding definitions of QR or QL. A focus on quantification and measurement as salient aspects of QR and QL could provide a way to forge links between mathematics and science education in this area.

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