RELATIONSHIP OF COVARIATIONAL REASONING ON COLLEGE ALGEBRA STUDENTS' INTERPRETATION OF FUNCTION NOTATION

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A dissertation submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Education and Human Development Program

2018

This dissertation for the Doctor of Philosophy degree by

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Date: December 15, 2018

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Dissertation directed by Associate Professor Heather L. Johnson

ABSTRACT

In this study, I examined links between College Algebra students' covariational reasoning and their conception of general function notation ($y=f(x)$). I investigated the following research questions: How might students' conceptions of function impact their conceptions of function notation? How might covariational reasoning related to function impact students' conceptions of function notation? How do students conceive of general function notation ($y=f(x)$)? I posit three levels of students' conceptions of function notation: *function notation as label*, *function notation as convention*, and *function notation as a relationship between variables*, and draw connections to students' engagement in quantitative, variational, and covariational reasoning, as well as their employment of a correspondence approach to function.

For this study, I report three cases of students, Jack, Dave, and Lisa, who demonstrated different conceptions of function notation and different forms of variational and covariational reasoning. These students were enrolled in a College Algebra course at a public university in a large US city. I conducted a sequence of four task-based clinical interviews with the first interview serving as the Pre interview and the last interview serving as the Post interview. I analyzed the data using Wolcott's (1994) constructs of *Description, Analysis*, and *Interpretation*. I used constant comparative analysis (Corbin & Strauss, 2008) to detect any differences in reasoning from the Pre interview to the Post interview.

I found a link between students' engagement in covariational reasoning and their conception of function notation: Students engaging in early levels of covariational reasoning could conceive of function notation as a relationship between variables. Furthermore, students' conceptions of the definition of function mitigated their conceptions of function notation. In addition, when engaging with different kinds of tasks, they demonstrated different conceptions of function and function notation, and engaged in different forms of covariational reasoning. To promote students' conceptions of function and general function notation ($y=f(x)$) expressing an invariant relationship between quantities, researchers/teachers should leverage technology-rich tasks incorporating two different graphs that represent the same relationship, and tasks providing opportunities for students to make sense of others' claims about graphs.

The form and content of this abstract are approved. I recommend its publication.

Approved: Heather L. Johnson

I dedicate this work to my sons Eliab Naveed Azeem, Alian Obed Azeem, and Nauman Gul Azeem.

ACKNOWLEDGEMENTS

I am thankful for the encouragement and support I received from so many people during the course of this dissertation. First of all, I am very thankful to my mother Jasmine Inayat who took care of my children while I was working on this dissertation. This would not have been possible without her support. I would also like to thank my friends who always supported me.

I would like to express my gratitude to my advisor, Professor Heather Johnson for being a great mentor. She continually supported me and guided me during the process of writing this dissertation. She provided me with valuable feedback on several drafts of my dissertation and challenged me to think deeper. I also extend thanks to the chair of my dissertation committee, Professor Alan Davis, and committee members Professor Ron Tzur, and Professor Michael Ferrara. Their profound mathematical insights and careful attention to detail helped me to improve the quality of my dissertation.

I would like to take the opportunity to thank all the participants who actively participated in this study. I would not have been able to conduct this study without their willingness to participate and talk about their mathematical work.

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CHAPTER I

INTRODUCTION

Science, Technology, Engineering, and Mathematics (STEM) education is important in the high-tech global economy (Drew, 2011; The President's Council of Advisors on Science & Technology (PCAST), 2012). According to the U.S. Department of Commerce (2011), STEM jobs grew three times as fast as the non-STEM jobs over the past 10 years. STEM workers play an important role in the growth of the U.S. economy. In order to have a stronger STEM workforce, it is critical to improve the teaching methods so that more students can have opportunities to choose STEM majors. Teachers need to create a welcoming atmosphere for STEM learners and help students overcome mathematical challenges (PCAST, 2012).

The Problem with College Algebra

According to the PCAST (2012) report, college students switch their majors because of uninspiring introductory courses, little help provided by the colleges, and an unwelcoming atmosphere from faculty in STEM courses. Mathematics professors need to improve the first two years of STEM education (PCAST, 2012). For example, College Algebra is a gate keeper course and students' experiences in College Algebra can be a reason why students may leave STEM majors (Chen, 2013; Gordon, 2008; Herriott & Dunbar, 2009). A College Algebra course is meant to prepare students for higher level mathematics courses, but the current design of College Algebra course only serves 5-10% of the students to be prepared for PreCalculus (Herriott & Dunbar, 2009). Even if students are successful in College Algebra, they can still experience challenges in PreCalculus. To retain students in College Algebra, students should be given opportunities to use innovative learning materials to help them learn difficult concepts in College Algebra (Johnson, McClintock, Kalir, & Olson, 2018).

The concept of function is central to undergraduate mathematics (Carlson & Oehrtman, 2005; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994c). Promoting an emphasis on covariational reasoning can impact students' reasoning with function and function notation in College Algebra, in particular. Researchers have shown that students in high school and students taking freshman college math courses have a weak understanding of function (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Cooney & Wilson, 1996; Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994b). High school and undergraduate students taking mathematics courses are not given opportunities to reason about quantities which impacts their reasoning about function (Dubinsky & Wilson, 2013; Oehrtman et al., 2008; Thompson, 1994b, 1994c). College students' lack of opportunities to employ covariational reasoning may account for some of their difficulties in understanding the concept of a function. If College Algebra students are given an opportunity to reason about quantities, it can develop their conceptions of function and general function notation ($y=f(x)$).

Researchers investigating undergraduate mathematics students' reasoning found that students reasoned in ways that were more static rather than dynamic. Undergraduate PreCalculus students have a static view of conceiving of functions simply as an operation or a procedure (Clement 2001; Oehrtman et al., 2008). Current textbooks are not helping to address this problem. Tasova, Stevens, and Moore (2018) examined calculus textbooks on lessons dealing with the topic of functions. Tasova et al. (2018) found that calculus textbooks did not support in developing students' conceptions of quantitative and covariational reasoning. I found something similar in a College Algebra textbook used by a public university in a large US city. In a seventh edition textbook titled *A Graphical Approach to Algebra & Trigonometry* by Hornsby, Lial, and Rockswold (2019), the emphasis is on the correspondence approach to function, which is static. For example, function notation is defined using a correspondence approach as: "To say that *y* is a function of *x* means that for each value of *x* from the domain of the function *f*, there is exactly one value of *y*. To emphasize that *y* is a function of *x*, or that *y* depends on *x*, it is common to write $y=f(x)$, *y* equals $f(x)$, with $f(x)$ read "*f* of *x*." This notation is called function notation" (p.18). With this definition, Hornsby et al. (2019) prepare students to complete the table of values, evaluate functions at a given *x*-value and to graph functions of the form $y= f(x)$ given *x* and *y* values. If College Algebra students had more opportunities to conceive of $y=f(x)$ in a dynamic way as a relationship between two quantities, *x* and *y*, they may develop stronger conceptions of function.

A Promising Possibility: A Focus on Covariation

Opportunities for students to engage in covariational reasoning can promote their success in College Algebra (Johnson et al., 2018) and other introductory and higher-level mathematics (Thompson & Carlson, 2017). Covariational reasoning entails conceiving of how two quantities' values change together. For example, consider a Ferris wheel situation with distance traveled around the Ferris wheel and the height of the Ferris wheel. An individual who conceives of both distance and height changing together, such that as the distance increases, the height increases and decreases, engages in covariational reasoning. College Algebra students can succeed in developing stronger conceptions of function and general function notation ($y=f(x)$) if they are given an opportunity to engage in covariational reasoning.

Researchers that investigate students' and teachers' engagement in covariational reasoning focus on their conceptions of graphs. Several researchers have focused on secondary and undergraduate students' graphing activity (e.g., Bell & Janvier, 1981; Carlson, 1998; Carlson et al., 2002; Johnson, 2015a, 2015b; Johnson, Hornbein & Azeem, 2016; Johnson, McClintock,

& Hornbein, 2017b; Leinhardt, Zaslavsky, and Stein, 1990) and other researchers focused on pre-service and in service teachers' conceptions of graphs (Moore, 2014; Moore, Silverman, Paoletti, & LaForest, 2014; Moore & Thompson, 2015; Moore, Silverman, Paoletti, Liss and Musgrave, in press). Bell and Janvier (1981), Carlson (1998), and Moore and Thompson (2015) found that students often reason about graphs based on physical characteristics such as the shape of a graph. Researchers also found that students could conceive of graphs as invariant relationships between quantities (Moore et al., 2014; Moore & Thompson, 2015; Moore et al., in press) and conceived of points on graphs as multiplicative objects (Johnson et al., 2016; Johnson et al., 2017b). College Algebra students' conceptions of graphs as relationships between quantities can impact their conceptions of general function notation ($y=f(x)$) as a representation of two quantities' values changing together.

Researchers have designed learning materials for university students enrolled in different mathematics classes to support their covariational reasoning. Researchers have provided Calculus and PreCalculus students' opportunities to engage in covariational reasoning (e.g., Carlson et al., 2002; Thompson & Ashbrook, 2016; Thompson & Carlson, 2017). Recently Johnson et al. (2018) designed dynamic computer activities to promote College Algebra students' covariational reasoning. Johnson et al. (2018) developed free, accessible computer activities linking dynamic animation and graphs. The tasks included the dynamic segments and the graphs that represented the same attributes on different axes. These tasks avoided the numerical amounts to foster students' covariational reasoning. In my study, I aim to contribute to research on college students' covariational reasoning by investigating and focusing on how covariational reasoning could impact their conceptions of general function notation ($y=f(x)$).

A Covariation Perspective on Function

Several researchers, whose focus has been on the concept of function, have argued for the powerfulness of covariational reasoning for developing secondary and college students' robust conceptions of function (Carlson, 1998; Carlson et al., 2002; Confrey & Smith, 1994, 1995; Johnson, 2012; Thompson, 1994b, 1994c; Thompson & Carlson, 2017; Zandieh, 2000). Thompson and Carlson (2017) define a function as: "A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (p.444). I interpret that this definition explains a function in terms of individual's conceptions such that a function represents two quantities that change together and the relationship between two quantities stays the same with one value of a quantity giving exactly one value of the other. Engaging in quantitative reasoning and covariational reasoning can develop College Algebra students' conceptions of function and general function notation ($y=f(x)$).

Quantitative reasoning and covariational reasoning is based on quantities. Quantity is defined as an individual's conception of a measurable attribute of an object (Thompson, 1993). For example, consider a Ferris wheel situation with distance traveled around the Ferris wheel and the height increasing and then decreasing. An individual who conceives of distance increasing and height increasing and then decreasing may put numbers along the axes to demonstrate that he/she conceives of both distance and height as possible to measure. Students' reasoning with quantities can support a covariational perspective on function (Ellis, 2011; Johnson et al., 2016, Thompson & Carlson, 2017). By engaging in covariational reasoning students can understand the concept of a function in a more meaningful way. I argue that a covariation perspective can make a difference in how College Algebra students interpret general function notation ($y=f(x)$).

Need for a Covariation Perspective on Function Notation

There is a need to provide students an opportunity to engage in covariational reasoning and to learn how it can impact their conceptions of general function notation ($y=f(x)$). I use the term general function notation for a function notation $y=f(x)$ to distinguish it from a function notation that includes a formula such as $f(x) = 3x+4$. Researchers have found that college students can think a function must be defined by a single algebraic formula (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992). I think that high school and college students find it easier to interpret $f(x) =3x+4$ because they can substitute values to evaluate a function, graph a function, and complete a table of values. Because there is no formula in general function notation ($y=f(x)$), it is difficult for students to interpret $y=f(x)$. Students conceive of general function notation ($y=f(x)$) as two expressions separated by an equal sign (Thompson, 1994c). In addition, students conceive of an equal sign as an operational (to do something) symbol (Kieran, 1981). A covariation perspective could provide students an opportunity to conceive of general function notation ($y=f(x)$) expressing a relationship between quantities.

Researchers/teachers need to learn more about students' engagement in covariational reasoning and general function notation ($y=f(x)$). Researchers focused on function notation involving specific rules such as $f(x) = 3x+4$ (e.g., Fonger, Ellis, & Dogan, 2016; Musgrave & Thompson, 2014; Thompson & Milner, 2017). Musgrave and Thompson (2014) as well as Thompson and Milner (2017) focused on high school teachers' meanings of function notation. Fonger et al. (2016) focused on middle school students' conceptions of quadratic function rules of the form $y=ax^2$. Few researchers have addressed how students conceive of function notation involving non-specific rules or more general function notation (e.g., Sajka, 2003). While researchers (e.g., Carlson, 1998; Thompson & Carlson, 2017) have alluded to general function notation ($y=f(x)$), the research community does not yet know how students' reasoning with quantities is related to their reasoning with general function notation ($y=f(x)$). If we encourage College Algebra students to engage in covariational reasoning, students may conceive of general function notation ($y=f(x)$) as a representation of two quantities changing together.

To learn more about how students conceive of general function notation ($y=f(x)$), it is important to learn how students engage in quantitative reasoning and covariational reasoning and relate it to a function. In other words, researchers/teachers need to attend to students' conceptions of quantities and variables, how quantities vary, and their conceptions of how quantities change together to make a function and function notation more meaningful to them. Teachers can help students to succeed in a College Algebra class by developing their conceptions of a function, the meaning of a variable, and their conceptual understanding of general function notation ($y=f(x)$).

Tasks as Opportunities

The type of tasks students work on can impact their learning opportunities (Johnson, Coles, & Clarke, 2017a). By task I mean more than just a written problem. A task involves students' conceptions of the task and how teachers can design tasks to support students' engagement with the task (Johnson et al., 2017a; Sierpinska, 2004). Tasks can provide opportunities to educators to learn how students' engagement with tasks can engender students' covariational reasoning and can impact their interpretation of general function notation ($y=f(x)$).

In recent efforts to improve mathematics teaching and learning, researchers have designed tasks often involving dynamic computer environments linking animations and graphs (e.g., Carlson et al., 2002; Ellis, Ozgur, Kulow, Dogan, & Amidon, 2016; Johnson, 2012b; Johnson, 2015b; Johnson et al., 2017b; Johnson et al., 2018; Kaput & Roschelle, 1999; Moore, 2014; Moore et al., in press; Saldanha & Thompson, 1998). Researchers have designed tasks with attributes on different axes (Johnson et al., 2016; Johnson et al. 2017b; Johnson et al., 2018; Moore et al., 2014; Moore et al., in press) and tasks with other students' claims about the graph (Johnson et al., 2018, August) to provide students opportunities to engage in quantitative reasoning and covariational reasoning. Moreover, the types of attributes used in tasks can provide educators an opportunity to learn about students' quantitative and covariational reasoning (Johnson et al., 2018; Moore et al., 2014; Moore et al., in press). Researchers/teachers can support students' mathematical reasoning by employing technology rich tasks that do not always include finding numerical answers (Thompson & Carlson, 2017; Johnson, 2013; Johnson, 2015b).

By engaging with tasks designed to foster students' quantitative and covariational reasoning, students can develop a flexible understanding of function. Fonger et al. (2016) suggested that tasks focused on covariational reasoning could allow students to have a flexible understanding of quadratic function rule of the form $y=ax^2$. Students reasoning covariationally may also conceive of general function notation ($y=f(x)$) as more than letters. Students may conceive of $y=f(x)$ more than two expressions separated by an equal sign and more than a "to do something" symbol. Supporting students' covariational reasoning through technology-rich tasks can develop an understanding of general function notation ($y=f(x)$) as a representation of two quantities changing together.

Research Questions

To learn about students' covariational reasoning and its relationship to general function notation ($y=f(x)$), I pose the following research questions:

1. How might students' conceptions of function impact their conceptions of function notation?
2. How might covariational reasoning related to function impact students' conceptions of function notation?
3. How do students conceive of a general function notation ($y=f(x)$)?

Overview of Chapters

In the chapters that follow, I articulate the theoretical grounding for my study, methodology and analysis methods, the results of the study, and implications /conclusion. Chapter 2 includes the theoretical and conceptual framework. Chapter 3 includes the review of the literature. Chapter 4 includes the methodology and analysis methods that I use for this study. Chapters 5, 6, and 7 describe the case studies of Jack, Dave, and Lisa's (pseudonyms) work respectively. Chapter 8 includes the cross case analysis and Chapter 9 includes discussion and implications of this study.

CHAPTER II

CONCEPTUAL AND THEORETICAL FRAMEWORK

In this chapter, I describe Piaget's (1970) constructivist theory, Thompson's theory of quantitative reasoning (Thompson, 1993; Thompson, 2011) and Thompson and Carlson's (2017) variation and covariation frameworks. I draw on Thompson's theory of quantitative reasoning (Thompson, 1993; Thompson, 2011) and Thompson and Carlson's (2017) framework to address how students conceive of functions, how they conceive of quantities, and how covariational reasoning impacts students' conceptions of a function. All theories presented here build on Piaget's (1970) constructivist theory. I also include images of change, tasks, representations and conceptions of function as part of my conceptual and theoretical framework.

Piaget's Constructivist Theory

According to Piaget's (1970, 1985) constructivist theory of learning, learning occurs through experiences and action, rather than through knowledge passed on by others. Individuals construct their own understandings of new information and those constructions depend on their current formulated knowledge and ideas. From a constructivist perspective, emphasis is on describing the mental images of an individual to learn the individual's ways of thinking and approaches to problem situations (Piaget, 1970). These mental representations refer to dynamic mental activity. Learners create their own meaning through the process of sense making (Piaget, 1970). Sense making refers to how a mathematical situation holds together from the perspective of an individual (Simon, 1996). Based on Piaget's theory, students may conceive of quantities and function notation by constructing their own understandings, and those constructions depend on their current conceptions.

Thompson's Theory of Quantitative Reasoning

In this section, I describe Thompson's theory of quantitative reasoning (Thompson, 1993; Thompson, 2011). This theory explicates how students may conceive of mathematical situations using the relationship between varying quantities and is characterized by three main parts that include quantity (Smith & Thompson, 2008), quantification (Thompson, 2011), and quantitative reasoning (Thompson, 1993; Thompson, 2011).

Quantity. Quantity is defined as an individual's conception of a measureable attribute of an object (Thompson, 1993). For example, an individual may conceive of measuring the length of the table by using their hand as a measurement tool.

Quantification. Quantification is a three-step process involving "settling what it means to measure a quantity, what one measures to do so, and what a measure means after getting one" (Thompson, 2011, p.38). This refers to how an individual conceives of an object, considers an attribute of an object, acknowledges that an attribute has a unit of measure, and conceives of the relationship between the attribute and the unit of measure (Thompson, 2011). For example, to quantify height, an individual would need to consider the height as an attribute of some object that could be measured, think of a unit with which to measure the height, and form a relationship between the unit of measure and the measure of height as an attribute. Thompson (2011) makes a clear distinction between quantification process and the process of measuring and describes that the process of quantification precedes measurement. In other words, the learner may engage in the process of quantification before doing actual calculations.

Quantitative reasoning. *Quantitative reasoning* refers to the mental activity that an individual engages in to conceive of the relationships among measurable attributes of the objects in a situation. (Thompson, 1993; Thompson, 1994a; Thompson, 2011). Quantitative reasoning is fundamental in developing students' reasoning about functions as relationships (Ellis, 2011; Johnson, 2012b; Moore, Silverman, Paoletti, & LaForest, 2014).

Covariational Reasoning

Several researchers, whose focus has been on the concept of function, have argued for the powerfulness of covariational reasoning for developing robust conceptions of function (Carlson, 1998; Carlson et al., 2002; Confrey & Smith, 1994, 1995; Johnson, 2012b; Thompson, 1994b, 1994c; Thompson & Carlson, 2017; Zandieh, 2000). In this section, I define covariational reasoning and then discuss images of change because Thompson's levels of variational and covariational reasoning were based on Castillo-Garsow's (2010, 2012) chunky and smooth images of change. I first present levels of variational reasoning, then the levels on covariational reasoning. Then I compare and contrast Thompson's covariational reasoning with other perspectives.

Thompson's definition of covariational reasoning. According to Thompson (2011), when the student imagines two quantities' values varying together so that the quantitative relationship remains unchanged, he/she engages in covariational reasoning. Thompson's covariational reasoning is based on conceiving of quantities' values varying, constructing multiplicative objects (combining quantities varying and forming a new quantity), and conceiving of invariant relationships.

Images of Change. Castillo-Garsow (2010, 2012) and Castillo-Garsow, Johnson, and Moore (2013) described that students might conceive of a quantity's value varying discretely or continuously. In the first image, the student conceives of the change as it already happened, so he coordinates two completed changes which Castillo-Garsow terms *chunky* images of change. Thus, when a student engages in chunky reasoning, he focuses on discrete values at the end of an interval. For example, a student engaged in chunky reasoning might think of a car traveling from one mile to two miles without thinking that the car moved between those values. In the second image, the student would conceive of the magnitude of each quantity passing through all possible values between the initial and final value, which Castillo-Garsow term *smooth* images of change. When a student engages in smooth reasoning, he/she conceives of change in progress. This means that a student engaging in smooth thinking conceives of a beginning point but no endpoint. In other words, if a student thinks in chunks and then also reasons covariationally, the student relates amounts of completed change.

Thompson's levels of variational reasoning. Thompson and Carlson (2017) developed a framework for explicating levels of students' variational reasoning, placing reasoning involving smooth images of change at the highest level of their framework. Table describes major levels of variational reasoning.

Table 1 (Thompson & Carlson, 2017, p.440)

Major Levels of Variational Reasoning

Level	Description
Smooth Continuous Variation	The person thinks of variation of a quantity's or variable's value (hereafter, variable's) as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-sized intervals of variation, but not necessarily.
Chunky Continuous Variation	The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be same sized, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 to (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, etc. "come along" by virtue of each being part of a chunk---like numbers on a ruler, but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, etc. as values. Chunky continuous variation is not just thinking that changes happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.
Gross Variation	The person envisions that the value of a variable increases or decreases but gives little or no thought that it might have values while changing.
Discrete Variation	The person envisions a variable as taking specific values. The person sees the variable's value changing from a to b by taking values a_1, a_2 , …, a_n, but does not envision the variable taking any value between a_i and a_{i+1}.
No Variation	The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.
Variable as Symbol	A variable is a symbol. It has nothing to do with variation.

I elaborate on the differences between these six levels of variation. At a *variable as symbol* level, a student conceives of variables as letters only and does not conceive of letters representing something changing. At a *no variation* level, a student may conceive of distance traveled by a car to be 6 miles. This student conceives of the attribute of an object to have a fixed measure. At a *discrete variation* level, a student may conceive of a car traveling different distances of 6 miles and 7 miles but does not conceive of distance attaining values within 6 miles and 7 miles. At a *gross variation* level, a student may attend to distance increasing, but does not attend to distance attaining the values while changing. At a *chunky continuous variation level*, a student will attend to the ends of the chunk such as distance traveled is six miles, then seven miles. Although the student may be aware of a car traveling between six and seven miles, he/she does not attend to how the quantity's measure varies within six and seven miles. At a *smooth continuous variation* level, a student conceives of change in progress and keeps track of distance traveled not only at six miles and seven miles, but also within six miles and seven miles.

Variational reasoning cannot be separated from covariational reasoning. In a previous study, Saldanha and Thompson (1998) stated that their images of covariation were developmental, beginning with covariation as two quantities varying separately (one then the other). Later, images of covariation involved time as a continuous quantity. They further explained:

> An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values" (Saldanha & Thompson, 1998, p. 2).

Saldanha and Thompson (1998) described a developmental conception of covariational reasoning where students conceived of one quantity, then the other to conceiving of both quantities varying together. Saldanha and Thompson suggested that a student constructed a variable as conceiving of the quantity's value always varying.

Thompson's levels of covariational reasoning. Thompson and Carlson (2017) proposed levels of covariational reasoning (see Table 2) and explained that these levels of covariation "retain emphases on quantitative reasoning and multiplicative object (Thompson) and coordination of changes in quantities' values (Confrey, Carlson) and add ways in which an individual conceives quantities to vary (Castillo-Garsow)" (ibid, p. 441). They explain that each level is intended to characterize an individual's capacity to engage in covariational reasoning. The tables do not provide a "learning progression in the sense that one level should be targeted instructionally before the next level" (Thompson & Carlson, 2017, p.440), but if a student reasons variationally or covariationally at the highest level, all other levels up to that level are included.

Table 2 (Thompson & Carlson, 2017, p.441)
Major Levels of Covariational Reasoning

Level	Description
Smooth Continuous Covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and they envision both variables varying smoothly and continuously.
Chunky Continuous Covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Coordination of Values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Gross Coordination of Values	The person forms a gross image of quantity's values varying together, such as "this quantity increases while that quantity decreases". The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, non-multiplicative link between the overall changes in two quantities' values.
Pre- Coordination of Values	The person envisions two variable's values varying, but asynchronously---one variable changes, then the second variable changes, then the first, etc. The person does not anticipate creating pairs of values as multiplicative objects.
No Coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.

To elaborate the differences between these six levels of covariation, consider a plane so that as it covers distance along the ground, its altitude changes. At a *no coordination* level, a student focuses on distance varying or the altitude changing but does not conceive of the two quantities changing together. At a *pre-coordination of values* level, a student conceives of distance changing and then the altitude changing but does not conceive of both distance and altitude changing together. At a *gross coordination of values* level, a student may say that as the distance increases, the altitude increases and then decreases, but may not conceive of quantities' values changing together. At a *coordination of values* level, a student conceives of coordinating the values of distance and altitude and conceives of ordered pairs. At a *chunky continuous covariation* level, a student conceives of values of distance and altitude changing simultaneously within a chunk. At a *smooth continuous covariation* level, a student conceives of distance and altitude as quantities that change together simultaneously and keeps track of distance and altitude changing together smoothly and continuously.

Multiplicative object. Constructing an invariant relationship between quantities and conceiving of continuous variation are part of Thompson's definition of covariational reasoning. However, in order to engage in covariational reasoning, the student must coordinate two images of variation and construct what Thompson term *multiplicative object*. Saldanha and Thompson (1998) explained multiplicative object as:

> Covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value (Saldanha & Thompson, 1998, pp. 1-2).

Saldanha and Thompson (1998) extended the work of Inhelder and Piaget (1964) and explained that when a student constructs a multiplicative object, he/she conceives of how two quantities' values vary together so that whenever he/she conceives of variation of one quantity, he/she necessarily imagines variation in the other. For example, if a student conceives of a point (x, y) in the Cartesian Coordinate system as a multiplicative object, then as he/she conceives of the value of x varying, he/she also conceives of the value of y varying with x. That is one way how an individual can conceive of a point on a graph as a multiplicative object so that an individual's conception of a point (x, y) is not dependent on a procedure to determine a location of a point (x, y).

Situating Thompson's Covariational Reasoning with Other Perspectives

There are different ways of thinking that constitute covariational reasoning. I elaborate on Confrey and Smith and Carlson's work because Thompson's definition of covariational reasoning contrasts with Confrey and Smith (1994, 1995) and builds on Carlson et al.'s (2002) framework.

Confrey and Smith's definition of covariational reasoning. Confrey and Smith (1994, 1995) defined covariational reasoning as a process of coordinating successive values of two variables. According to Confrey and Smith (1994), students are "able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (p.137). When engaged in this activity, students focused on the repeated action of change in the value of x and change in the value of y. For example, consider the relationship represented in Figure 1. A student might construct the pattern that in each subsequent row, the value of x always increases by 2 and the value of y increases by 4. He/she can coordinate these changes and conceive of the value of x increasing by 1, ½ of 2, then the value of y must increase by 2, ½ of 4. A student determines a new pair of values that satisfies the relationship between *x* and *y* by identifying patterns of change in the value of *x*, patterns of change in the value of *y*, and then coordinating these patterns of change.

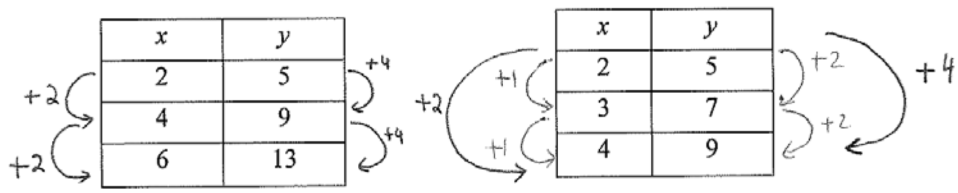


Figure 1 : Constructing patterns of change in *x* and *y* values

Confrey and Smith reported that students found the *covariation approach to function* more intuitive than the arbitrary correspondence from *x* to *f(x)*. Confrey and Smith's conception of covariational reasoning emerged from their research on how students conceived of exponential functions (Confrey & Smith, 1994, 1995). They proposed that once a student constructed an additive conception of counting and a multiplicative conception of splitting, he/she could conceive of exponential functions by coordinating changes in the value of *x* with changes in the value of *y*. Confrey and Smith (1995) described a notion of covariation where students coordinate a completed change in the value of *x* with a completed change in the value of *y*, which contrasts with Thompson's definition of covariational reasoning. Thompson (2011) described covariational reasoning where students track two quantities' values changing simultaneously.

Carlson's framework of covariational reasoning. Thompson and Carlson's (2017) covariation framework builds from Carlson et al.'s (2002) framework, but the difference is that Carlson et al. (2002) included rate of change as part of their framework whereas Thompson and

Carlson (2017) revised their framework and kept rate of change separate. Rate is no longer integrated into the covariation framework.

Carlson et al. (2002) framework involved an image defined as “dynamics of mental operations” (Thompson, 1994a, p. 231). By an individual’s image of a quantity, I refer to “dynamics of mental operations” (Thompson, 1994a, p.231) which are the same as Castillo-Garsow’s (2010, 2012) images of change. We can learn how students may conceive of a quantity by examining and comparing different ways in which they represent their mental images or conceptions of a quantity.

Representations of Functions

A representation can be defined as “a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else” (Goldin & Kaput, 1996, p. 398). Representations can be either internal or external, where an internal representation can be defined as “possible mental configurations of individuals, such as learners or problem solvers” (Goldin & Kaput, 1996, p.399), and an external representation can be defined as “physically embodied, observable configurations such as words, graphs, pictures, equations, or computer micro worlds” (Goldin & Kaput, 1996, 400). For example, when a student conceives of distance increasing and writes numbers along the x-axis, he/she writes numbers to represent distance as an increasing quantity. Students’ internal and external representations can inform how they conceive of function and general function notation ($y=f(x)$).

The representation a mathematical relationship takes can profoundly influence how one understands the underlying relationship (Leinhardt et al., 1990). There are several researchers who have highlighted the importance of different representations of the same thing (e.g., Dubinsky & Wilson, 2013; Duval, 1987; Even, 1998; Gagatsis & Shiakalli, 2004; Habre & Abboud (2006); Janvier, 1987; Hitt, 1998; Moschkovich, Schnoenfeld, & Arcavi, 1993). Duval (1987) defined the notion of natural interpretation as understanding the meaning of a concept and identifying equivalent meanings of different representations of the same concept. Janvier (1987) used the term *translation* and suggested that to develop students' translation ability, the students should be asked to perform translations both from the source (initial representation) to the target (final representation) and from the target to the source (Janvier, 1987). For example, given an equation and a graph, there are two translations- from a graph to an equation and from an equation to a graph. Translations between representations and transformations within them are also very important (Lesh, Post & Behr, 1987a, 1987b; Moschkovich et al., 1993).

Conceptions of Function

I will address Action, Process, Object, Schema (APOS) theory (Dubinsky, 1991) which builds on Piaget's (1985) theory of learning. Dubinsky considered that reflective abstraction is the mental mechanism by which all logico-mathematical structures are developed in the mind of an individual (Piaget, 1985). The APOS stages of Dubinsky (1991) are one way of evaluating a student's understanding of a mathematical concept (e.g., function) and provide a road map for helping the student in that development. Action and process conceptions of function by Dubinsky and Harel (1992) build on the theoretical perspective of APOS theory. According to Dubinsky and Harel (1992), an action conception of function involves substituting numbers and evaluating algebraic expressions. For example, given $f(x)= 3x+2$, and $x=2$, students can evaluate $f(2)$. In contrast to an action view, Dubinsky and Harel (1992) state that:

> A process conception of function involves a dynamic transformation of quantities according to some repeatable means that, given the same original quantity, will always produce the same transformed quantity. The subject is able to think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. When the subject has a process conception, he or she will be able, for example, to combine it with other processes, or even reverse it. Notions such as 1-1 or onto become more accessible as the subject's process conception strengthens (p.85)

For example, given the function $f(x) = 3x + 2$, the student can imagine a set of input values that are mapped to a set of output values by the defining expression for *f*. A student with a process view could engage in the cognitive activity of "running through a continuum of input values" without actually determining specific values (Oehrtman, Carlson, & Thompson, 2008, p.33). This ability of moving from an action to process view of function is particularly important when reasoning covariationally (Carlson & Oehrtman, 2005; Oehrtman et al., 2008). For the purposes of this study, I used action-process distinctions to discuss students' conceptions of function and general function notation (*y=f(x)*).

Student Perspective on Tasks

Tasks are more than just written problems; they include how students conceive of the tasks (Johnson et al., 2017a; Sierpinska, 2004). Taking a student perspective on tasks, Johnson et al. (2017a) defined mathematical tasks as "a designer's intended purpose for the task, a teacher's intentions in implementing a task, students' activity in undertaking a task, and artifacts (problem statement, tools and constructed objects, including student written materials) employed in and generated by the actions of teachers and students during the process of task completion" (p.814).

By this definition I mean that teachers and students both have certain goals with the task and students' actions may depend on teachers' actions. Teachers can pose questions in a way that brings student perspectives and provide opportunities for students to learn.

To provide students opportunities to engage in quantitative reasoning and covariational reasoning and interpret function notation, tasks matter. If students are given cognitively demanding non-routine tasks, the students develop their cognitive abilities and also participate in classroom discussions. In recent efforts to improve mathematics teaching and learning, mathematical tasks have been given considerable attention. The National Council of Teachers of Mathematics (2000) and the more recent Common Core State Standards for mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) describe a set of goals for students' mathematical learning that include both procedural fluency and conceptual understanding in a range of mathematical domains.

Scholars laying groundwork for function represented relationships between varying quantities (e.g., Carlson, 1998; Thompson & Carlson, 2017) which may help students to expand their current conceptions of functions. When students engage in problem solving, they attempt to relate their task to the concepts, meanings, and approaches they acquired in their past experiences of mathematics and problem solving (Hung, 2000). Moreover, the tasks allow students to engage in mathematical activity below or beyond their current level of understanding and impact their learning in the process (Nagle & Styers, 2015). Therefore, it is desirable to design tasks that will provide students' opportunities to develop their conceptions of function and function notation.

Concluding Remarks

Mathematics is a product of human activity which implies that mathematics concepts are not static ideas that can be passed on from one person to the next (Freudenthal, 1973). Thompson's theory of quantitative reasoning and covariational reasoning as well as Dubinsky's conceptions of function are built on Piaget's constructivist theory. From Piaget's stance, students construct their own understandings, and those constructions depend on their current understandings. How students represent something internally or externally as well as the tasks students engage in helps them to expand on their current conceptions. As a result, students construct new knowledge based on those current conceptions. Thompson's theory, Dubinsky's conceptions of function, students' internal and external representations, and tasks students engage in can inform how students conceive of quantities and how their covariational reasoning is related to their conceptions of function and general function notation ($y=f(x)$).

CHAPTER III

LITERATURE REVIEW

In this Literature review, I focused on research related to function and function notation. In the following sections, I elaborate on the research done on the concept of a function, covariational reasoning, variables, and function notation. What is missing from the literature is an empirical study that links quantitative reasoning and covariational reasoning to general function notation $y=f(x)$. In my study, I seek to fill that gap between reasoning with quantities and reasoning with function notation $y=f(x)$ based on quantities.

First, I present various definitions of a function and students' conceptions of a function. Second, I address students' interpretation of function notation. Third, I address tasks involving graphs because researchers investigating secondary students' and pre-service and in-service teachers' quantitative and covariational reasoning have made extensive use of graphs (e.g., Johnson, 2012a; Johnson, 2012b; Johnson, 2015a, 2015b; Leinhardt et al., 1990; Moore & Thompson, 2015; Moore et al., in press). I use tasks in my interviews and the tasks build on the literature. I conclude with key ideas emerging from the literature review.

Functions

I provide a brief historical perspective of function to describe how scholars developed the notion of a function. Mathematics is a human activity, which implies that functions are not static ideas that can be passed on from one person to the next (Freudenthal, 1973). The definition of a function was shaped over time. I focus on the definitions of function because the definitions provide a way to investigate what students think when they hear the word *function*. The definitions include defining a function as *correspondence* and *covariation* (Thompson & Carlson, 2017). Definitions also encompass how students conceive of different representations of functions and the connections they make between representations, such as a graph and function notation.

Brief historical perspective. Function is a concept that scholars developed, and its definition evolved over time (Kleiner, 1989; O' Connor & Robertson, 2005; Sfard, 1992; Sierpinska, 1992). Historically, envisioning variables as varying continuously (*variation*) has been foundational to the development of important mathematics (Kaput, 1994; Thompson & Carlson, 2017). Kaput (1994) argued that emerging conceptions of quantities varying continuously were central to the emergence of calculus. Similarly, Euler and Leibniz spoke of change in one variable producing change in another variable. However, mathematicians considered reasoning about how variables changed together to be an intuitive understanding of function. As a result, covariational reasoning got undervalued and Dirichlet's definition on an arbitrary correspondence of function became central and Dirichlet's definition is prevalent today (Kaput, 1994; Thompson & Carlson, 2017).

Correspondence definition. Functions are usually defined as a *correspondence* between x and y values. A function can be defined as "A function $f: S \to T$ consists of two sets S and T together with a "rule" that assigns to each $x \in S$ a specific element of T, denoted $f(x)$" (Marsden & Hoffman, 1993, p.3). This definition of a function is known as the Dirichlet-Bourbaki definition of a function, which is consistent with the correspondence definition. Correspondence definition is more codified in curricular materials. The *correspondence* definition has two distinct characteristics namely *arbitrariness* and *univalence* (Freudenthal, 1983). The arbitrariness of a function refers to "both the relationship between the two sets on which the function is defined and the sets themselves" (Even, 1993, p.96). This means there does not need to be a specific symbolic expression, a set pattern in a table of values, or a graph with a specific shape. I describe univalence property below because I use that in my analysis.

Univalence. This characteristic of a function refers to the part of the definition that states that for each element in the domain, there is only one element in the range (Even, 1990; Even, 1993). The univalence characteristic of a function implies both one-to-one and onto notions of a function as shown in Figure 2 below:

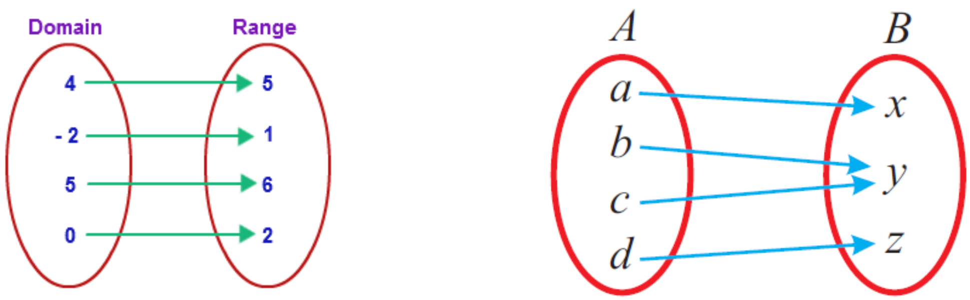


Figure 2: Both diagrams satisfy univalence property. Retrieved from http://images.google.com

In this Venn diagram, the image shown on the left shows a one-to-one mapping where each input has a different output. The image shown on the right represents an onto function where two different inputs in A are mapping to the same output in B.

Defining function in terms of quantity (covariation). An alternative way to define a function is by expressing a relationship between two quantities that change together (covariation). Chazan (2000) and Thompson and Carlson (2017) defined a function taking this covariational perspective, but I use Thompson and Carlson's (2017) definition of a function. Chazan (2000) defined a function as an expression of the relationship between quantities where the "output variables depend unambiguously on input variables" (p. 84). This definition of function makes clear the covarying relationship of quantities involved. Thompson and Carlson

(2017) define a function as: “A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person’s conception, every value of one quantity determines exactly one value of the other” (p.444). The definition by Thompson and Carlson (2017) is similar to Chazan’s (2000) quantity-based definition, but it further explicates meaning of function without mentioning input and output variables. Thompson and Carlson’s (2017) definition of function combines the prior definition of function (for every x, there is an output y) to the covariational aspect and offers a complete definition of function based on quantities. Furthermore, this definition does not specify variables as dependent and independent or as variables x and y, so a person can envision simply two quantities that covary.

I distinguish two different parts in Thompson and Carlson’s (2017) definition of a function. A student conceiving of function in a way consistent with Thompson and Carlson’s (2017) definition of function could engage in covariational reasoning and employ a correspondence approach. For the first part of the definition (p. 444), "a conception of two quantities varying simultaneously such that there is an invariant relationship between them", students could engage in covariational reasoning. Then for the second part (p. 444), "every value of one quantity determines the other" they could employ a correspondence approach. I explain this with an example. Consider a Ferris wheel situation with distance traveled around the Ferris wheel and the height of the Ferris wheel. An individual who conceives of height as a function of distance such that as distance increases, the height increases and then decreases, conceives of an invariant relationship between quantities. If an individual conceives of one distance value corresponding to a value of height, he/she employs a correspondence approach. So, I refer to

Thompson and Carlson's (2017) definition as a combination of quantitative reasoning and covariational reasoning, and a correspondence approach.

Both covariation and correspondence approaches to function may be useful to learn about students' reasoning of function and function notation. In a correspondence approach, a student has a more static view of function, in which a function has a fixed relationship between the members of two sets. In a covariation approach, a student conceives of a function as a relationship between two quantities, so covariation definition is more intuitive (Thompson & Carlson, 2017). A combination of covariation and correspondence approaches can be helpful for student to learn about function and general function notation ($y=f(x)$).

Empirical studies on function. In this section, I present studies that emphasize the importance of covariational reasoning and quantitative reasoning for students' understanding of function. Ellis and colleagues (e.g., Ellis, 2007; Ellis, 2011; Ellis, Ozgur, Kulow, Williams, & Amidon, 2013; Ellis, Ozgur, Kulow, Williams, & Amidon, 2015; Ellis, Ozgur, Kulow, Dogan & Amidon, 2016) have focused on middle school students' reasoning with function. Ellis (2007) reported results of a teaching experiment on a group of seventh-graders' reasoning with linear function where students explored constant rates of change by investigating gear ratios and constant speeds. Ellis (2011) reported results on middle school students' understanding of functional relationships, who worked with gear ratios and growing rectangles tasks which supported students' abilities to conceive of functions from a quantitatively meaningful stance. Ellis (2011) suggested that beginning with a covariation approach, a student had a more flexible view of function that had the correspondence perspective embedded in it. In other words, the students coordinated both the correspondence and covariation approaches.

Ellis and colleagues (Ellis et al., 2013, Ellis et al., 2015, Ellis et al., 2016) focused on exponential functions and reported the results of teaching experiments with middle school students. Ellis et al. (2013) proposed a learning trajectory (LT) related to exponential function. The LT showed three types of reasoning involved in conceiving of exponential function: pre-function, covariation and correspondence. Similarly, Ellis et al. (2016) presented an Exponential growth learning trajectory (EGLT) and reported the results of a teaching experiment with middle school students who explored the growth of a plant. Ellis et al. (2013) and Ellis et al. (2016) suggested that pre-functional reasoning developed prior to covariational reasoning and correspondence reasoning and the development of correspondence and covariation may occur in a non-sequential manner. The results of these studies also suggested that covariational reasoning influenced students' interpretation of correspondence rules and they moved back and forth between correspondence and covariation perspectives. Ellis et al. (2016) reported that students' reasoning about exponential growth as a correspondence emerged simultaneously with their covariational reasoning. Ellis et al. (2015) reported the results of a teaching experiment with middle school students who explored the growth of a plant, emphasizing three conceptual shifts in coordination of multiplicative and additive growth for exponentiation that supported students' abilities to flexibly move between the covariation and correspondence approaches to function. In all these studies, the authors suggested that students who engaged in covariational reasoning began to develop sophisticated covariational reasoning and a correspondence perspective in tandem.

Empirical studies on the definition of a function. The definition of a function, including its characteristics, appears to be straightforward to mathematicians and scholars. This is not necessarily the case for students who are learning about functions. It is, therefore, important to determine how one defines a function in order to learn how one conceives of a function. In this section, I include empirical studies that address students' reasoning regarding different definitions of a function. Researchers have found that high-performing college students may possess weak function conceptions (Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994b). Students can think a function must be defined by a single algebraic formula (Breidenbach et al., 1992; Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992).

I specifically address the study by Even (1993) because it offers a qualitative way for interpreting students' conceptions of the definition of a function. Even (1993) reported a qualitative study conducted on prospective teachers' pedagogical and subject matter knowledge of the characteristics of a function called *arbitrariness* and *univalence*. The participants' responses as they related to arbitrariness were typically found to be (a) functions are (or can always be represented as) formulas or equations; (b) graphs of functions should be nice (smooth and continuous); (c) functions are "known" (Even, 1993, p.104). The participants responses related to univalence (every element in the domain corresponds to exactly one element in the range) were such that most of them knew this requirement for a relation to be a function, but most of them did not know why this was a requirement. I use Even (1990, 1993) categories for students' responses on the essential characteristics of a function. The above-mentioned study served as a tool that was used in describing students' conceptions of the definition of a function.

Students' Interpretation of Variables and Function Notation

Variables. Because variables have different meanings in different contexts, the notion of a variable is very important to consider in school algebra (Dogbey & Keraint, 2012; Philipp, 1992; Usiskin, 1988; Wagner, 1999). Researchers have found that students as well as many teachers find it difficult to work effectively with variables (e.g., Clement, 1982; Schoenfeld & Arcavi, 1988). These difficulties with variables have been attributed to many factors, including the historical development of variables (Stacey & MacGregor, 1997; Philipp, 1992; Usiskin, 1988) as well as different ways variables are used in different contexts (e.g., x as an unknown in $2x=1$ and x as in $y=f(x)$). Function notation has been reported to be an enormous obstacle for undergraduate students as they develop and refine their understanding of the function concept (e.g., Carlson, 1998).

Complexity of variables. Variables are complex to interpret because of what they mean in different contexts (e.g., Philipp, 1992; Stacey & MacGregor, 1997; Kieran, 1992; Kinzel, 1999; Usiskin, 1988). Researchers describe how variables are used in many different ways today: as labels (h, s in $1h = 3600\ s$; 3600 seconds in 1 hour), specific unknowns (x in $2x+5=7$), continuous unknowns (x in $x+2$ where x=1, 2, 3, …), varying quantities (x, y in $x+y=y+x$), parameters (m, b in $y=mx+b$), and generalized numbers (x, y in $x+y=y+x$) in school mathematics (Philipp, 1992; Usiskin, 1988). Both Stacey and MacGregor (1997) and Kinzel (1999) have observed that students have a limited understanding of algebraic symbols used in different contexts. Kieran (1992) reported that the majority of students think of variables as representing just specific unknowns. Usiskin (1988) pointed out that many students think of variables as letters that stand for only numbers. Yet, the roles of a variable are not always to represent numbers. For example, the variable f often stands for a function. Because same variables are used in different ways, it is difficult for students to interpret them.

Due to different uses of variables, students intertwine different meanings of variables (Booth, 1988; Chazan, 2000; Stacey & MacGregor, 1997; Usiskin, 1988; Wagner, 1981). The variables in mathematics are used to solve for specific unknowns (e.g., $2x+3=5$) and the variables also imply relationships between quantities (e.g., $f(x)=2x+3$). Usiskin (1998) and Chazan (2000) have addressed the dual nature of variables (as an unknown and as a quantity that varies). For example, the variable x is both an unknown and a variable quantity at the same time. Stacey and MacGregor (1997) found that Pre-Algebra students assigned values to variables by turning to other symbol systems. For example, given a variable h representing the height of a boy, some students assigned a value of 8 to h because h is the eighth letter of the alphabet. In addition, Wagner (1981) examined U.S high school students' understanding of letter variables and found that the students considered different letters as always representing different numbers in equations. Students treated n and w in the equations $7n +22= 109$ and $7w +22= 109$ as representing two different numbers. In their view, n represented a smaller value than w, because they thought that the alphabet that came before in an alphabetic order represented a smaller value in an equation. Booth (1988) found that secondary students in UK interpreted symbol as a label of something. For example, in a task "add 3 to 5*y*", one student referred to *y* as yacht, yogurt or yam. Although experts categorize the use of variables differently, a student may not be able to distinguish the use of variable from one another. In all those studies, researchers highlighted different ways students conceived of variables. There is a need to support students' learning of symbols so that they are able to distinguish between different uses of variables.

Function Notation. Function notation is a product of human activity (Freudenthal, 1973) and has evolved over time. Cajori's history of mathematical notations (Cajori, 1928, 1929) gave examples of different attempts and the difficulty to express the relationship between two values. In 1734, Euler created our modern notation and adjusted his notation, but textbooks began using his notation in early 1800's. As I reviewed the literature, I found more studies that focused on specific function notation such as $f(x)=3x+4$ and very few that focused on non-specific rules or a variant of general function notation ($y=f(x)$) such as $f(x+y) = f(x)+f(y)$, and I describe them in the sections below.

Research on function notation involving specific rules. Researchers have addressed function notation involving specific rules or formulas (e.g., Fonger et al., 2016; Musgrave & Thompson, 2014; Thompson, 2013; Thompson & Milner, 2017). Fonger et al. (2016) focused on middle school students' conceptions of quadratic function rules of the form $y=ax^2$ and ways of reasoning supporting their abilities to write notation rules. Fonger and colleagues suggested that tasks focused on covariational reasoning could allow students to have a flexible understanding of function rules. They found that students who engaged in covariational reasoning conceived of function rules as statements of covariation. The students did not have to be flexible going back and forth between the correspondence and covariation approaches. Covariation alone was more powerful to conceive of the quadratic function rule $y=ax^2$.

Thompson (2013) stated that individuals should use function notation to express a relationship between two quantities. Musgrave and Thompson (2014) as well as Thompson and Milner (2017) focused on high school teachers' meanings of function notation and included similar tasks that were taken from *the Mathematical Meanings for Teaching secondary mathematics (MMTsm)* instrument. Musgrave and Thompson (2014) reported that teachers had limited conceptions of function notation and the teachers were not bothered by an ill-defined function. By an ill-defined function the authors meant that the variable on the right-hand side was different than the variable on the left-hand side (e.g., $f(x)= n+1$). Musgrave and Thompson (2014) explained that some teachers conceived of the left-hand side of function notation (e.g., $f(x)=3m+5$) as a "four-character idiom consisting of function name, parenthesis, variable, and parenthesis" (p. 281). In the case of $f(x)=3m+5$, students may evaluate $f(2)$, because to them, the left-hand side is an idiom and is less relevant , so they only pay attention to the right-hand side and get an output of 11. Thompson and Milner (2017) reported results on teachers' meanings of function notation in South Korea and the United States. They found that middle and high school teachers in South Korea had more productive meanings of function notation than the U.S high school teachers. Teachers/students should be provided opportunities so that they interpret function notation as representing a relationship between quantities.

Research on function notation involving non-specific rules. In this section, I address results from studies on function notation involving non-specific rules. In contrast to research on specific rules that focused on students' covariational reasoning, Sajka (2003) as well as Tall, Gray, Ali, Crowley, DeMarois, McGowen, Pitta, Pinto, Thomas and Yusof (2001) employed Gray and Tall (1994) *procept* theory to address the dual use of symbols. According to the procept theory, symbols could be interpreted as both a *process* and a *concept* which they termed procept. For example, multiplication of 8x4 represents multiplication as a process (successive addition) and the result of the process (multiplication of 8x4=32). Sajka (2003) presented a work of a secondary school student named Kasia who interpreted $f(x+y) = f(x)+f(y)$. Sajka explained that Kasia's interpretation of function notation consisted of four procepts. Sajka described how Kasia interpreted $f(x)$ and $f(y)$. At a process stage, Kasia computed values and drew a graph. At the concept stage, she conceived of a function as a formula alone, something that determines the formula, graph alone, and formulas and graphs together.

Sajka (2003) focused on function notation involving non-specific rules or a variant of the general function notation ($y=f(x)$) such as $f(x+y)=f(x)+f(y)$. Tall et al. (2001) focused on the development of symbols in arithmetic, algebra, calculus, and undergraduate mathematics and addressed the dual use of symbols as a process and concept.   Using Gray and Tall (1994) procept theory, Tall et al. (2001) explained that symbols could be interpreted using both the process and the concept or the *procept*. Tall and colleagues explained that concept was the output of the process. For example, Tall et al. (2001) explained that $y=f(x)$ could be interpreted as an assignment as a process and a function as a concept. As a process, by assignment, they probably mean x is an input, f is the name of a function, and y is the output. As a concept, x as an input, f as the name of a function, and y as an output defines a function. In contrast to Sajka (2003) and Tall et al. (2001), in my study, I focus on the impact of students' covariational reasoning on their interpretation of general function notation ($y=f(x)$).

Tasks

In this section, I discuss different tasks that have been used to foster students' variational and covariational reasoing. I draw on tasks involving non-dynamic graphs where attributes were represented on different axes that researchers such as Moore et al. (2013), Moore and Thompson (2015), Moore et al. (2014), and Moore et al. (in press) used in their studies and I adapt those tasks to include function notation $y=f(x)$. I build on Johnson's work (e.g., Johnson, 2015b; Johnson, et al., 2016; Johnson, et al., 2017b) related to the Ferris wheel tasks that involved a dynamic computer environment and extend the Ferris wheel tasks to function notation $y=f(x)$. I also draw on tasks where students responded to other students' claims about graphs (Johnson, Olson, Gardner, & Smith, 2018, August) to learn more about students' reasoning with function notation $y=f(x)$.

Tasks related to graphs. Graphs are prevalent in studies where researchers investigated students' covariational reasoning and conceptions of function. Teachers/researchers cannot treat functions and graphs as isolated concepts, as they serve as a "bridge between reasoning from the concrete to the abstract and reasoning among abstractions" (Leinhardt et al., 1990, p. 3). In other words, graphical representations may allow researchers to investigate students' quantitative reasoning and covariational reasoning. Different ways in which students label the axes and how they reason about the shape of a graph can inform how students form relationships between varying quantities. Students conceive of graphs in different ways and it matters.

Moore and Thompson (2015) introduced the constructs of *static shape thinking* and *emergent shape thinking* to make distinctions regarding students' interpretation of graphs. In *static shape thinking* (Moore & Thompson, 2015), a particular shape or property of shape implies something about quantities. For example, an individual engaged in static shape thinking would conceive of a parabola as U-shaped. In *emergent shape thinking* (Moore & Thompson, 2015), the students view a graph in terms of a trace and how it is made. In other words, a student engaging in *emergent shape thinking* would interpret the graph as expressing the same relationship between quantities. To develop students' reasoning with function and function notation, students should be provided opportunities so that they can conceive of graphs as emergent traces expressing the same relationship between quantities.

Research focused on shape of graphs. Researchers (e.g., Bell & Janvier, 1981; Carlson, 1998; Leinhardt et al., 1990) have explicated impoverished ways in which students interpreted graphs. For example, students may conceive of graphs iconically, such that the shape of a graph would resemble an object.

Bell and Janvier (1981) found that students' reasoning about graphs was based on their perception of the shape of the graph. Bell and Janvier (1981) used terms such as situational distractions and pictorial distractions. They explained that situational distractions occur when the student's experience of the situation interferes with his/her ability to attend to the meanings of the features of the graphs. They also explained that pictorial distractions occur when the student confuses the aspects of the situation. To illustrate these types of distractions, consider the Ferris wheel task (see Figures 4 and 5). Students experiencing a pictorial distraction would graph a circle to represent the relationship between distance and height because the shape of the Ferris wheel should match the shape of the graph. Students experiencing a situational distraction may say that the Ferris wheel is a circle so both distance and height should be increasing and then decreasing. The shape of a graph impacted students' interpretation of graphs.

Similarly, Carlson (1998) found that students' interpretation of graphs depended on the shape of a graph. Carlson (1998) studied how College Algebra and second semester Calculus students reasoned about the position of two cars after one hour given graphs of each car's velocity with respect to time. Carlson found that 88% of students who completed College Algebra with an A and 29% of students who completed Calculus with an A interpreted graphs as pictures of the paths of the cars. Bell and Janvier (1981) and Carlson's (1998) findings provide evidence that students often reason about graphs based on physical characteristics such as the shape of a graph.

Empirical studies related to multiplicative objects. Several researchers have argued for the importance of conceiving of graphs as multiplicative objects (e.g., Ellis et al., 2015; Johnson et al., 2016; Johnson et al., 2017b; Stalvey & Vidakovic, 2015; Thompson, Hatfield, Yoon, Joshua, & Byerly, 2017). Ellis et al. (2015) found that conceiving of multiplicative objects was difficult for middle school students, yet they conceived of quantities at higher levels of variational reasoning. Ellis et al. (2015) reported the results of a teaching experiment with middle school students and explained that students' exponential function reasoning involved conceiving of a next value as the product of a prior value and the growth factor. In particular, a student named Uditi conceived of the variation of a quantity's value as increasing by intervals of a fixed size, which is consistent with what Thompson and Carlson (2017) term *chunky continuous variation*. Uditi did not conceive of quantities as multiplicative objects because her reasoning was at a variational level.

Johnson et al. (2016) and Johnson et al. (2017b) found that students at more advanced levels of covariational reasoning demonstrated that they could conceive of a relationship between the values of individual quantities as a *multiplicative object*, which entailed transforming individual quantities to create a new, joint quantity that comprised the individual quantities. Johnson shared the work of Ana, who interacted with the dynamic Ferris wheel activity. Ana first envisioned the quantities as changing separately (*variation*) and then changing together (*covariation*). Ana's reasoning was consistent with the highest level of covariational reasoning by Thompson and Carlson (2017) called *smooth continuous covariation*, so she conceived of quantities as a multiplicative object.

Stalvey and Vidakovic's (2015) emphasize the centrality of multiplicative objects in a study of 15 Calculus 2 students' responses to the task. Stalvey and Vidakovic (2015) asked students to do these activities: a) graph a relationship between time and the volume of water for both coolers on the same coordinate plane, (b) graph a relationship between time and the height of the water for both coolers on the same coordinate plane, (c) graph a relationship between the volume of the water and the height of the water for both coolers on the same coordinate plane, and (d) indicate the orientation of your graph in (c). Students could answer questions (a) and (b) in terms of an understanding of constant rate of change with respect to time and linearity and envisioned either quantity varying with the passage of time. But questions (c) and (d) demanded that students covary height and volume as each varied with respect to time. Students needed to create volume versus height as a multiplicative object. Stalvey and Vidakovic (2015) reported that a majority of students struggled to envision values of height and values of volume as varying simultaneously in order to sketch a graph. However, one student conceived of the relationship between volume and height as a multiplicative object, such that both height and volume changed simultaneously over small intervals of change.

In another study, Thompson, Hatfield, Yoon, Joshua, and Byerly (2017) reported results from their investigations of in-service mathematics teachers' covariational reasoning. The task consisted of an animation that presented two bars (labeled *v* and *u*) of varying length on the un-numbered axes. The teachers were asked to sketch a graph that captured the values of *u* in relation to the values of *v*. The task focused on teachers' placement of the graph's initial point, and their graphs' shape. As the animation played, the lengths of the bars varied together each having one end fixed at the origin. Like Stalvey and Vidakovic (2015), Thompson et al. (2017) argued that their results highlighted the importance of conceiving of a point as a *multiplicative object* in order to conceive of a graph as a representation of two quantities changing together.

Researcher moves with task design. Having students examine and construct graphs without numbers can help them to focus on the relationship shown by the graph (Van de Walle and Lovin, 2006; Johnson, 2015b; Johnson et al., 2017b). Researchers such as Moore and Thompson (2015), Moore et al. (2014), Moore et al. (in press), Johnson (2015b), Johnson et al. (2016), and Johnson et al. (2017b) used tasks with same attributes represented on different axes and addressed the importance of quantitative reasoning and covariational reasoning.

Moore's Tasks. Moore, Paoletti, and Musgrave (2013), Moore and Thompson (2015), Moore et al. (2014), and Moore et al. (in press) used non-dynamic graphs in their study and suggested the importance of covariational reasoning. Moore and Thompson (2015) conducted a study to learn about students, teachers, and undergraduate students' reasoning about graphs. The tasks used in this study involved the same attributes represented on different axes in a non-dynamic Cartesian coordinate system. Moore et al. (2014) conducted a teaching experiment that consisted of fifteen 75-minute teaching sessions. The participants were prospective secondary teachers (PST) and they were given two tasks. The first task involved the graph of $y=3x$ with unlabeled axes, and the second graph had a different orientation of the axes (y was along the horizontal axis and the x was along the vertical axis). The PSTs had difficulty accepting that both graphs represented $y=3x$, because their concept of slope entailed perceptual cues and global properties of change (static) rather than on the graphs representing quantities that changed together. The second task in Moore et al. (2014) presented the distance and time relationship with different axis orientations. The teachers were asked in an interview if each graph represented a function. Since the relationship between the distance and time was given as a parabola, the PSTs applied the vertical line test and concluded that the graph did not represent a function. The findings of Moore et al. (2014) were similar to Breidenbach et al. (1992) findings who also stated that the vertical line test inhibited students' ability to conceive of x as a function of y.

Researchers/teachers need to "break conventions" to support students' quantitative reasoning (Moore et al., 2014). By "breaking conventions", I mean representing quantities in graphs and function rules differently than what is a common practice in our mathematics community. For example, suppose we have a function given by $h=f(d)$, where d is the distance and h is the height. If we represent the attribute of height on the horizontal axis and the attribute of distance on the vertical axis, a student using a convention of the vertical line test will say that the graph does not pass the vertical line test, so it does not represent a function. But, from a quantitative reasoning perspective, representing the attributes on different axes does not change the relationship between two quantities, so we can express this relationship between quantities as *h* equals *f* of *d* ($h=f(d)$). It is desirable to promote students' use of emergent shape thinking when interpreting graphs (Moore & Thompson, 2015). Moore et al. (in press) found that some pre-service and in-service teachers conceived of graphs as emergent traces and conceived of graphs representing *x* as a function of *y* with *x* along the vertical axis and *y* along the horizontal axis.

Moore et al. (2013) conducted a study to explore service teachers reasoning with graphs in both the Cartesian Coordinate System (CCS) and the Polar Coordinate System (PCS). They asked the participants to graph functions, such as $f(\theta)=2\theta +1$. The teachers began by plotting points and then considered how the values of quantities changed together. Moore and colleagues explained that one of the teachers was able to reason that two graphs that looked different represented the same relationship between quantities. The results from this study suggest that teachers construct invariant relationships by engaging in covariational reasoning. I see a need to extend students' engagement in covariational reasoning to learn more about general function notation ($y=f(x)$).

Johnson's Tasks. Johnson (2015b), Johnson et al. (2016) and Johnson et al. (2017b) used Geometer's Sketchpad Software (Jackiw, 2009) to develop a dynamic computer environment namely the Ferris wheel task. The Ferris wheel environment linked an animation of a Ferris wheel to a dynamic graph relating the quantities of height and distance (see Figures 4 and 5). Johnson (e.g., Johnson, 2015b; Johnson et al., 2016) designed the tasks to study shifts in students' reasoning within variational reasoning, from variational reasoning to covariational reasoning, and within covariational reasoning. Johnson included tasks with attributes represented on different axes. Johnson et al. (2018) designed dynamic covariation techtivities (e.g., Cannon man situation) to promote College Algebra students' covariational reasoning. Johnson and colleagues incorporated Cartesian graphs that represented the same attributes in different ways to provide students opportunities to conceive of graphs representing relationships between quantities.

Johnson et al. (2016) shared the work of Ana, who interacted with the dynamic Ferris wheel activity. Ana first envisioned the quantities as changing separately (*variation*) and then changing together (*covariation*). Her graphs also showed different features when representing the changing distance and height. In particular, investigation of function with the Ferris wheel activity by Johnson et al. (2016) revealed that changing the orientation of the axes provided opportunities for Ana to conceive of height and distance as varying together within different intervals. In the first case, Ana conceived of change occurring in different directions, such as the quantities of height and distance were increasing or decreasing. Her graph showed a single type of curvature to represent change occurring in different directions. In the second case, she conceived of height and distance as varying together within different intervals. In the case of change occurring in an interval, such as from zero to half the total distance, she noticed that the height increases slowly at first, then more quickly, then more slowly as the car reached the maximum height. Her graph depicted different types of curvature to show change occurring in an interval.

The results of Johnson and colleagues (Johnson, 2015b; Johnson et al., 2016; Johnson et al., 2017b) and Moore and colleagues (Moore et al., 2013; Moore et al., 2014; Moore & Thompson, 2015; Moore et al., in press) studies suggested that engaging in quantitative reasoning and covariational reasoning shifted students' and teachers' reasoning about graphs as images of quantities that changed together. I adapted Johnson's Ferris wheel tasks, and Moore's as well as Johnson's tasks with attributes on different axes to include function notation to learn about students' conceptions of general function notation (*y=f(x)*).

Connecting Levels of Function Notation Connected to Students' Variational and Covariational Reasoning

A covariation perspective is useful for students' interpretation of function notation. In my study, I focus on general function notation (*y=f(x)*). A function given by *y=f(x)* is difficult to interpret for two reasons. The students do not engage in any reasoning in their classrooms to interpret what *y=f(x)* really is. As a hypothetical example, if I ask a College Algebra student to explain what *y=f(x)* means to him/her, he/she may say that there is not enough information and he/she wants to see a formula. Second, *y=f(x)* can be problematic because prior studies have shown that elementary and middle school students think of an equal sign as an operator or to do something symbol (e.g., Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; Kunth, Alibali, McNeil, Weinberg, & Stephens, 2011). To promote students' reasoning of *y=f(x)* as a relational symbol, students' engagement in variational and covariational reasoning is important. In this study, I investigated how students' variational and covariational reasoning impacted their conceptions of general function notation (*y=f(x)*).

As I engaged in my literature review, I found a few sources that alluded to general function notation (e.g., Sajka, 2003; Thompson & Carlson, 2017), yet I did not find an empirical study on students' covariational reasoning impacting their conceptions of general function notation (*y=f(x)*). In the table below (see Table 3), I have linked covariational reasoning to my classification categories of general function notation (*y=f(x)*) from a lower level to a higher level. The left column categorizes different conceptions of function notation and the right column provides a description of how students may link conceptions of function notation to their variational or covariational reasoning in a Cartesian coordinate system. My conjecture is that as students engage in higher levels of covariational reasoning, they would also conceive of function notation at a higher level.

Table 3

Students' conceptions of general function notation (*y=f(x)*)

Name	Description of what students do in a Cartesian coordinate system
Function notation as label	At this level, students match a label to a graph. Students associate function notation *y=f(x)* or *x=f(y)* with the shape of a graph. Students may employ a correspondence approach without engaging in covariational reasoning.
Function notation as convention	Student may engage in variational or covariational reasoning and may use convention of matching the axes labels to function notation. Function notation can be written either as *x=f(y)* or *y=f(x).* If the horizontal axis is labeled as *x*, then function notation should be written as *y=f(x)*, so that the variable in the parentheses should match the variable on the horizontal axis. If the horizontal axis is labeled as *y*, then function notation should be written as *x=f(y)*, so that the variable in the parentheses should match the variable on the horizontal axis.
Function notation as a relationship between variables	Thompson and Carlson (2017) definition of function: When students employ a correspondence approach and engage in covariational reasoning, function notation is not just how quantities change together, but also a special way (function) in which they are related. The independent variable can be represented along the horizontal axis or the vertical axis, and the function notation can be written as *y=f(x)* or *x=f(y)* as long as value of one quantity *x* satisfies the value of the other quantity *y* or the value of one quantity *y* satisfies one value of the other quantity *x*.

I elaborate on the levels of students' conceptions of general function notation (*y=f(x)*) and links between students' covariational reasoning and their conceptions of function notation in a Cartesian coordinate system. Suppose that in a swing situation, a child has been swinging and the graph represents the total distance traveled and the height of the swing. A student engaging at a *function notation as label* level may conceive of variables as letters that have nothing to do with distance and height of the swing changing and may conceive of general function notation representing different letters *y*, *f*, and *x*. Students at this level may draw on the shape of a graph to conceive of function notation. At a *function notation as convention* level, a student may engage in variational reasoning or covariational reasoning. When engaging in variational reasoning, a student may conceive of height of the swing increasing or decreasing. Similarly, a student may conceive of distance as increasing if he/she engages in variational reasoning. When engaging in covariational reasoning, a student may conceive of both distance and height changing such that as the distance increases, the height of the swing increases and then decreases. In terms of the general function notation ($y=f(x)$), a student engaged in variational reasoning or covariational reasoning may write a function rule depending on how axes are labeled. By that I mean, if the horizontal axis is labeled x and the vertical axis is labeled y, the student may write $y=f(x)$. If the horizontal axis is labeled as y, then function notation should be written as $x=f(y)$, so that the variable in the parentheses should match the variable on the horizontal axis. At a *function as a relationship between variables* level, students may engage in covariational reasoning and employ a correspondence approach together. To classify conception of function notation at a *function as a relationship between variables* level, I adapted Thompson and Carlson (2017) definition of function to function notation. I refer to Thompson and Carlson's (2017) definition as a combination of correspondence and covariation approaches. Function notation can be written as $y=f(x)$ or $x=f(y)$ as long as value of one quantity x satisfies the value of the other quantity y or the value of one quantity y satisfies one value of the other quantity x.

Conclusion

When a student conceives of a function as a relationship between two quantities, he /she may conceive of function notation $y=f(x)$ based on quantities as well. It is useful for students to view function notation and graphs as a relationship between varying quantities (Thompson & Carlson, 2017). Students' quantitative reasoning (Thompson, 1993; Thompson, 2011) and covariational reasoning (Thompson & Carlson, 2017) also plays an important role to know how they conceive of function situations. For example, given a graph, do the students see dots or do they see graphs as *multiplicative objects*? (Thompson & Carlson, 2017; Thompson, et al., 2017). The tasks researchers use to examine mathematical sense making impact students' opportunities to engage in mathematics reasoning. The table I presented earlier (see Table 3) served to link students' reasoning with quantities and general function notation ($y=f(x)$) and I discuss this in the next chapter.

CHAPTER IV

METHODS

In this chapter, I will articulate the research methodology that I used in conducting the study. First, I describe how I selected the participants and collected data, followed by a description of the tasks that I used for the study, explicating the goal of each task. Finally, I include the methods that I used for data analysis. In addition, I include some data excerpts to illustrate my data analysis.

Research Questions

To study students' reasoning about varying quantities and its impact on the function notation, I pose the following research questions:

1. How might students' conceptions of function impact their conceptions of function notation?
2. How might covariational reasoning related to function impact students' conceptions of function notation?
3. How do students conceive of general function notation ($y=f(x)$)?

Research Methodology

Grounded Theory

I employed grounded theory by Corbin and Strauss (2008) to answer my research questions. Grounded theory assumes a pragmatic philosophy of knowledge (Corbin & Strauss, 2008), which means that knowledge is unique to individuals, and cannot be transmitted from one to another. The methodology of grounded theory will enable me to develop rich descriptions of students' reasoning about changing quantities and function notation in my analysis. For my theoretical framework, I am using Piaget's (1985) constructivism, which asserts that individuals construct their own knowledge from what they already know. I anticipate that students will reason about quantities and function notation in different ways, and I am consistent with the grounded theory (Corbin & Strauss, 2008) that reasoning across students vary and no connections exist among different forms of reasoning.

Case Study & Clinical, Task -Based Interviews

Yin (2006) identified three steps to design a case study: defining the case, justifying selection of case studies, and stating how theoretical perspectives inform the case. I report three case studies (Stake, 2005) of a student's impact of covariational reasoning on function notation.

Clinical interviews are open-ended and allow gathering information on individuals' mental processes and uncovering their underlying thought processes (Clement, 2000). Task-based interviews involve a participant taking part in strategically designed mathematical tasks. These tasks provide a setting to examine individuals' mathematical reasoning (Goldin, 2000). Clinical interviews (Clement, 2000) and task-based interviews (Goldin, 2000) were useful to investigate my research question. I used clinical interviews to explore an individual's thinking without promoting shifts. Clinical interviews allow researchers to pay attention to differences among individuals' mathematical conceptions. I conducted semi-structured interviews (Zazkis & Hazzan, 1999), which are defined as the questions that are planned in advance but may be modified based on the participants' response (Zazkis & Hazzan, 1999). I used semi-structured because I modified or clarified the questions based on a participant's response.

Pilot Study

I conducted a small pilot study at an urban university in Spring 2016. I started out with three participants, but two of them were unable to complete the study. I had only one student Jenna (pseudonym) who completed the entire study. The purpose of the study was to learn how students responded to my tasks, and how I could develop my tasks based on the results of my pilot study.

In my pilot study, I asked about both function notations $y=f(x)$ and $f(x) = 3x+4$. I asked about function notation from multiple perspectives (representations) such as from a function notation to a graph, from graphs to a function notation, and then from a situation to a graph and function notation. After conducting my pilot study, I discussed with my advisor, Dr. Johnson, and modified my tasks. I wanted to be more purposeful about investigating students' conceptions of function. So, I decided to ask about general function notation such as $y=f(x)$. The tasks that I present in this chapter are the modified versions from my pilot study. For example, I added tasks 9 and 11 (see Table 5) to gather evidence of any links between students' variational or covariational reasoning and function notation.

Participants

To select the participants, I visited five sections of College Algebra classes and announced my study. I asked students to fill out consent forms and information sheet, so I could contact them either by e-mail or phone to schedule the interviews. I selected students who were enrolled in a College Algebra class based on their willingness to participate in the study. I expected College Algebra students to be familiar with graphs and function notation, which would help them to make sense of task aspects.

I interviewed six participants for my study who were enrolled in a College Algebra class at a University at the time of the interviews. It is an urban university with a large commuter population. Out of the six participants, I selected three students, Jack, Dave, and Lisa (pseudonyms) who had different forms of reasoning and provided a richer set of data to analyze. I selected students who demonstrated different forms of reasoning and different conceptions of function notation in the Pre interview and the Post interview. In this study, I identified when students demonstrated shifts in their conceptions of function notation, their conceptions of function, and their variational and covariational reasoning. I had a range of reasoning in this group and I was able to learn how students conceived of the changing quantities and general function notation. I used those shifts to organize each case study chapter and the cross case chapter to highlight links between students' covariational reasoning and their conceptions of function notation. While I identified shifts in students' conceptions and reasoning, explaining the mechanism of how those shifts happened is beyond the scope of this study.

Data Collection

I conducted a sequence of four task-based clinical interviews with each student in Spring 2017 during the months of April, May, & June. The timeline for each student was once a week. When the student needed to reschedule, the gap between the interviews was longer or shorter than a week (see Table 4).

Table 4

Schedule of Participant's Interviews

Date	Participant	Event
04/05/2017	Sam	Pre interview
04/12/2017	Sam Lisa	Ferris wheel task 1 Pre interview
04/13/2017	John	Pre interview
04/16/2017	Sam	Ferris wheel task 2
04/21/2017	John Lisa Sam	Ferris wheel task 1 Ferris wheel task 1 Post interview
04/28/2017	John Jack Jane Lisa	Ferris wheel task 2 Pre interview Pre interview Ferris wheel task 2
05/11/2017	John Lisa Dave	Post interview Post interview Pre interview
05/12/2017	Jane	Ferris wheel task 1
05/16/2017	Jane Jack Dave	Ferris wheel task 2 Ferris wheel task 1 Ferris wheel task 1
05/22/2017	Dave Jane	Ferris wheel task 2 Post interview
05/26/2017	Dave Jack	Post interview Ferris wheel task 2
06/06/2017	Jack	Post interview

Each interview had its own set of tasks (see Appendices E-H). A series of four interviews allowed me to use a variety of tasks and gave me an opportunity to further investigate students' work from prior interviews. I had another graduate student, Peter, who helped me video-record the interviews. In addition, I collected students' written responses and took digital photographs of the written material. First, I transcribed the interviews and then annotated the transcripts. If a student drew diagrams, made gestures, or manipulated a computer representation, I took screen shots to capture what the student did at that moment. I placed those screen shots in an annotated transcript. As an example, I included a screen shot from my study (see Figure 3). In this screen shot, Jack moved the cursor counterclockwise from the starting point and stopped mid halfway from the starting point to show that it was the zero point for the distance.

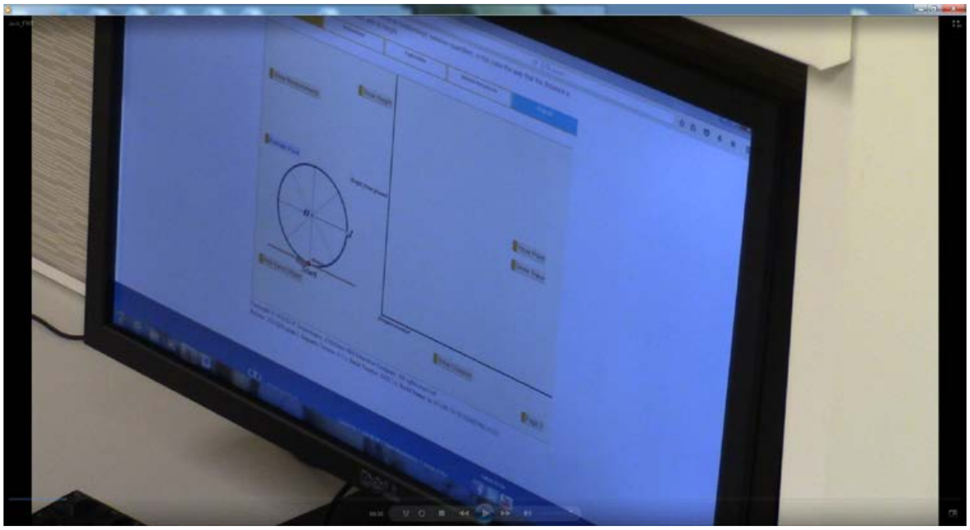


Figure 3: Jack's starting point mid halfway (where the cursor is) from the actual starting point (bottom of the Ferris wheel) in *Ferris Wheel* Interview1

Additions to the interview sequence during data collection. Midway through data collection, I met with my advisor, Dr. Johnson, who provided me an idea to have students respond to others' claims about a graph (see also Johnson, Olson, Gardner, & Smith, 2018, August). As a result, we modified a few tasks. I was concerned that I was not getting enough data about students' conceptions of function notation. I wanted to make sure that my interview schedule would make it likely that I would gather evidence that would count as data. For example, in the Pre interview and the Post interview, the task of matching four sets of graph with function rules had a graph of a linear function. That linear graph could be written using two function notations, but within the Pre interview, Lisa, whom I interviewed first, followed the convention of matching the axes of graph with variables in function notation. I wanted to learn if she could conceive of using two function notations for that linear graph. So, I added a task in the

Post interview where I asked Lisa to interpret a response from another student named Max who said that both $r=s(m)$ and $m=s(r)$ could be used to describe a linear graph. This task was very useful to provide the evidence that I could count as data.

Interview Sequence

The interview sequence consisted of a Pre interview, two Ferris wheel interviews, and a Post interview. Each interview included multiple tasks, which I describe in detail in this section. The first interview served as a Pre interview. In the Pre interview, each student worked on several tasks about functions and function notation. I used those tasks to diagnose where the students were in terms of the function concept. Then students had a second interview, in which I used a *Ferris wheel task* that I adapted from Johnson (2015b). I used the Ferris wheel task to investigate students' covariational reasoning and their interpretation of function notation. In the third interview, I used the Ferris wheel task, but this time, I had the attributes on different axes, which means that the quantity that was represented on the horizontal axis was now represented on the vertical axis. The fourth interview served as a Post interview. I conducted the fourth interview to investigate any changes that occurred in students' reasoning and interpretation of function notation from the Pre interview to the Post interview. During each interview, I asked questions to better understand how students reason about the quantities and how they connect it to function notation. In this section, I describe the overarching tasks for each interview. The tasks that I present below include the additions to the interview sequence. See Appendices A-H for a complete description of tasks and interview schedules.

Pre interview and Post interview. The Pre interview and the Post interview had a similar structure with slightly different graphs and different variables in general function rules. Both the Pre interview and the Post interview had the same situation tasks. There were 11 overarching tasks in each. The tasks consisted of functions, graphs, tables, and function rules. I also had tasks that involved a description of a situation, which I refer to as situation tasks. I decided to put functions, graphs, tables, and function rules tasks first, because I wanted to learn how students interpreted general function notation ($y=f(x)$). Then I had situation tasks to learn if there were any connections between students' covariational reasoning and function notation. I used the term one-to-one, where one input mapped to one output. For example, in the case of linear graphs, one input maps to one output. I used the term onto, where different inputs mapped to the same output. For example, in the case of parabolas opened upward, two different inputs map to the same output.

Table 5 (left column) shows overarching tasks for the Pre interview and Table 5 (right column) shows the tasks for the Post interview.

Table 5
Brief Overview of Pre interview and Post interview Tasks

Task	Pre interview (italics)	Post interview (bold)
1	What comes to your mind when you think of a function?	What comes to your mind when you think of a function?
2	Please read each statement out loud and explain what each statement means. a. *Given y=f(x), for every input x, there is exactly one output y.* b. *Given x=f(y), for every input y, there is exactly one output x.* c. *Given g=r(y), as y increases, g decreases.* d. *Given y=g(r), as r increases, y increases and then decreases.*	Please read each statement out loud and explain what each statement means. a. **Given g=f(y), for every input y, there is exactly one output g.** b. **Given x=t(y), for every input y, there is exactly one output x.** c. **Given m=r(y), as y increases, m decreases.** d. **Given y=g(p), as p increases, y increases and then decreases.**
3	Given two tables one representing y as a function of x and the other representing x as a function of y and a *one-to-one and an onto graph*, do these represent function?	Given two tables one representing y as a function of x and the other representing x as a function of y and a one-to-one **and an onto graph rotated clockwise**, do these represent function?
4	What does *u=r(s)* mean? How do you make sense of it?	What does **g=r(m)** mean? How do you make sense of it?
5	Given a set of four graphs *including parabola opening downward along the x-axis, a circle, a line, and a parabola opened sideways along the y-axis*, which represent functions? Which do not? Why?	Given a set of four graphs **including a parabola opening upward, an ellipse, a line, a parabola opened** sideways which represent functions? Which do not? Why?
6	Use the general function rules provided (with independent variable represented by the x-axis or y-axis) to describe graphs in 4.	Use the general function rules provided (with independent variable represented by the x-axis or y-axis) **(used different letters)** to describe graphs in 4.
7	Sam said that both *m=t(p)* and *p=t(m)* can be used to describe the following graph. Why that made sense to that person? What do you think?	Max said that both **m=s(r)** and **r=s(m)** can be used to describe the following graph. Why that made sense to that person? What do you think?
8	a. Given a description of a plane's altitude and distance traveled along the ground, please interpret the graph. b. Given the plane situation with attributes on different axes, please interpret the graph. c. Is it possible to write the notation as *a=f(d)* or *d=f(a)*?	a. Given a description of a plane's altitude and distance traveled along the ground, please interpret the graph. b. Given the plane situation with attributes on different axes, please interpret the graph. c. Is it possible to write the notation as **a=f(d)** or **d=f(a)**?
9	*Nat* said that for the first plane situation, the graph can be written as both *a=f(d)* and *d=f(a)*. What do you think?	**Chris** said that for the first plane situation, the graph can be written as both **a=f(d) and d=f(a)**. What do you think?
10	a. Given a description of a swing representing the total distance traveled and the height of the swing, please interpret the graph. b. Given the swing situation with attributes on different axes, please interpret the graph. c. Is it possible to write the notation as *h=f(d)* or *d=f(h)*?	a. Given a description of a swing representing the total distance traveled and the height of the swing, please interpret the graph. b. Given the swing situation with attributes on different axes, please interpret the graph. c. Is it possible to write the notation as **h=f(d)** or **d=f(h)**?
11	For the swing situation, *Pat* said that both graphs can be written as *h=f(d)*. What do you think?	For the swing situation, **Sam** said that both graphs can be written as *h=f(d)*. What do you think?

For the Pre interview and the Post interview, the first task in Table 5 was an open ended question about a function. The goal of this question was to learn how students conceived of a function and different representations of a function such as a graph or a function notation.

The second task in Table 5 asked students about each statement and what each statement meant to them. The goal of this task was to learn how students interpreted the definition of a function. A student may say that statement 2a defines a function because we have $y=f(x)$. Students are familiar with the notation $y=f(x)$ because that is how textbooks typically present it. For statement 2b, the notation is different, and participants may think that this does not represent a function because the order of the variables is different in the notation $x=f(y)$. It is also possible that students may pick different points to explain what function notation means. Statements 2c and 2d are definitions of function from a covariation perspective. Students are less familiar with this definition and may not consider these as function definitions. Students may express each statement graphically and try to match the variable on the parentheses to match with the variable on the horizontal axis and the variable on the other side of the equal sign to be on the vertical axis.

The third task in Table 5 asked if given two tables, one representing y as a function of x and the other representing x as a function of y, and a one-to-one and an onto graph, which of those represented a function. Students are familiar with tables and a student may say that the first table defines a function because it is given as $y=f(x)$. A participant may say that y equals f of x does not represent a function because it violates the characteristic of univalence and is not one-to-one because $f(-1)=2$ and $f(3)=2$. The other table represented x as a function of y, but the students may still consider y as a function of x and say that the same input of -3 gives two outputs 1 and 2. I asked them to read the statement first and then students may say that different inputs have an output. Some may say that two different inputs of 1 and 2 map to the same output of -3, which does not represent a function because they may conceive of one-to-one functions as functions. In addition to the tables, I had a one-to-one and an onto graph, as I explained in interview sequence, to learn how students conceived of a function. Students may say that a one-to-one graph represents a function but an onto graph does not represent a function, because they may conceive of "every input has an output" as one input mapping to one output but may not consider different inputs mapping to the same output to be a function.

In the fourth task in Table 5, I asked the students what general function notation meant to them. I had function notation $u=r(s)$ in the Pre interview and function notation $g=r(m)$ for the Post interview. Through this task, my goal was to learn what each variable meant to the students and how they connected the rule to a scenario or a graph.

The fifth task in Table 5 asked students to explain which graphs represented functions, which graphs did not represent functions and why. I chose a set of four graphs for both the Pre interview and the Post interview with slightly different graphs. The first graph was a parabola opening downward /opening upward (not one-to-one). The second graph represented a circle/ellipse, which did not represent a function. The third graph was a linear graph and it represented a function. The fourth graph represented parabolas opening sideways and it did not represent a function if students used the vertical line test. But, if a student engaged in quantitative reasoning, then the parabolas opening sideways represented a function. The goal of this task was to learn how students conceived of a function and if they engaged in quantitative reasoning specifically for the graphs with parabolas opening sideways.

The sixth task in Table 5 asked students to match the graphs above with the function rules. This question was to learn if the students chose the rule for all graphs or only for the graphs that represented functions. Students may choose their rules based on how the axes are labeled. For example, if the horizontal axis is labeled *m* and the vertical axis is labeled *g*, then they may say that *g=f(m)* is the correct notation. It is also possible that they may employ a correspondence approach to function and decide which rule should work. For example, for the linear graph, they may say that one input maps to one output, so it represents a function.

The seventh task in Table 5 asked students if the linear graphs could be written by both *m=t(p)* and *p=t(m)* in the Pre interview and by *m=s(r)* and *r=s(m)* in the Post interview. The goal of this task was to learn if students matched the variable on the horizontal axis to the variable in the parentheses and the variable on the vertical axis to the variable on the other side of equal sign to conceive of function notation, because that is how they learn it in school. Thinking covariationally, as one quantity increases the other also increases, so both function notations can be used. A student may combine covariational reasoning to the correspondence approach and say that one input has one output and as one quantity increases, the other also increases and therefore we can write notation both ways. Overall, the goal of tasks 2,3, 4, 5, 6, and 7 was to provide students multiple representations of a function and to learn if they reasoned differently about function notation and function notation rules describing the graphs.

The eighth task in Table 5 asked about a plane's distance traveled along the ground and the altitude. The students were given a graph and they had to interpret the plane situation. The goal of this task was to learn how students conceived of quantities. I also asked students to have the distance traveled along the ground represented on the vertical axis and the altitude on the horizontal axis. The goal of this task was to learn if they engaged in covariational reasoning. Then I asked them about general function notation and if they could write both *a=f(d)* or *d= f(a)* for each graph. The goal of this task was to learn if students paid attention to the labels or engaged in quantitative reasoning and covariational reasoning and combined the covariational perspective to the correspondence approach to conceive of function notation. If students pay attention to how the axes are labeled, then the graph in the first situation can be written as $a=f(d)$ and the graph with distance along the vertical axis can be written as $d=f(a)$. If students engage in quantitative reasoning and covariational reasoning, then both graphs represent distance, one along the horizontal axis and the other along the vertical axis. A student may say that each distance value corresponds to an altitude, so $a=f(d)$ is the function notation. If the students say that the altitude depends on the distance and also each distance value corresponds to an altitude value and $a=f(d)$ is the correct function notation, then they have combined the covariational approach to the correspondence approach to conceive of function notation.

The ninth task in Table 5 asked that for the first plane situation with distance along the horizontal axis, was it possible that the graph could be written as both $a=f(d)$ and $d=f(a)$. The goal of this task was to gather evidence if students engaged in covariational reasoning or employed a correspondence approach, or if they combined both to justify function notation.

The tenth task in Table 5 asked about a swing situation representing the total distance traveled and the height of the swing. The goal of this task was to learn how students conceived of quantities. I also asked students to have the total distance traveled represented on the vertical axis and the height of the swing on the horizontal axis. The goal of this task was to learn if they engaged in covariational reasoning. Then I asked them about general function notation ($y=f(x)$) and if they could write both $h=f(d)$ and $d=f(h)$ for this situation. The goal of this task was to learn if students paid attention to the labels or engaged in quantitative reasoning and covariational reasoning and combined covariational reasoning to a correspondence approach to conceive of function notation. If students pay attention to how the axes are labeled, then the graph in the first situation can be written as *h=f(d)* and the graph with distance along the vertical axis can be written as *d=f(h)*. If students engage in quantitative reasoning and covariational reasoning, then both graphs represent distance, one along the horizontal and the other along the vertical axis. A student may say that each distance value corresponds to a height, so *h=f(d)* is the function notation. If the students say that the height depends on the distance and each distance value corresponds to a height value and *h=f(d)* is the correct notation, then they have combined the covariational approach to a correspondence approach to conceive of function notation.

The eleventh task in Table 5 asked if for the swing situation, both graphs could be written as *h=f(d)*. The goal of this task was to learn if students paid attention to the labels or engaged in quantitative reasoning and covariational reasoning and combined covariational reasoning to a correspondence approach to conceive of function notation. If students engage in quantitative reasoning and covariational reasoning, then both graphs represent distance increasing, one along the horizontal axis and the other along the vertical axis. The height increases and decreases while the distance keeps increasing. Thinking quantitatively, in both cases *h* depends on *d* and also employing a correspondence approach, a student may say that each distance value corresponds to a height, so *h=f(d)* is the correct notation. In other words, they may combine covariational reasoning and a correspondence approach to conceive of function notation.

Ferris wheel Interview 1. The Ferris wheel tasks involved a dynamic computer environment that Johnson created in Geometer's Sketchpad (Version 5.0). Ferris wheel interview 1 incorporated a sequence of tasks, each following the task template shown in Table 6. Johnson (2015b) used Ferris wheel tasks where two changing quantities were other than time. Students worked with the Ferris wheel and graphs without numbers. The students also reasoned about quantities when the attributes were represented on different axes. The Ferris wheel tasks were adapted from Johnson (2015b) and included the changing distance and height of the Ferris wheel (see Figure 4).

Table 6

Ferris Wheel Interview 1 Template

Part	Description
1	Identify changing attributes.
2	Sketch a graph that represents a relationship between given attributes.
3	Investigate general function notation
4	Investigate how vertical and horizontal segments on graph relate to Ferris wheel
5	Compare graph sketched in Part 2 with trace shown on dynamic graph
6	Investigate general function notation again.
7	Pat said that the graph below can be written as either *d=f(h)* or *h=f(d)*. What do you think?

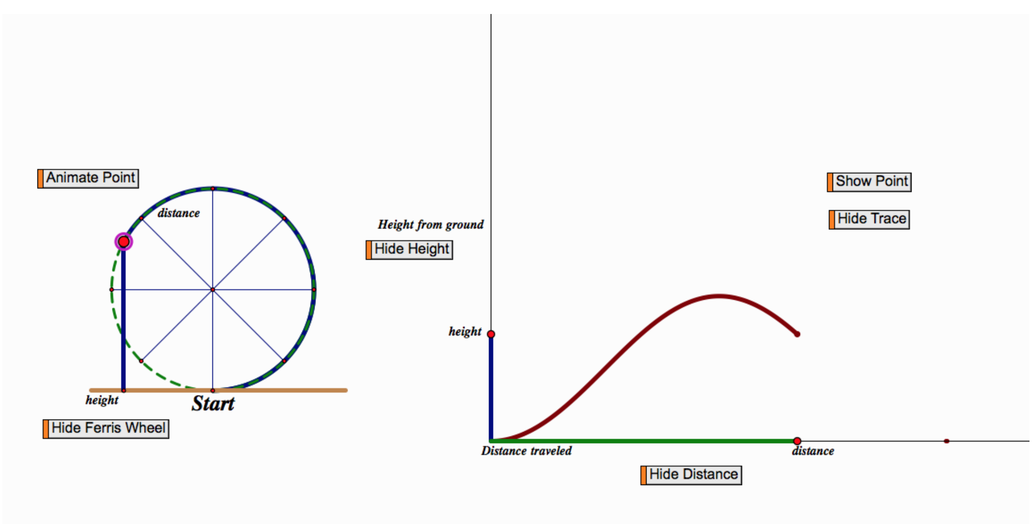


Figure 4: A Dynamic Ferris Wheel Computer Activity Relating Height and Distance

Johnson (2015b) used quantities of distance, width, and height. I only used two quantities distance and height to keep the task simple and to learn more about students' conceptions of function. The task used in this study was similar to Johnson (2015b) where I asked students to explain how the distance and the height changed. I also asked them to graph the relationship between distance and height. I also asked about the moving point and the trace. I added a part on general function notation which is different than Johnson (2015b) and examined students' reasoning about general function notation.

The first task in Table 6 asked about the changing attributes. First, I asked the students to talk about their experience of being on a Ferris wheel. The students saw the Ferris wheel without any measurements (see Figure 4). After the students watched the animated Ferris wheel, I asked them to explain how distance changed and how height changed to learn how they conceived of distance and height.

The second task in Table 6 asked the participants to predict and graph the relationship between distance and height. The goal of this task was to learn how students' reasoning impacted how they graphed two quantities. Another important thing I wanted to note was how students labeled their axes.

The third task in Table 6 asked the students to express the relationship between two quantities in a function notation. By viewing variable as something that can vary, rather than only as an unknown, students can make more meaning from function notation (Chazan, 2000; Usiskin, 1988). For example, when students solve an equation such as $2x+3=5$, the variable x has the meaning of an unknown. When students interpret general function notation such as $y=f(x)$, students need to think of a variable x as something that varies. I asked students about each variable in general function notation to learn how they conceived of function notation and if they conceived of function notation idiomatically (Musgrave & Thompson, 2014) as I discussed in Chapter 3.

The fourth task in Table 6 investigated how vertical and horizontal segments on a graph related to the *Ferris wheel*. Students saw the animated distance segment along the x-axis (see Figure 4). I asked them to explain what the length on the x -axis meant in terms of the Ferris wheel. This helped me to learn how students conceived of distance represented by a segment. Then, I showed students the height segment along the y-axis. Then the students saw the animation of both distance and height segments together. This task was designed to help the participants to think about two changing quantities.

At the end, the students saw full animation of the Ferris wheel, the trace and the moving point. The fifth task in Table 6 asked students to compare their graph predictions to the animation. Their explanation helped me to learn how they reasoned about two quantities.

The sixth task in Table 6 asked the students about function notation once again. The goal of this task was to learn if students conceived of the relationship between the two changing quantities notationally, and if they could express that in a function notation form at the end of this task.

The seventh task in Table 6 asked students if the same graph could be written as $h=f(d)$ or $d=f(h)$. The goal of this task was to learn if students paid attention to the labels or engaged in quantitative reasoning and covariational reasoning and combined the covariational approach to a correspondence approach to conceive of general function notation. If students pay attention to the labels, then the graph on the left can be written as $h=f(d)$ and the graph on the right can be written as $d=f(h)$. If students engage in quantitative reasoning and covariational reasoning, then both graphs represent distance along the horizontal axis and we cannot interchange variables. A student may say that each distance value corresponds to a height, so *h=f(d)* is the function notation. If the students say that height depends on the distance and each distance value corresponds to a height value and *h=f(d)* is the correct function notation, then they have combined covariational reasoning to a correspondence approach to conceive of function notation.

Ferris wheel Interview 2. Table 7 shows Ferris wheel interview 2 template. The first task in Table 7 asked the students to run the animation again to explain how distance changed and how height changed. This served as a quick overview of what they did in their previous interview (Ferris wheel interview 1).

Table 7

Ferris Wheel Interview 2 Template

Part	Description
1	Identify changing attributes.
2	Sketch a graph by having the attributes on different axes.
3	Investigate general function notation
4	Investigate how vertical and horizontal segments on graph relate to Ferris wheel
5	Compare graph sketched in Part 2 with trace shown on dynamic graph
6	Investigate general function notation again.
7	Nat said that both graphs can be written as *h=f(d)*. What do you think?

The second task in Table 7 asked students to predict and then graph the relationship between distance and height with attributes represented on different axes (see Figure 5). Distance was now on the vertical axis and the height was on the horizontal axis.

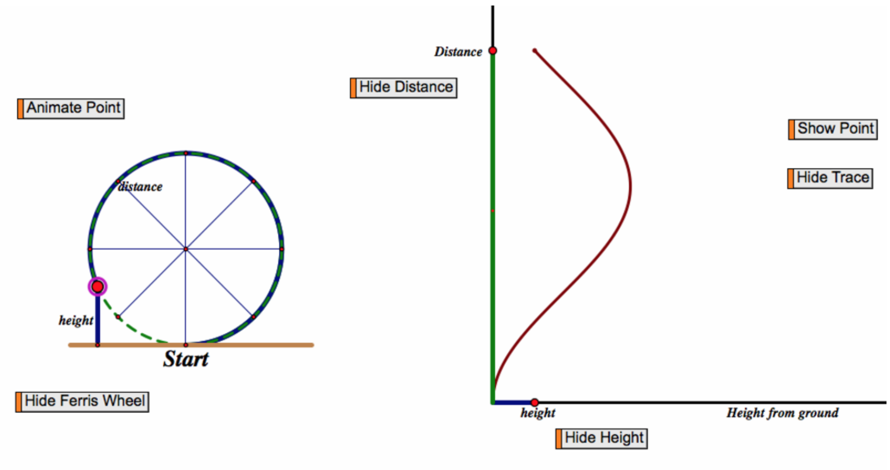


Figure 5: Ferris Wheel Activity Relating Distance and Height

Graphical representation allowed me to learn how students interpreted the two changing quantities and how they demonstrated their thinking in terms of the graph. I also wanted to learn if switching attributes on different axes changed how students conceived of quantities. This task helped me to explore students' reasoning about function.

The third task in Table 7 investigated general function notation by asking students to express the relationship between two quantities in a function notation. By viewing variable as something that can vary, rather than only as an unknown, student can make more meaning from function notation (Chazan, 2000; Usiskin, 1988). For example, when students solve an equation such as $2x+3=5$, the variable x has the meaning of an unknown. When students interpret general function notation such as $y=f(x)$, I expect that students should be able to conceive of x and y as two varying quantities. I asked students about each variable in general function notation to learn how they conceived of general function notation and if they conceived of function notation idiomatically (Musgrave & Thompson, 2014) as I discussed in Chapter 3.

In the fourth task in Table 7, I investigated how students thought about vertical and horizontal segments on the graph in relation to the Ferris wheel. In this task, students saw the animated distance segment along the vertical axis, and I asked them to explain what the length on the vertical axis meant in terms of the Ferris wheel. This helped me to learn how students conceived of distance represented by a segment. They saw the height segment along the horizontal axis. The student may be bothered that we now have height along the horizontal axis instead of distance. It is possible that they may say that the height keeps going just because they had distance along the *x*-axis before and they may still be thinking about the distance. Then, I showed them the animation of both distance and height segments together. This task was designed to help the participants to think about two changing quantities.

At the end, the students saw full animation of the Ferris wheel, the trace and the moving point. The fifth task in Table 7 asked students to compare their graph predictions to the animation. Their explanation helped me to learn how they reasoned about two quantities.

The sixth task in Table 7 asked the students about function notation once again. The goal of this task was to learn if students conceived of function notation expressing the relationship between two changing quantities, and if they could express that relationship in a function notation at the end of this task. This task (switching axes) allowed me to explore how students related quantities to function notation.

The seventh task in Table 7 asked students if two different graphs could both be written as $h=f(d)$. The goal of this task was to learn if students paid attention to the labels or engaged in quantitative reasoning and covariational reasoning and combined covariational reasoning to a correspondence approach to conceive of function notation. If students pay attention to the labels, then the graph on the left can be written as $h=f(d)$ and the graph on the right can be written as $d=f(h)$. A student only looking at the label for a graph on the right may say that one height input corresponds to two different distance outputs, so the graph does not represent a function and therefore cannot be written as $d=f(h)$.

If students engage in quantitative reasoning and covariational reasoning, then both graphs represent distance increasing, one along the horizontal axis and the other along the vertical axis. The height increases and decreases while the distance keeps increasing. Thinking quantitatively, in both cases h depends on d and employing a correspondence approach, a student may say that each distance value corresponds to a height, so $h=f(d)$ is the correct notation. In other words, they may combine covariational reasoning and a correspondence approach to conceive of function notation.

Students and tasks. The table given below (see Table 8) describes which additional tasks Jack, Lisa, and Dave worked on and which additional tasks they did not work on.

Table 8

Tasks given to students

Students	Pre interview	Ferris wheel interview 1	Ferris wheel interview 2	Post interview
Lisa	Did not work on modified tasks 4,7,9,11 (see Table 4)	Did not work on task 7 (see Table 6)	Did not work on task 7 (see Table 7)	All tasks
Jack	Did not work on tasks 4,7,9,11 (see Table 4)	All tasks	All tasks	All tasks
Dave	All tasks	All tasks	All tasks	All tasks

Interview Schedules

For each of the interviews, I created an interview schedule to guide the flow of the interview. The interview schedules allowed me to plan questions that helped me to better understand student's reasoning and its impact on general function notation. I used interview schedules as a guide to anticipate possible student responses and probing questions prior to the interview (see appendices A-D).

Each interview schedule used a two-column format. The left-hand column contained researcher questions and potential student responses. After each interviewer question, I included potential student responses and probing questions to further investigate those responses as appropriate.

The right-hand column contained annotated notes. The notes focused on my thinking about the interview. The notes included rationale for asking a particular question, what I think I might infer about a students' reasoning from a particular response, and connections between researcher questions or potential student responses and existing literature. The notes also included statements regarding what to ask the student and how to proceed with the interview. I illustrate these using an example.

I am including an example relating to the definition of function because it is critical to learn how students conceive of the definition of a function. It is also important to learn what students consider to be a function and what students do not consider to be a function. For example, I include a snippet of a table from my appendix (see Table 9).

Table 9

Task and Potential Student Responses

1. What comes to your mind when you think of a function? P1: I don't know. Follow up: Would it help if I gave you something more specific? Sometimes in math, we have graphs, tables, and equations. Would any of these help? P1: yes. Prompt: I'll ask them to read each statement and explain what it means. I will also provide tables and graphs and ask them if tables/graphs helped to clarify the statements.	This question is to know what students think about functions and what do they mean by a function and general function notation. Students could define the function as a correspondence (For every *x*, there is an output *y*). They may define a function using a graphical representation or a symbolic representation. Students often think a function must be defined by a single algebraic formula (Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992). The students may graph a function and write a notation. They may pick points or just graph a function without picking numbers.

I asked the students why the given set of graphs represented functions and which rules described those graphs. By asking these questions, I was able to gather evidence of students' reasoning about functions and how it impacted their conceptions of general function notation.

Analysis Methods

The transcripts, students' written materials, and video recordings of students' interviews served as a source of data. I determined which sections were important by looking at where the student provided evidence of attending to quantities (variation/covariation), function notation, or both changing quantities (covariation) and function notation. My data analysis consisted of ongoing analysis and retrospective analysis. The ongoing analysis took the form of reflective notes, which I compiled after each interview. Ongoing analysis informed future task-based interviews conducted with individual students. In retrospective analysis, I made different passes of data analysis. The first pass was *description* (Wolcott, 1994), then *analysis* (Wolcott, 1994), and then *interpretation* (Wolcott, 1994). I also used Thompson and Carlson (2017) levels of variational and covariational reasoning to analyze students' covariational reasoning. The students may not be at a very advanced level of covariation, but it still may have some impact on how they conceive of function notation.

By *description*, Wolcott (1994) meant telling the story. The *description* included explaining a critical or key event more thoroughly (Wolcott, 1994). "*Analysis* refers quite specifically and narrowly to systematic procedures followed in order to identify essential features and relationships" (Wolcott, 1994, p. 24). In this phase, the researchers should stay close to the descriptive account. "*Interpretation* is not bound to the descriptive account as tightly as analysis" (Wolcott, 1994, p. 37), but it should still have links to the cases under study. One way to approach *interpretation* is to turn to theory (Wolcott, 1994). I give examples later in this section. Then I used the *constant comparative analysis* (Corbin & Strauss, 2008) for each case to detect any similarities or differences from the Pre interview to the Post interview. I also did a cross case analysis to learn about students' conceptions of function and function notation.

I used the categories of descriptive, theoretical and methodological notes to organize my reflection of the study as described below. Some of my sample analyses come from the pilot study, which I described earlier in this section.

Ongoing analysis: reflective notes. "Field notes are data that may contain some conceptualization and analytic remarks" (Corbin & Strauss, 2008, pp.123-124). After each interview, I wrote reflective notes to allow me to reflect on what happened during the interview. This reflection also helped me to make decisions regarding future interviews with the same student as well as iterations of the same interview with different students. The reflective notes provided me further ideas on what to ask the students. Reflective notes took three forms: Descriptive, Methodological, and Theoretical.

Descriptive notes. Descriptive notes captured what happened during the interview. Descriptive notes included how a participant interacted with the tasks during the interview or responded to my questions. My goal for descriptive notes was to describe what students did in an interview, staying as close as possible to what the student said during the interview. The example given below is from my pilot study.

Example. Jenna talked about how the distance and height were changing in a Ferris wheel task. She said the distance increases. When I asked her about the height, she said "the height is going as well until it reaches its max point which is as high as the Ferris wheel goes…. as high as the Ferris wheel is and then starts to decrease again". I also wrote descriptions of gestures she made in conjunction with her explanation. For example, she pointed to the screen, and when she moved to the top of the Ferris wheel, she moved her finger in a circular motion right at the top to emphasize that the height reached the maximum at that point (right at the top).

Methodological notes. Methodological notes relate to the procedural aspects of my research study. These notes included ideas for probing questions to address particular kinds of student responses. If a student responded to a task in a way I did not anticipate, I wrote how I responded to that student and specified how I will respond to other students in future interviews. My goal was to refine my interviewing techniques and to phrase questions so that I could learn more about a students' reasoning.

Example. Suppose given $g=r(m)$, I ask a student what the variable m means to them. The student responds by saying that he does not know, because there is no expression. I will make a note of how I will follow up to address this kind of response. For example, I will ask the student what the expression should be like and if they could write down the expression. Then I can use the same follow up question if another individual responds the same way.

Theoretical notes. Theoretical notes included my thoughts about what students did during an interview and significant decisions I made during the interview. For example, I wrote short descriptions of my thoughts regarding what a student may have meant by a particular word he or she used during the interview. My goal of using theoretical notes was to record my hypotheses regarding students' reasoning, which I further investigated through questions. I also made connections between researcher questions or potential student responses and existing literature.

Example. I have chosen a theoretical note about the function notation $g=r(m)$. If I ask a student how they think about function notation $g=r(m)$, they may say they do not know because there is no formula. Based on results of literature (Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992), I hypothesize that the student wants to see a formula like $r(m) = 3m+4$. I will further test my hypothesis by asking the student to write down what formula they were thinking about and what they expected to see to make sense of $g=r(m)$.

Retrospective Analysis. My retrospective analysis used the constructs of *Description, Analysis, and Interpretation* (Wolcott, 1994). The retrospective analysis included multiple passes through the data. In the first pass, I used open coding (Corbin & Strauss, 2008) to identify chunks of data when students form relationships between quantities and notation. I also used the construct of *description* (Wolcott, 1994). In the second pass, I used the construct of *Analysis* (Wolcott, 1994) and *Interpretation* (Wolcott, 1994). In the third pass, I used constant comparative analysis (Corbin & Strauss, 2008) to detect any differences in reasoning from the Pre interview to the Post interview.

I include one example that highlights Wolcott's (1994) constructs of *description, analysis*, and *interpretation*.

Example. I present Lisa's work from Ferris wheel interview 2. I am using the construct of *description* (Wolcott, 1994) by describing a thick description of a key event. At the *analysis* (Wolcott, 1994) level, I still say close to the data to gather evidence of Lisa's reasoning through her gestures and words. At an *interpretation* (Wolcott, 1994) level, I am turning to theory (Wolcott, 1994), and also taking the approach of casual and *unbounded* (Wolcott, 1994).

Description. I showed Lisa both a dynamic Ferris wheel and a dynamic trace together. I include an excerpt below to provide evidence that she conceived of distance and height as quantities that changed together.

Excerpt 1: Lisa Ferris wheel interview 2

Lisa: (watching the dynamic trace and Ferris wheel) silent.
Azeem: So, does it make sense that the graph is doing what it is doing?
Lisa: (silent, pause). I don't know yet. Umm (watching the dynamic trace and Ferris wheel together). Yes! Because the further you are traveling along the distance, then the height from the ground is going to peak at a certain point, which is right there (pointed with her pen at the max, see Figure 6) and then you are traveling back down height from the ground.

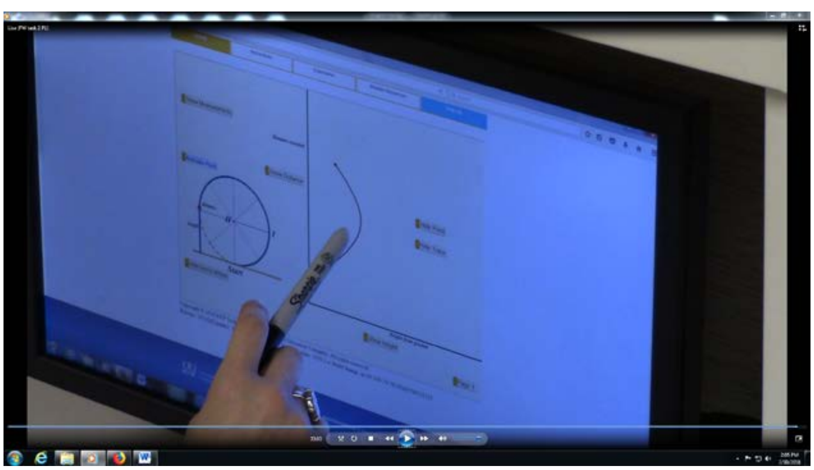


Figure 6: Lisa pointing to height reaching its maximum

Analysis. In this episode, after watching the dynamic trace and dynamic Ferris wheel together, Lisa said that as the distance increased continually, the height increased and then decreased. She also pointed to the max with her pen (see Figure 6) and said that the height reached at its highest and then decreased. I interpret that she conceived of distance as a quantity that increased and conceived of height as a quantity that increased up to the maximum and then decreased.

Interpretation. I interpret that Lisa engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson & Carlson, 2017) because she conceived of distance and height as quantities that changed together such that as the distance increased, the height increased and then decreased.

Appendices

Appendices A-D include researcher questions and potential student responses in the left-hand column. After each interviewer question, I included potential student responses and questions to further investigate those responses as appropriate. The right-hand column contains annotated notes including the rationale for asking a particular question, what I think I might infer about a students' reasoning from a particular response, and connections between researcher questions or potential student responses and existing literature. Appendices E-H include the interview questionnaires.

CHAPTER V

CASE STUDY OF JACK

In this chapter, I present a case study of Jack who demonstrated that within the Pre interview, he first conceived of function notation at *function notation as convention* level and then conceived of function notation at a *function notation as a relationship between variables* level. After a change in his conception of function notation within the Pre interview from *function notation as convention* level to *function notation as a relationship between variables* level, his conception of function notation remained consistent throughout all interviews. At a *function notation as convention* level, he engaged in variational reasoning and covariational reasoning and conceived of function notation using convention of Cartesian coordinate system such that the horizontal axis represented the independent variable. At a *function notation as a relationship between variables* level, he engaged in variational reasoning, quantitative reasoning, and covariational reasoning and also employed a correspondence approach to function.

I interviewed Jack in the middle right after Lisa, but before Dave. Some of the tasks that Jack worked on were not exactly the same as what others worked on. Midway through data collection, I met with my advisor, Dr. Johnson, to discuss how tasks were providing opportunities for me to gather evidence of students' conceptions of function notation. Dr. Johnson provided me an idea to have students respond to others' claims about a graph (see also Johnson et al., 2018, August). As a result, we added a few tasks (see Table 8 in Chapter 4). Because I interviewed Jack in the middle, he worked on the modified tasks in all interviews except the Pre interview.

I included a selection of tasks that Jack worked on during the set of interviews. I selected seven tasks from the set of interviews. There is one task from Ferris wheel interview 1 and two tasks from Ferris wheel interview 2. In the Pre interview and the Post interview, I include two tasks from tasks involving functions, graphs, tables, and function rules and one task from the situation tasks. I also include an additional situation task from the Post interview only because Jack did not work on this task in the Pre interview. I selected these tasks because they provided strongest evidence of Jack's individual forms of reasoning and his conceptions of function notation. Excerpts are representative of Jack's broader work across tasks. I have merged Wolcott's (1994) constructs of *description, analysis* (my interpretation), and *interpretation* (connections to literature) in the results. I use the term *interpret* to refer to Wolcott's (1994) analysis and interpretation levels. When I make connections to extant literature, I move from analysis to interpretation.

I organized this chapter in such a way as to make it easy to see Jack's growth in understanding function and function notation after intervention. The Ferris wheel tasks are presented in chronological order. The Pre interview and the Post interview tasks are not presented in chronological order, because I present a task from the Pre interview and then a similar task from the Post interview. I present Ferris wheel tasks first, and then tasks from the Pre interview and the Post interview, to provide readers an opportunity to follow the impact of intervention on Jack's reasoning with function and function notation from the Pre interview to the Post interview.

At the conclusion of this chapter, I present a summary that addresses my research questions. I include each research question and describe how Jack's work answered each of my research questions. Within the Pre interview, Jack shifted from *function notation as convention* level to *function notation as a relationship between variables* level and his reasoning remained consistent throughout all interviews, where quantities satisfied the correspondence definition of function. He was flexible in his reasoning that either axis could represent the independent variable.

Ferris Wheel Interviews

In this section, I describe Jack's reasoning with quantities and his conception of function notation in the Ferris wheel tasks. I include one task from Ferris wheel interview 1 and two tasks from Ferris wheel interview 2, where he conceived of function notation at a *function notation as a relationship between variables* level. He provided evidence that he engaged in quantitative reasoning, variational reasoning, and covariational reasoning in those tasks.

Ferris Wheel Interview 1, Task 1: Function Notation as a Relationship between Variables

I present a task from the end of Ferris wheel interview 1 below, because in this task, Jack conceived of function notation at a *function notation as a relationship between variables* level. I asked him to interpret a response from a student named Pat who said that the graph below which Jack annotated (see Figure 7) could be written as either *d=f(h)* or *h=f(d).*

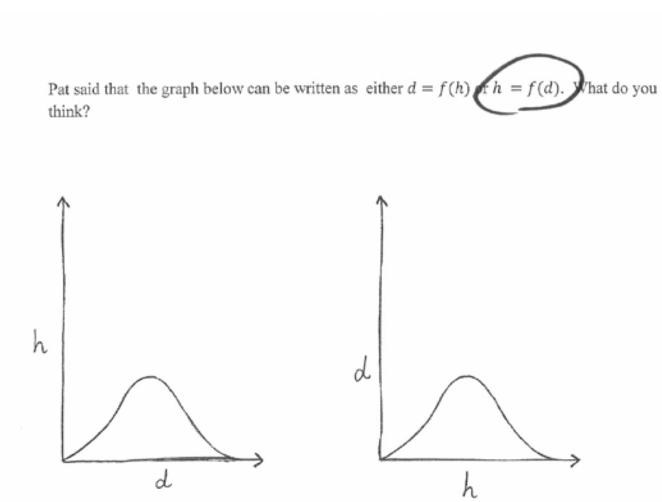


Figure 7: Distance along the horizontal axis (left), distance along the vertical axis (right)

Excerpt 1: Jack Ferris wheel interview 1

Azeem: Ok, alright. So, my next question to you is again about the notation. So, Pat said that this graph can be written as either *d* equals *f* of *h* ($d=f(h)$) or *h* equals *f* of *d* ($h=f(d)$). What do you think?

Jack: I don't think so, because if that was the distance [moving his finger along the vertical axis, see Fig. 7 right], you know that would not change the fact that it still has to be constantly increasing [moved his finger along the vertical axis]. And so like you know basically if you had this be the distance [vertical axis in Fig.7 right], you would have to transpose the graph like over there [moving his hands to show transpose, see Fig. 8].

Azeem: So, which one makes sense to you?

Jack: That one does [moved his finger to the vertical axis, see Fig.7 left], that one does [pointed to the horizontal axis, see Fig.7 left] and that one does not [pointed to graph on the right, see Fig.7 right].

Azeem: So, in terms of the notation, can you say which one is right?

Jack: Yeah, that would be the, um.

Azeem: Because Pat said that you could write it both ways, but you are saying no I cannot write both ways, so which way do you think we should be writing it?

Jack: The *h* equals *f* of *d* I think [circled $h=f\ (d)$] is the right way for like that [pointed to graph on the left, see fig. 7 left]. I don't know if I should draw an arrow, but does that make sense.

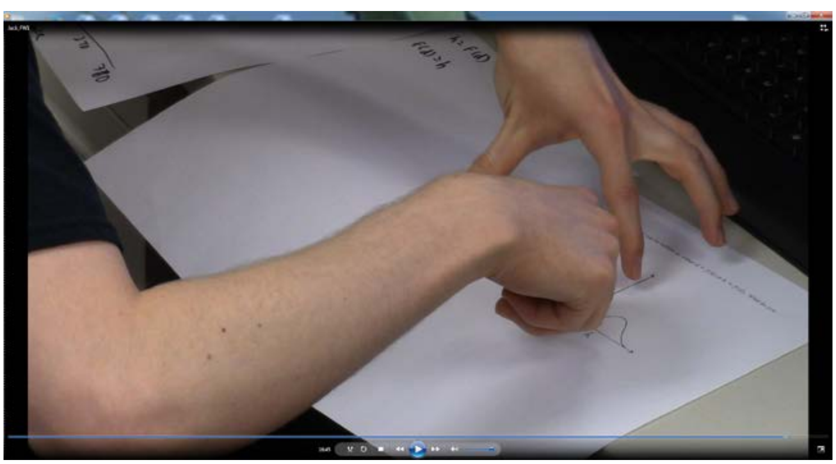


Figure 8: Jack showing that graph needs to be transposed.

Jack provided evidence that he conceived of both graphs representing the same information. He moved his finger along the vertical axis (see Figure 7 right) and said, "because if that was the distance, you know that would not change the fact that it still has to be constantly increasing." I interpret that he meant that the vertical axis represented the height going up and down, so the vertical axis could not be labeled as‘*d*’. In other words, he provided evidence that Pat could not just interchange variables along the axes. He moved his hands to show that if the graph was transposed (see Figure 8), then distance would keep going. He engaged in *emergent shape thinking* (Moore & Thompson, 2015) when interpreting the graphs, because he conceived of distance always increasing. Jack engaged in variational reasoning at a *gross variation* level by Thompson and Carlson (2017) because he said that the distance increased. Jack had opportunities to *break convention* (Moore et al., 2013; Moore et al., 2014) such that quantity representing the independent variable should always be on the horizontal axis.

Jack conceived of changing quantities when reasoning with function notation. I interpret that he conceived of quantities because he moved his finger to the vertical axis and pointed to the horizontal axis (see Figure 7 left) and said that it made sense but pointed to the graph on the right (see Figure 7 right) and said that it did not make sense. Based on what he did, I interpret that he meant that the distance kept going and the height went up and down in Figure 7 left, and that made sense, but in Figure 7 right, it did not make sense to have height on the horizontal axis, because height should have been fluctuating and the distance should have kept going. He conceived of distance increasing along the horizontal axis in both graphs and the height fluctuating along the vertical axis, so he circled function notation $h=f(d)$. To Jack, the left-hand side ‘*h*’ of function notation represented a quantity that increased up to a point and then decreased. In other words, Jack interpreted function notation as something more than what Musgrave and Thompson (2014) term idiomatic expression. Jack conceived of function notation at a *function notation as a relationship between variables* level because he engaged in quantitative reasoning and variational reasoning.

Ferris Wheel Interview 2

In this section, I report on Jack's work in Ferris wheel interview 2. I only presented one task from Ferris wheel interview 1, because Jack's reasoning in that task was representative of the entire Ferris wheel interview 1. In Ferris wheel interview 2, I present two tasks to capture a broader range of reasoning. I first asked Jack how distance and height changed. Then I asked him to graph the relationship between distance and height. Next, I asked him how the distance and height segments (see Figure 5 right) related to the Ferris wheel. Then I asked him about the dynamic point (see Figure 5 right). I present tasks from Ferris wheel interview 2 in chronological order. First, I present a task related to a moving point, because he engaged in covariational reasoning and conceived of the moving point as a multiplicative object. Then I present a task where he conceived of function notation at a *function notation as a relationship between variables* level because he engaged in covariational reasoning and also employed a correspondence approach.

Ferris wheel interview 2, task 2: Jack's conception of the moving point: Covariational reasoning and multiplicative object. Right after Jack worked on the animated segments tasks, I asked him how the animated point on the graph related to the Ferris wheel. In the excerpt below, Jack provided strong evidence that he conceived of the moving point as a combination of quantities represented on each axis.

Excerpt 2: Jack Ferris wheel interview 2 task 2

Azeem: Now I am going to show you this point [moving point on the graph] and how does this point relate to the FW?

Jack: I am gonna guess probably like they are combining both axes into like one point so it is going that way which is the distance axis [moved his finger vertically along the vertical axis] and then it is also going up which is like the height axis [moved his finger up along with the moving point], so it is gonna make like I think it is a parabola.

In this excerpt, Jack conceived of the moving point expressing both the changing height and distance together. His finger movements and his words explained how he conceived of the moving point. He said that “they are combining both axes into like one point.” He moved his finger vertically along the vertical axis and then moved his finger up along with the moving point and said that the shape of the graph would be “a parabola.” I interpret that the moving point not only expressed the increasing distance but also the height going up and then down which he called a “combination of two axes.” He conceived of the moving point as a multiplicative object (Thompson & Carlson, 2017) where the distance and height united to form a single entity. He said something similar in Ferris wheel interview 1, but the evidence was not too strong within Ferris wheel interview 1.

Ferris Wheel Interview 2, Task 3: Function Notation as a Relationship Between Variables

In this task, Jack conceived of both graphs representing distance along different axes such that the distance increased. He conceived of function notation at a *function notation as a relationship between variables* level, because he conceived of quantities and employed a correspondence approach to function. I gave him the graphs which he annotated (see Figure 9). I asked Jack to interpret a response from a student named Nat who said that both graphs could be written as $h=f(d)$.

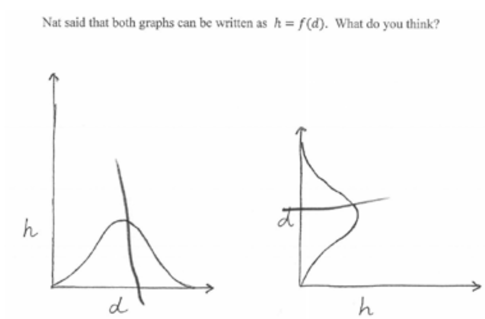


Figure 9: Jack’s annotation of the graphs with distance on the horizontal axis (left) and distance on the vertical axis (right)

The excerpt given below is taken from the end of the interview. In the excerpt below, Jack provided evidence that he conceived of how distance and height changed and then employed a correspondence approach to interpret function notation.

Excerpt 3: Jack Ferris wheel interview 2 task 3

Azeem:	Alright. Ok, so my next question is also about the notation. There is a student named Nat and Nat said that both graphs can be written as *h* equals *f* of *d* ($h=f(d)$) this one and this one [pointing to one then the other]. What do you think about that?
Jack:	Yeah, I think so.
Azeem:	And why?
Jack:	Um because like the distance basically it is the same graph, it is just the axes switched but it is almost like I don't know a transpose that is the word to describe for it. It is like even though the graphs are the different axes it's like the same thing, because that is still like the highest point [pointing to the max on both graphs left and right, see Figure 9].
Azeem:	So, why does it make sense to write them both as that notation [referring to $h=f(d)$]?
Jack:	Um because even though the distance is increasing, it's like you still have it where like each point on the, I guess, *d*-axis still has like one correlated point on the *h*-axis [pointing to graph on the right, see Fig.9]. There is no point where there is like you know you have the same point but different heights.

Jack conceived of both graphs representing the same information and then employed a correspondence approach to interpret function notation. He said, "the distance basically it is the same graph, it is just the axes switched." He also pointed to the max on both graphs (see Figure 9) and said, "it's like the same thing, because that is still like the highest point," so I interpret that he conceived of both graphs showing distance as increasing and the height attaining its max in both cases but represented along different axes. When I asked Jack why both graphs could be written as $h=f(d)$, he said, "I guess *d*-axis still has like one correlated point on the *h*-axis." He also pointed to the graph on the right (see Figure 9). I interpret that by pointing to the graph on the right, he meant that even though the graph looked sideways, but the vertical axis represented the distance, and so one distance had one corresponding height regardless of the shape of the graph. He conceived of function notation at a *function notation as a relationship between variables* level, where he engaged in quantitative reasoning and employed a correspondence approach.

Function notation as a relationship between variables. In the previous excerpt, Jack provided evidence that he was doing more than just engaging in variational reasoning. To learn more about how he conceived of a function and function notation, I asked Jack if he could point to the graph or show what he meant, by picking points. The excerpt given below came right after the excerpt I presented earlier and is important because he conceived of function notation at a *function notation as a relationship between variables* level, where he engaged in covariational reasoning and employed a correspondence approach.

Excerpt 4: Jack Ferris wheel interview 2

Azeem:	So, can you show me by actually pointing to the graph or just making two points on there and then showing what you are exactly saying?
Jack:	Um basically like even though the *d* is increasing this way [moved pen along the horizontal axis left to right, fig.9 on left] the *h* is still [moved pen over the curved part and then down on graphs, Figure 9 left and right] it still works. I don't know how to describe but [pause]. Basically, like I think it is called the vertical line test, where you can do that [drew a vertical line over Figure 9 left] and it's a function if there is no overlap between any point of the graph and then for this [sketched a horizontal line over Figure 9 right] it will be horizontal but it is the same thing.
Azeem:	[Pause]. Ok, so why is it horizontal there?
Jack:	Because it is still the same function. Even though its different axes, it is still *h* is *f* of *d* ($h=f(d)$), so it's still like h is a function of d. So, this is still like the [pointing to Figure 9 right] like the independent variable.
Azeem;	So, shouldn't there be a vertical line there (pointed to Figure 9 right)?
Jack:	No, because it's basically the same graph, you just took it and did that to it [moved his hand and rotated it].

According to Jack, both graphs represented distance that increased regardless of which axis it was represented by. He conceived of how the independent variable changed (in this case distance increasing). He drew a vertical line over the left graph (see Figure 9 left) and drew a horizontal line over the graph on the right (see Figure 9 right), but by both of those lines he meant that distance had a one-to-one correspondence to height. He said, "*d* is increasing this way," and moved his pen along the horizontal axis left to right. He moved his pen over the curved part and then down on both graphs (see Figure 9) and said, "*h* still works." I interpret that he conceived of an invariant relationship between distance and height, where distance increased, and height depended on the distance. He engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level, because he moved his pen over different parts of graphs to show that as distance increased, the height increased and then decreased.

Jack conceived of function notation by attending to quantities first and then employed a correspondence approach. For function notation, $h = f(d)$, he still thought that the independent variable should be in parentheses and the dependent variable should be on the other side of the equal sign, but graphically, the independent variable could be represented by any axis (horizontal or vertical). He then employed a correspondence approach to justify why $h=f(d)$ made sense. His definition of function was consistent with univalence (Even, 1990; Even, 1993), where the same input could not have different outputs. I interpret that he conceived of function notation at a *function notation as a relationship between variables* level. Jack employed Thompson and Carlson's (2017) definition of function such that there was an invariant relationship between quantities and one value of a quantity determined one value of the other quantity. In other words, he engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. He attended to the other part of function definition (value of one quantity determined the value of the other quantity) when he employed a correspondence approach. For him, both *h* and *d* meant something, so function notation $h=f(d)$ was something more than what Musgrave and Thompson (2014) term idiomatic expression.

Function and Notation in Tasks Involving Functions, Graphs, Tables, and Function Rules

In this section, I include a statement task to show that within the Pre interview and the Post interview, Jack conceived of function notation at a *function notation as convention* level in conjunction with engaging in covariational reasoning. Then I include a graphs and matching rules task where Jack provided evidence of a shift from the Pre interview to the Post interview in his conception of function notation from *function notation as convention* level to *function notation as a relationship between variables* level. I argue that his shift in reasoning within the Pre interview plane situation task and his reasoning with function notation in the Ferris wheel tasks can explain his shift in his conception of function notation across the Pre interview to the Post interview in this task.

Statement Task: Function Notation as Convention across the Pre Interview to the Post Interview

I compare Jack's reasoning with function and function notation given a statement task. The excerpt below is taken from the beginning of the interview. I included the third statement task because it provided evidence of Jack's engagement in covariational reasoning across the Pre interview to the Post interview. Across the Pre interview to the Post interview, Jack's conception of function and function notation remained consistent and he conceived of function notation at a *function notation as convention* level because he labeled the axes of his graph based on function notation, such that the independent variable was along the horizontal axis and the dependent variable was along the vertical axis.

Pre interview: Covariational Reasoning. I asked Jack to read the third statement out loud. He also sketched a graph (see Figure 10).

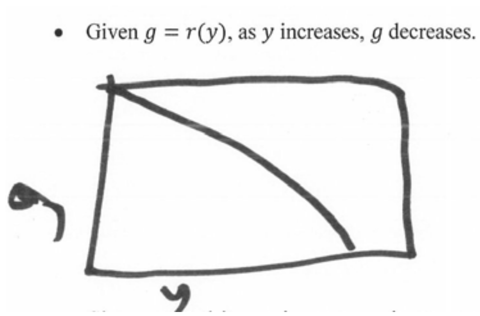


Figure 10: Jack's graphical representation of *g=r(y)*

Excerpt 5: Jack Pre interview

Jack:	Given *g=r(y)*, as *y* increases, *g* decreases [reading out loud]. Um I'm gonna guess they are inversely related or not really say that so that would mean I think then basically what it is saying is like as the first variable gets larger the second variable gets smaller every single time.
Azeem:	ok, could you like express it graphically or any other way?
Jack:	So basically, every time the *y* gets greater the *g* value gets smaller until you- its they are inversely related. I don't know how else to really explain it but.
Azeem:	So, you are saying as y is increasing, *y* values are going up
Jack:	And *g*s are going down.

In the Pre interview, Jack conceived of how quantities changed and followed convention of matching the variables in function notation with the axes labels. He said, "every time the *y* gets greater the *g* value gets smaller". Jack also sketched a graph (see Figure 10) to show that the *y* values increased and the *g* values decreased with it. I interpret that Jack engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level because he talked about *y* and *g* changing together earlier. Jack used the term *inversely related* to mean that as values of one variable increased, the other decreased. He labeled the axes of his graph based on function notation, such that the independent variable along the horizontal axis was *y* and the dependent variable along the vertical axis was *g*. Because he labeled the axes of his graph based on variables in function notation, he followed convention (Moore et al., 2014). He matched the variable on the horizontal axis to be the independent variable, so he conceived of function notation at a *function notation as convention* level in conjunction with engaging in covariational reasoning.

Post interview: Covariational Reasoning

Jack read the statement, 'Given *m=r(y)*, as y increases, m decreases' out loud. He sketched the coordinate plane, labeled the horizontal axis *y*, labeled the vertical axis *x*, and then sketched a line (see Figure 11 below).

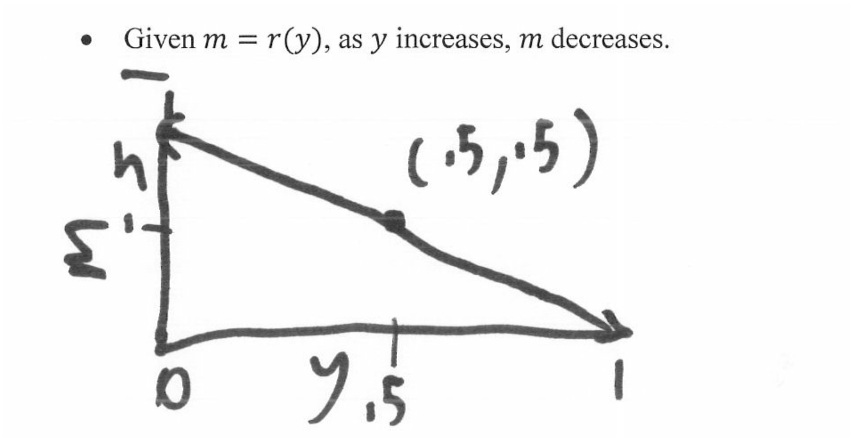


Figure 11: Jack's graphical representation of *m=r(y)*

In the excerpt given below, he explained how *y* increased and how *m* decreased. He also stated how he conceived of his graph.

Excerpt 6: Jack Post interview

Azeem:	Ok, so can you explain what is going on there?
Jack:	Yeah, basically like as *y* increases which is on this axis [moving finger along the horizontal axis left to right], on the other axis *m* decreases so it just keeps going down [moved finger along his sketched line].
Azeem:	So, where is *y* increasing? Can you just point to it?
Jack:	Well, just like right is considered an increase [moving finger along the horizontal axis left to right]

Azeem: Ok then what about *m*? How is *m* decreasing?
Jack: *m* is going down on that axis [moving pen top to the origin along the vertical axis]
Azeem: Ok, so what do these points represent here? You have several points there
Jack: Um I don't know, let's just say like 0 to1 [labeled horizontal axis going from 0 to 1] I guess it is sort of generic [labeled vertical axis going from 0 to 1], like that you mean?
Azeem: No. Like points on the line [moving finger on the line]
Jack: Oh, um oh I guess for the sake of doing this thing so it would be like *y* goes from 0 to like 0.5 and then it would be increasing that way and then *m* would go from 1 to 0.5, so it will be decreasing.
Azeem: So, what is that number right there?
Jack: 0.5. I tried to write it sideways.
Azeem: Oh, 0.5 ok [referring to vertical axis labeled 0.5]. So, 0.5 here [referring to label on horizontal axis]. So where is that on the graph?
Jack: So, essentially half way through like in this *y* starts at 0 and then *m* would start at 1.
Azeem: Yeah, could you label a point on the graph exactly?
Jack: Yeah let's call it like [labeled (0.5, 0.5)]

In the Post interview, Jack explained and clarified that he conceived of *y* increasing and *m* decreasing together. He moved his finger along the horizontal axis left to right and stated that *y* increased "on this axis." He also said, "on the other axis *m* decreases so it just keeps going down," but he moved finger along his sketched line (not along the vertical axis). To clarify how he conceived of *y* increasing and *m* decreasing, I asked him about *y* and *m* separately. When I asked where *y* was increasing, he moved his finger along the horizontal axis left to right. When I asked how *m* decreased, he moved his pen top to the origin along the vertical axis and said, "*m* is going down on that axis." This clarified for me that he conceived of *y* increasing along the horizontal axis and *m* decreasing along the vertical axis. I interpret that Jack conceived of *y* as something that increased and *m* that decreased, so he engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carson, 2017). At this point in the interview, I only had evidence of *Gross Coordination of Values* (Thompson & Carson, 2017), and then the numbers provided me with more evidence to claim *Coordination of Values* (Thompson & Carson, 2017).

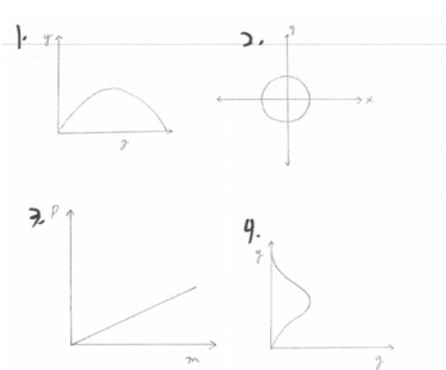
Jack provided evidence that he conceived of a graph representing *y* and *m* changing together using numbers. When I asked him what the points represented, I wanted to learn how he conceived of the points on the line, but he interpreted it differently and labeled the axes with numbers 0 and 1 instead (see Figure 11). I interpret that with the numbers, Jack demonstrated that he conceived of *m* and *y* as possible to measure. In other words, he conceived of *m* and *y* as quantities. Later, when I asked him about points on the line, he provided evidence of a different level of covariational reasoning. Jack said, "*y* goes from 0 to like 0.5 and then it would be increasing that way and then *m* would go from 1 to 0.5, so it will be decreasing." He wrote 0.5 on both axes. Then I asked Jack to label a point on the graph to see how he was thinking about the increasing *y* and decreasing *m* together. Jack labeled the point (0.5, 0.5) to show that *y* went up from 0 to 0.5 and *m* went down from 1 to 0.5, and the new point was (0.5, 0.5). Jack's point (0.5, 0.5) showed that he conceived of the variables *y* and *m* changing together, so he engaged in covariational reasoning at a *Coordination of Values* (Thompson & Carlson, 2017) level.

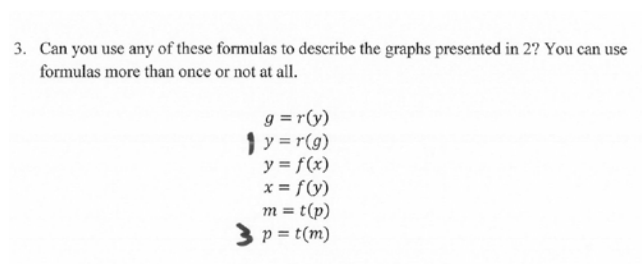
Jack labeled his graph following convention of labeling the axes based on the variables in function notation. He labeled the horizontal axis as '*y*' because the variable in the parentheses was also *y* and the independent variable is usually represented on the horizontal axis. He labeled the vertical axis as '*m*' because the variable on the other side of an equal sign or the dependent variable is represented along the vertical axis. Because he labeled the axes of his graph based on variables in function notation, he followed convention (Moore et al., 2014). He conceived of function notation at a *function notation as convention* level in conjunction with engaging in covariational reasoning.

Graphs and Rules task: Function Notation as Convention in the Pre Interview to a Relationship between Variables in the Post Interview

In this section, I compare Jack's reasoning with function and function notation given four sets of graphs. Jack provided evidence that his reasoning with function notation shifted within the Post interview and across the Pre interview to the Post interview.

Pre interview: Function notation as convention. In the Pre interview, I gave him a set of graphs and asked him which graphs represented functions, which graphs did not represent functions, and why? I also asked him which rules described those graphs. The excerpt given below is taken from the beginning of the interview. In the excerpt below, Jack demonstrated that he conceived of a function and function notation based on following convention such that the horizontal axis represented the independent variable.





3. Can you use any of these formulas to describe the graphs presented in 2? You can use formulas more than once or not at all.

$g = r(y)$
1 $y = r(g)$
$y = f(x)$
$x = f(y)$
$m = t(p)$
3 $p = t(m)$

Figure 12: Set of graphs given to Jack (left) and Jack's annotation of rules describing graphs (right)

Excerpt 7: Jack's Pre interview

Azeem:	Now I will give you a set of graphs and could you tell me which ones are functions and which are not and why?
Jack:	These two are (1, 3) and these are not (2, 4) because as I said earlier like the vertical line test that goes through both of these [pointing to graphs 2 and 4], which would imply that at this value [a point along the horizontal axis] you could

have two possible answers for this graph [moving finger over two points along the 4th graph] and the circle.

Azeem: Ok. So, can you use any of these formulas here to describe the graphs presented here [pointed to graphs] and you can use formulas more than once or not at all.

Jack: [pause] um, so I think *y* equals *r* of *g* ($y=r(g)$) would describe 1. *y* is on the y-axis and then *g* even though it's *r* of *g*, I think that is how that works.

Azeem: Ok.

Jack: Let's see. Does it matter since these are not functions? Would it like do you know what I mean.

Azeem: So, what do you think if it's not a function?

Jack: So, probably not then.

Azeem: Ok. How about the 3rd one [referring to graph 3]?

Jack: I think *p* equals *t* of *m* ($p=t(m)$) would work for 3.

Azeem: And what about 4? [referring to graph 4]

Jack: Umm it's not a function or else yeah.

Azeem: Oh, ok [pause]

Azeem: Could you use any other formulas to express graph 3 or 1?

Jack: I mean if you had different like actual number value as long as it still had the *m* and the *p* you know the *g* and the *y*, I think so.

Azeem: ok. So, like what is given here [pointing to rules] would any of these rules define graphs 1 and 3.

Jack: No, because they are not the same. Like if it was not labeled [pointed with his pen to graph 3 horizontal axis label and then the vertical axis label] you probably could, but since it is labeled here [pointed his pen to the vertical axis label and then the horizontal label for graph 3], you can't do that.

When reasoning with function, Jack provided evidence that he followed convention such that the horizontal axis represented the independent variable. Jack said that graphs 1 and 3 represented functions and graphs 2 are 4 did not represent functions. Because he pointed to a point along the horizontal axis and moved his finger over two points along graph 4 and graph 2 (circle) (see Figure 12) and also said that graph 4 and graph 2 failed the vertical line test because one input had two outputs, so I interpret that he conceived of the horizontal axis representing the independent variable. Jack demonstrated that he only considered the variable along the horizontal axis to be the independent variable.

Jack conceived of function notation at a *function notation as convention* level where his conception of function notation was separate from his covariational reasoning. He said that "*y* equals *r* of *g*" (*y=r(g)*) described graph 1, because "*y* is on the *y*-axis and then *g* even though it's *r* of *g* I think that is how that works". Jack moved his finger along the vertical axis when he was talking about the variable *y* and moved his finger along the horizontal axis when he talked about the variable *g*. So, I interpret that Jack chose function notation *y=r(g)* for graph 1, because the independent variable *g* on the horizontal axis and the dependent variable *y* on the vertical axis matched with the variables in function notation. He had similar reasoning with the function rule describing graph 3. He said that "*p* equals *t* of *m*" (*p=t(m)*) described graph 3. When I asked him if he could use any other rule to describe graphs 1 and 3, he said, "No, because they are not the same." He pointed to the horizontal axis label and then pointed to the vertical axis label of graph 3 with his pen and said, "since it is labeled here, you can't do that." Jack's pen movements along with his words confirmed that to Jack, the variables in the general function notation should match with the variables along the axes, where the variable on the left-hand side of the equal sign should be the dependent variable and the variable on the right-hand side within the parentheses should be the independent variable. He said that rules should not be used for graphs that did not represent functions, and therefore, he did not write function rules for graphs that did not represent functions. I interpret that Jack conceived of function notation at a *function notation as convention* level because he conceived of the horizontal axis representing the independent variable and did not provide evidence of engaging in covariational reasoning.

Post interview: Function notation as convention to a relationship between variables. In the Post interview, I asked Jack which graphs represented functions or did not represent functions and why? (see Figure 13). The excerpt given below is taken from the beginning of the interview. In the excerpt below, Jack provided evidence that he checked two function notations. Comparing his work in this task to other tasks, I can say that he conceived of function notation at a *function notation as a relationship between variables* level.

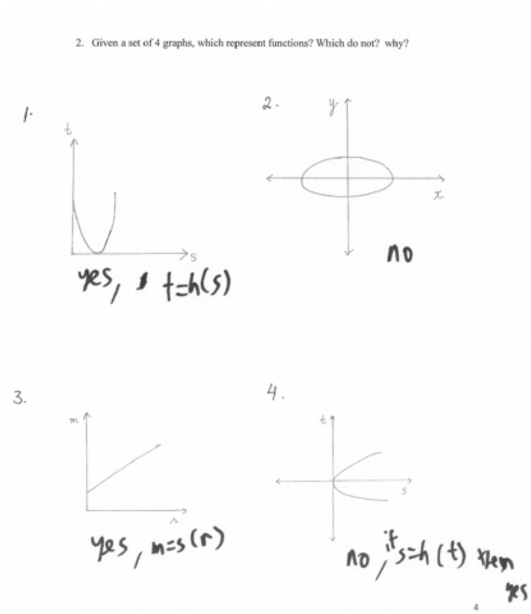


Figure 13: Jack's annotation of 4 sets of graph and rules describing graphs

Excerpt 8: Jack's Post interview

Azeem: You have 4 sets of graphs and I labeled them 1, 2, 3, 4. Can you tell me which one of those represent functions and which do not and why?

Jack: That one does (graph 1) that one does not (graph2). Um [wrote yes for graph 3). That one (graph 4) does not it is kind of like the last situation where if that is supposed to be the input axis [moving fingers along the horizontal axis] yeah that one would not be.

Azeem: Ok.

Jack: So maybe

Azeem: Ok, so maybe next question. Next, I will ask you can you use any of these formulas to describe those graphs presented in that problem and you can use formulas more than one time or not at all.

Jack: [pause] yeah you could use the *s* is *h* of *t* for this one, wait no that would be the other way around. It will be *t* is *h* of *s* (*t=h(s)*) for graph 1 and then *m* equals *s* of *r* (*m=s(r)*) for graph 3. Yeah, I think that's it.

Azeem: So, what about this one [pointed to graph 4]?

Jack: If there was one that was like *s* equal *f* of *t* (*s=f(t)*) or something, but that one is not on there, so you cannot.

Azeem: Ummm maybe look at it closely (moved the paper with the rules page closer to Jack).

Jack: Oh, yeah. I guess *s* is *h* of *t* ($s=h(t)$) never mind. I was looking for *f* I guess, so yes, *s* equals *h* of *t* [wrote $s=h(t)$ below graph 4] that would work.
Azeem: Ok and again you said for # 3 you said *m* equals *s* of *r*, ok.
Jack: Amhum.
Azeem: And you cannot write it [pointing to graph 2] any of those ways?
Jack: No, because since it is a circle it will be the same no matter which axis you flipped it to.

Within the Post interview, Jack provided evidence that he conceived of function notation at a *function notation as convention* level. Jack first stated which graphs represented functions and which graphs did not represent functions. He said that graph 1 and graph 3 (see Figure 13) represented functions, but graphs 2 and 4 did not. He only explained why graph 4 would not represent a function. He moved his finger along the horizontal axis of graph 4 and said, "if that is supposed to be the input axis, yeah that one would not be," so I interpret that he let the variable along the horizontal axis as an input to decide that graph 4 did not represent a function. I conjectured that Jack might be conceiving of *t* along the vertical axis as a possibility for an input, but I did not have evidence of that until later when I asked him to write function rules describing these graphs.

Within the Post interview, Jack shifted from conceiving of function notation from *function notation as convention* level to *function notation as a relationship between variables* level. He wrote one notation for graphs 1 and 3 that represented functions. Graph 3 could be written in two ways, but he only wrote $m=s(r)$, so I interpret that graph 3 represented a one-to-one function, so he did not feel the need to switch the axes labels because it represented a function either way. But, for graph 4 Jack provided evidence of a shift in his conception of function notation. He checked two function notations, but nothing more than that based on what he said in this particular task. Comparing his work in this task to his conception of function and function notation within situation tasks and the Ferris wheel tasks with attributes switched, I can say that he conceived of function notation at a *function notation as a relationship between variables* level. He conceived of function as one input mapping to one output and different inputs mapping to the same output. He checked both notations to show that he conceived of both quantities to see which quantity will satisfy his conception of function. When Jack said, "if there was one that was like *s* equal *f* of *t* (*s=f(t)*)," he provided evidence that he conceived of *t* along the vertical axis as an input such that different inputs *t* could map to the same output *s*. Then he also said that graph 4 could be written as "*s* equals *h* of *t*" (*s=h(t)*). Jack had difficulty finding *s=h(t)* in the function rules provided, because he said that he was looking for *f* instead of *h*. I interpret that he wanted to see *f*, because he was used to seeing an *f*. Jack provided evidence of flexibility in his reasoning that variable on any axis could represent the independent variable.

Within the Post interview, Jack provided evidence of switching axes labels and connecting it to the orientation of a graph. Earlier Jack said, "if there was one that was like *s=f(t),*" so he provided evidence that he switched the axes labels. When I asked him if graph 2 (circle) could be written as a function notation, he said, "since it is a circle it will be the same no matter which axis you flipped it to," so I interpret that by "flipping" he meant rotating a graph. Based on what he said, my conjecture is that he might be conceiving of both switching the axes labels and rotating a graph when he reasoned with graph 4 as well. Within the Post interview and across the Pre interview to the Post interview, Jack shifted from conceiving of function notation at a *function notation as convention* level to *function notation as a relationship between variables* level, because he shifted from matching the axes labels to function notation to checking both notations that would satisfy his conception of function based on quantities.

Situation Task: Function Notation as Convention to a Relationship between Variables from the Pre Interview to the Post Interview

In this section, I describe Jack's reasoning with function and function notation given a plane situation task in both the Pre interview and the Post interview. Jack's engagement with this task provided evidence of how he conceived of quantities and how his covariational reasoning impacted his conception of general function notation (*y=f(x)*).

Pre interview. In the Pre interview, I gave Jack a plane situation task such that as the plane covered the distance along the ground, its altitude changed. I asked him to read the situation out loud and interpret the graph.

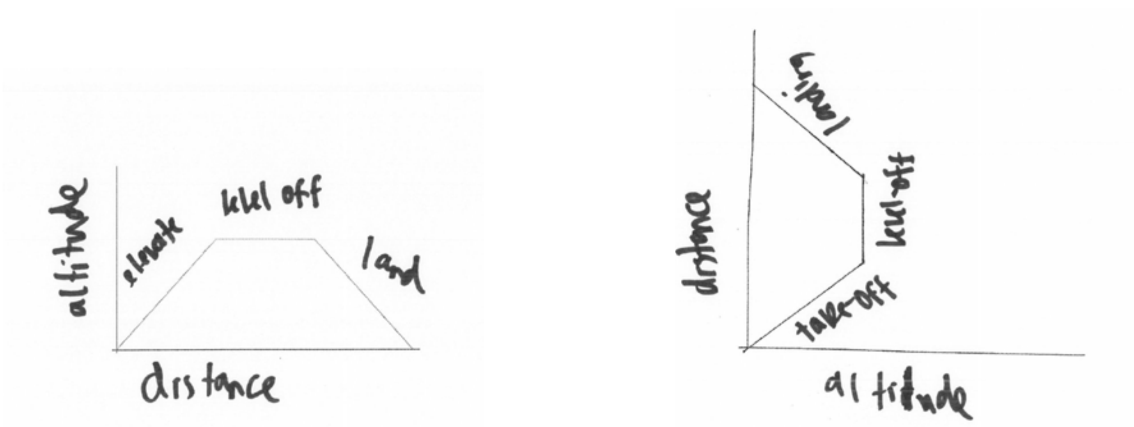


Figure 14: Jack's annotation of the situation with distance on the horizontal axis (left) and attributes switched (right)

Reasoning with quantities: Variational reasoning to covariational reasoning. Jack first conceived of altitude as a quantity that changed. He first talked verbally and pointed or moved his finger along different parts of the graph to show that the plane's altitude changed. I asked him to write down what he said. Then he wrote the words elevate, level off, and land over different parts of the graph (see Figure 14 left) to show that the altitude changed, so he engaged in variational reasoning at a *gross variation* (Thompson & Carlson, 2017) level. Then I asked him to have the same situation, but attributes represented on different axes. I asked him how distance and altitude were changing. He only mentioned the starting point at the origin (see Figure 14, right) where, to Jack, distance and altitude were both zero.

To learn more about how he conceived of distance and altitude, I asked him how the distance and the altitude changed once again. In the excerpt below, Jack provided evidence that he engaged in covariational reasoning and conceived of rotating a graph as well.

Excerpt 9: Jack's Pre interview

Azeem:	Ok [pause] So, how are distance and altitude changing?
Jack:	Um basically um distance it is kind of confusing I don't think it really works very well but have to picture the graph as a mirror [referred to part a graph], that is what I have been doing like you know manually flip it.
Azeem:	So, like what is distance doing in here? Is it going; is it decreasing?
Jack:	I am gonna guess this would be increase [moved his pen on the y-axis from 0 and up] so it would be increasing, and this would be altitude increasing [moved his pen from left to right along the horizontal axis] some point but.
Azeem:	So then if altitude is increasing what do these things represent then [moved finger over the increasing, leveling off, and the decreasing part of the graph]
Jack:	Basically, the same thing where like it's like it gets greater in altitude [moved pen over the increasing part] and then it levels off [moved pen over the horizontal part] and then it goes back down the altitude the landing [moved pen over the decreasing part of the graph] that's with the distance [moving pen vertically along the vertical axis].
Azeem:	Ok. So, is it doing anything different than it was doing before?
Jack:	No.

Jack provided evidence that he conceived of rotating the graph counterclockwise and provided evidence that he conceived of distance and altitude changing together. When I asked him first how distance and altitude changed, he said, "like you know manually flip it," so I interpret that by the word "flip", he meant rotating a graph counterclockwise so that distance still increased, and altitude varied with distance. When I asked him about distance and altitude again, Jack said the altitude increased, remained the same, and then decreased and also moved his pen over the graph showing changes in altitude with changing distance. He showed that distance increased by moving his pen vertically along the vertical axis. He was now conceiving of both distance and height changing together. He engaged in covariational reasoning at a *Gross Coordination of Values* level (Thompson & Carlson, 2017), because he said that altitude increased, remained the same, and then decreased as the distance increased while moving his pen along the vertical axis and moving his pen over different parts of a graph.

Jack's conception of function notation as convention for a graph shown in Figure 14 on the left. I present an excerpt below where Jack provided evidence that he conceived of function notations $a=f(d)$ and $d=f(a)$ at a *function notation as convention* level.

Excerpt 10: Jack's Pre interview

Azeem:	Ok, so what will it (function notation) be for 'a' (graph on the left)? Which one (referring to notation)?
Jack:	For 'a' (graph on the left) that would be the um let's see [pause] yeah so 'a' (graph on the left) is probably the first one (referring to $a=f(d)$) because altitude is a function of distance which means you know every time distance like distance is the independent variable and the altitude is the dependent variable so altitude is dependent on the distance for the sake of this one. So it will be like *d* [labeled horizontal axis'*d*'] and *f* of *d* ($f(d)$) [labeled vertical axis '$f(d)$'] and then [sketching another graph, see Figure 15 left] you have the same thing.
Azeem:	Could we say *d* equals *f* of *a* ($d=f(a)$) for graph a?
Jack:	For this one.
Azeem:	Umhum.
Jack:	I don't think so, because usually like you usually have the you know like since this is *f* of *d* ($f(d)$), that would imply that you have the independent variable on the *x*-axis. That is usually how that works. I don't think you can have the independent variable on the *y*-axis. Or you know what I mean
Azeem:	Ok.
Jack:	Yeah. It is kind of hard to explain but basically, I think like *f* of anything has to be on the *y*-axis and then the original thing has to be on *x*.

Jack conceived of function notation $a=f(d)$ at a *function notation as convention* level. When I asked him, which rule would go with Figure 14 left, he said, "*a* equals *f* of *d* (a=$f(d)$) because it is the altitude is a function of distance." He first pointed to the horizontal axis which was labeled as '*d*', and then said, "distance is the independent variable and the altitude is the dependent variable," so he matched the variables in function notation a=*f(d)* to the labels of axes. Then he sketched a graph (see Figure 15 left) and said, "it will be like *d*" and then put'*d*' along the horizontal axis and *f(d)* along the vertical axis, and also said, "you have the same thing," so he was labeling the axes of his graph very carefully based on function notation a=*f(d),* and he conceived of function notation at a *function notation as convention* level.

Jack provided evidence that he conceived of function notation *d=f(a)* at a *function notation as convention* level. When I asked him if we could write *d=f(a)* to represent Figure 14 left, he said, "I don't think you can have the independent variable on the *y*-axis." Jack moved his finger along the vertical axis (see Figure 15 left) and said, "I think like *f* of anything has to be on the *y*-axis." He also moved pen back and forth along the horizontal axis and said, "the original thing has to be on *x*." I was not surprised when Jack said that the independent variable should always be on the horizontal axis, because that is what he was used to and that is how graphs are presented to students. Because he said that *x* had to be along the horizontal axis, I interpret that he conceived of function notation at a *function notation as convention* level such that the horizontal axis represented the independent variable.

Jack's conception of function notation as convention to a relationship between variables for a graph shown in Figure 14 on the right. In the excerpt below, Jack provided evidence that he began to shift from conceiving of function notations *a=f(d)* and *d=f(a)* at a *function notation as convention* level to *function notation as a relationship between variables* level.

Excerpt 11: Jack's Pre interview

Azeem:	Ok, how about the other graph? (referring to figure 14 shown on the right)
Jack:	[pause] I don't think you could really call that a function because that would imply that I mean well *d* is *f* of *a*, but since *a* (altitude) would give you two values at certain point, I don't think you could really use that as a function.

Azeem: Ok. How about *a* equals *f* of *d* (*a=f(d)*) for the second situation (referring to figure 14 shown on the right)?
Jack: Um [pause] Maybe, I'm not sure.
Azeem: Let's look at it closely and carefully and see if it is possible to write it as *a* equals *f* of *d* (*a=f(d)*).
Jack: The thing is you like basically you want to switch to this axis (pointing to top half Figure 15 left) if you want to do that, that will make it the easiest I mean.
Azeem: But even though, the way that it is, is it possible to write it as *a* equals *f* of *d* (*a=f(d)*)?
Jack: I guess so.
Azeem: And why?
Jack: I mean I don't know about tradition or anything like if you have to have *f* of *d* (*f(d)*) on the *y*-axis, but if you don't then I guess you could just flip it as long as like, it would be the same thing, it would just be rotated.
Azeem: So, then is it okay for us to say *a* equals *f* of *d* (*a=f(d)*)?
Jack: I guess so, yes. Well, wait.
Azeem: Do you want to rotate it and see?
Jack: Yes. I think, yes, it would still work then, yeah, a mirror [smiles].

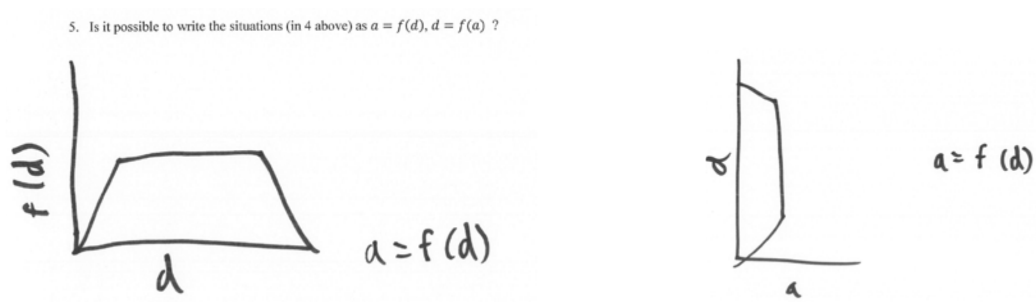


Figure 15: Jack's annotation of general function notation (Pre interview)

Jack was beginning to conceive of *d=f(a)* at a *function notation as a relationship between variables* level. When I asked him if he could express Figure 14 right as *d=f(a)*, he said "I mean well *d* is *f* of *a*," so I interpret that he meant that a graph had the horizontal axis labeled,'*d*', and the vertical axis labeled, '*a*', which matched with the notation *d=f(a)*, so it was okay to say *d=f(a)*. At the same time he also said, "since *a* (altitude) would give you two values at certain point, I don't think you could really use that as a function," so Jack meant that writing *d=f(a)* for figure 15 shown on the right did not define a function (one input '*a*' had two outputs '*d*'). Then Jack pointed to the horizontal axis and then moved his finger vertically on two different parts of the graph to show that one altitude input would give two distances as outputs and that violated the definition of a function. Because Jack moved his finger to show that one input had two outputs, so he employed a correspondence approach to justify that Figure 14 shown on the right could not be expressed as *d=f(a)*. Jack was beginning to shift from conceiving of function notation *d=f(a)* at a *function notation as convention* level to *function notation as a relationship between variables* level.

Jack provided evidence of what the word "flip" meant to him. When I asked him if he could express Figure 14 right as *a=f(d),* he first said he was not sure. Then I asked him to look at the graph carefully. He said, "you basically want to switch it to this axis (pointing to Figure 15, left) if you want to do that, that will make it the easiest I mean," so I interpret that he was thinking about rotating a graph (figure 15 on the right) counterclockwise. When he said, "you could just flip it …, it would just be rotated," so I interpret that by the word "flip" he meant rotating a graph counterclockwise so that distance was along the horizontal axis and altitude along the vertical axis. Jack shifted from conceiving of function notation at a *function notation as convention* level such that the independent variable always had to be along the horizontal axis to a shift that the vertical axis could also represent the independent variable.

Within this task in the Pre interview, Jack shifted from conceiving of function notation at a *function notation as convention* level to *function notation as a relationship between variables* level. Jack provided evidence that figure 14 shown on the right could be written as *a=f(d)*. He sketched a graph (see Figure 15 right) and also said, "So it would be the same *a* equals *f* of *d*" (*a=f(d)*) [wrote *a =f (d),* see Figure 15 right]. I asked him again if it was okay to say *a* equals *f* of *d* (*a =f (d)*) for Figure 14 right. He said, "yes", but said, "well, wait." So, I rotated Figure 14 right counterclockwise, and he said, "Yes. I think, yes, it would still work then, yeah, a mirror [smiles].” He talked about mirror earlier too, and I interpret that he conceived of rotating a graph counterclockwise earlier as well. He not only said that distance could be on the vertical axis, but also rotated his graph to show that distance increased in both cases. He engaged in variational reasoning and covariational reasoning earlier in this task when I asked him how height and distance changed, and he conceived of function notation at a *function notation as a relationship between variables* level.

Post interview. I gave Jack a plane situation task such that as the plane covered the distance along the ground, its altitude changed. This is the same task he worked on in the Pre interview. I asked him to read the situation out loud and interpret the graph. In the excerpt below, Jack provided evidence that he conceived of altitude as a quantity that was capable of changing.

Excerpt 12: Jack’s Pre interview

Jack:	I am assuming this is the distance traveled so that would be *d* [labeled the horizontal axis‘*d*’] and that would be altitude *a* [labeled the vertical axis ‘*a*’].
Azeem:	Ok.
Jack:	And so basically it is just the altitude increases at first [moved pen from the origin to the increasing part of the graph] as the plane is taking off and then it levels off [moved pen over the horizontal part of the graph], then it lands again [moved his pen over the decreasing part of the graph].

Reasoning with quantities: Variational reasoning. After reading the situation out loud and labeling axes of a graph (see Figure 16 left), he said, “the altitude increases at first as the plane is taking off,” and moved his pen from the origin to the increasing part of the graph to show that the altitude increased. Then he said, “it levels off”, and moved his pen over the horizontal part of a graph to show that altitude stayed the same. Then he said, “it lands”, and moved his pen over the decreasing part of a graph to show that altitude decreased. Because he moved his pen over different parts of a graph and said that altitude increased, remained the same, and then decreased, so I interpret that he conceived of an altitude as a quantity and engaged in variational reasoning at a level called *gross variation* (Thompson & Carlson, 2017).

Reasoning with quantities with attributes on different axes: Variational reasoning and conception of function. Then I asked Jack to have the attributes on different axes. In the excerpt below, Jack provided strong evidence that he conceived of distance as a quantity and employed a correspondence approach to conceive of function.

Excerpt 13: Jack's Post interview

Azeem:	What if you have the same situation as before [handed another paper] but now your attributes are represented on different axes. What would happen? How would you interpret that?
Jack:	It would work as long as you had to like put '*a*' there [labeled the horizontal axis '*a*', see fig. on right] and *d* there [labeled the vertical axis'*d*', see figure 16 right]
Azeem:	So now is that any different from what you had before? [turned the graph counterclockwise]
Jack:	I mean in terms of axes, yes, but in terms of what it is actually depicting, no.
Azeem:	What is it not depicting which it was not before?
Jack:	well, like I mean it is the same did I flip the question around? It is different in terms of axes, but it's the same in terms of what it is depicting.
Azeem:	Ok, ok. Alright. So, what is going on then. Can you explain one more time?
Jack:	Yes. I mean since it does not technically matter the axis, I don't think, so it would just be like I would say this is 0 to 500 for the distance (put 0 and 500 along the vertical axis) as long as it (distance) increases here, it equals one altitude value every single time, then it would work.

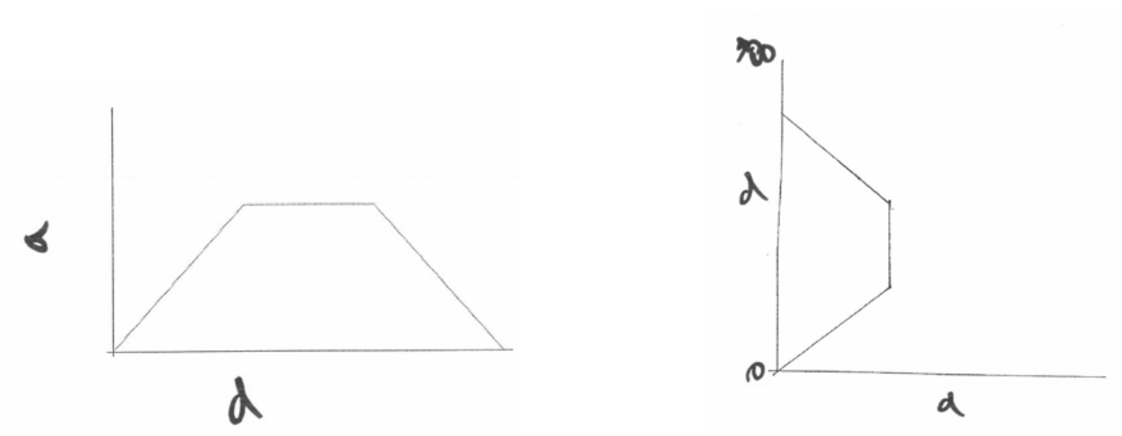


Figure 16: Jack's annotation of the plane situation with distance on the horizontal axis (left) and distance on the vertical axis (right)

I asked Jack to represent the attributes on different axes. He labeled the vertical axis '*d*' and the horizontal axis '*a*'. Then I asked him if it was different than before. He responded differently than the Pre interview and said, "It is different in terms of axes, but it's the same in terms of what it is depicting," so I interpret that to Jack, a graph represented the same relationship between distance and altitude regardless of which axis represented distance and altitude.

Jack put numbers 0 and 500 along the vertical axis, so I interpret that with the numbers he demonstrated that distance was possible to measure. He also said that the axis did not matter, so he conceived of distance and altitude such that the vertical axis could represent a quantity representing the independent variable, and the horizontal axis could represent a quantity representing the dependent variable. He said that as distance increased along the vertical axis, "it equals one altitude value every single time," which I take as evidence that he conceived of distance and altitude representing a function.

Jack conceiving of function notation as a relationship between variables for a graph shown in Figure 17 (top graph). Jack provided evidence that he conceived of function notation *a=f(d)* at a *function notation as a relationship between variables* level. Jack sketched his own graph (see Figure 17 top graph) and wrote *a=f(d).* When I asked him to explain why he could say *a=f(d)*, he wrote 1 *d* input = 1 *a* output, so I interpret that he employed a correspondence approach to show that a graph shown in Figure 17 on the top could be expressed as *a=f(d).*

Jack conceiving of function notation as a relationship between variables for a graph shown in Figure 17 (bottom graph). In the excerpt below, Jack provided evidence that a graph shown in Figure 17 on the bottom could only be expressed as *a=f(d).*

Excerpt 14: Jack Post interview

Jack: And, yeah for this one it would not work because you would have points like right here where you have one altitude [pointing to a point on the horizontal axis, see figure 17 bottom] but you will have two different distances [pointed to two points on either side of the horizontal part]

Azeem: Ok.

Jack: So, that would be [writing 1 *a* input ≠ 1 *d* output]

Azeem: So, one altitude input does not equal one distance output

Jack: Yes, because there are points where you have multiple outputs for one input.

Azeem: So, which notation we can say.

Jack: You can say *a* equals *f* of *d* (*a=f(d*) but not the *d* equals *f* of *a* (*d=f(a)*).

Azeem: Ok, just write it down.

Jack: I guess that would be yes that is *d* equals *f* of *a* (*d=f(a)*) [wrote *d =f(a)*] does not work or something?

Azeem: Yeah, say that.

Jack: [wrote 'doesn't work']

Azeem: So, which one works for that?

Jack: [wrote 'works' next to the graph on the top]

Azeem: For this one [pointed to the top graph] or for that one [pointed to the bottom graph]?

Jack: I mean for that one you really couldn't (pointed to graph at the top). For that one (pointed to graph on the bottom) it would have to be the same where it is the *a* equals *f* of *d* (*a=f(d)*) one [wrote *a =f(d)* next to the graph on the bottom]

Azeem: That works or that does not work.

Jack: Yeah that one would work [wrote 'works' next to *a=f(d)* bottom graph]

Azeem: And again, why is that?

Jack: Because it would be just basically the same graph as that graph [pointing to the graph at the top] just flipped axes wise.

Azeem: So still you are getting the one input one output.

Jack: Umhumm.

Jack first explained that *d=f(a)* did not work because one input had more than one output, so I interpret that he employed a correspondence approach. He wrote *a=f(d)* works. When I asked him why *a=f(d)* worked, he said, "because it would be just basically the same graph as that graph [pointing fig.17, top], just flipped axes wise." Jack engaged in quantitative reasoning and employed a correspondence approach, so he conceived of function notation at a *function notation as a relationship between variables* level. He conceived of both graphs representing distance and altitude related in a way such that altitude depended on distance, regardless of which axis represented distance and altitude. When I asked him if this graph satisfied the definition of a function “one input one output”, he agreed. Again, Jack provided evidence that to him, function notation *a=f(d)* meant more than what Musgrave and Thompson (2014) termed idiomatic expression.

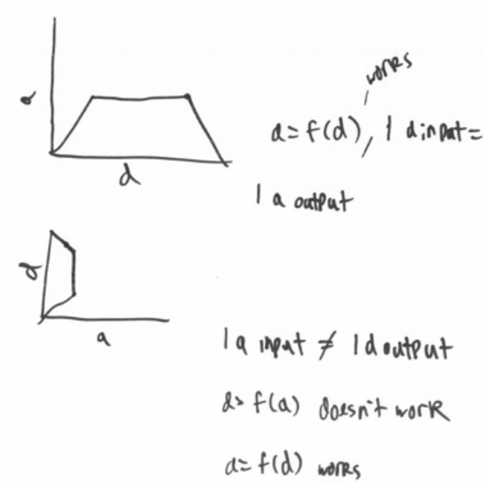


Figure 17: Jack’s annotation of his sketched graphs

Post interview task: Function Notation as a Relationship between Variables

I present the modified task from the Post interview. Jack did not work on this task in the Pre interview. To learn more about how he conceived of function notation, I asked him to interpret a response from a student named Sam who said that both graphs (see Figure 18) could be written as *h=f(d).*

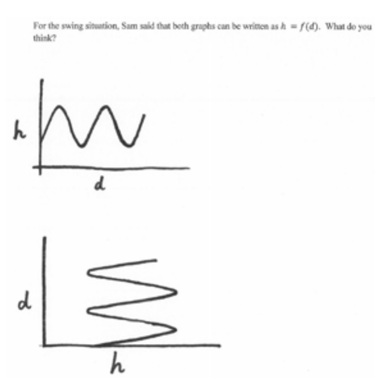


Figure 18: Distance on the horizontal axis (top) and distance along the vertical axis (bottom)

The excerpt below is taken from the end of the Post interview. In the excerpt below, Jack provided evidence that he engaged in covariational reasoning and conceived of function notation at a *function notation as a relationship between variables* level.

Excerpt 15: Jack Post interview

Azeem:	Ok. My last question is for the swing situation. Sam said that both graphs can be written as *h* equals *f* of *d* (*h=f(d)*). What do you think?
Jack:	Um, yeah, I think so. It would work.
Azeem:	And why?
Jack:	Um because both times it is the same thing it is flipping the axes, but the distance is always increasing both times [moved pen along the vertical axis from origin going up [bottom graph] and moved pen along the horizontal axis left to right [top graph] and the height is fluctuating [traced the graph with a closed pen top graph and then traced the bottom graph with a closed pen] based on the distance.

Jack conceived of function notation at a *function notation as a relationship between variables* level. In this excerpt, Jack said that both graphs could be represented as *h=f(d)*. When I asked why, he said, "Um because both times it is the same thing it is flipping the axes, but the distance is always increasing both times and the height is fluctuating based on the distance." Even though he mentioned "flipping the axes", he conceived of how distance and height changed. He also moved his pen along the vertical axis from the origin going up (see Figure 18 bottom graph) to show that the distance increased along the vertical axis. He also moved his pen along the horizontal axis left to right (see Figure 18 top) to show that the distance increased along the horizontal axis. When he said, "the height is fluctuating," he also traced both graphs with a closed pen. I interpret that he meant that the height fluctuated along the vertical axis (see Figure 18 top) as well as along the horizontal axis (see Figure 18 bottom), and in both cases, the height depended on the distance. He engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson & Carlson, 2017), because with his pen movements he showed that as distance increased, the height increased and then decreased. When he said, "it is the same thing," he conceived of both graphs representing distance and height just along different axes, such that height depended on distance. So, Jack conceived of function notation at a *function notation as a relationship between variables* level. To Jack, function notation $h=f(d)$ expressed a relationship between two quantities, which is something different than what Musgrave and Thompson (2014) term idiomatic expression.

Summary

Within the Pre interview, Jack first conceived of function notation at a *function notation as convention* level and then conceived of function notation at a *function notation as a relationship between variables* level. After a change in his conception of function notation within the Pre interview from *function notation as convention* level to *function notation as a relationship between variables* level, his conception of function notation remained consistent throughout all interviews. At a *function notation as convention* level, he conceived of function notation in conjunction with variational reasoning and covariational reasoning. At this level, he conceived of function notation using convention of Cartesian coordinate system such that the horizontal axis represented the independent variable. At a *function notation as a relationship between variables* level, he engaged in variational reasoning, quantitative reasoning, and covariational reasoning and employed a correspondence approach to function. Next, I briefly summarize how Jack's case answers my three research questions.

How Might Students' Conceptions of Function Impact Their Conceptions of Function Notation?

Jack's conceptions of function remained consistent across the Pre interview to the Post interview. He engaged in variational reasoning and covariational reasoning and employed a correspondence approach to function. He conceived of a function as every input mapping to an output. He was consistent throughout the tasks in the Pre interview, Ferris wheel interviews, and the Post interview, that the same input could not map to different outputs, but different inputs could map to the same output. He also engaged in covariational reasoning across all tasks involving situation tasks as well as the Ferris wheel tasks.

Within the Pre interview plane situation task, Jack demonstrated shifts in his conception of function notation from *function notation as convention* level to *function notation as a relationship between variables* level. After a shift within the Pre interview that any axis can represent the independent variable, his reasoning with function notation remained consistent throughout all interviews. Within the Post interview in a graphs and matching rules task for graph 4, Jack checked two function notations, but nothing more than that based on what he said in this particular task. Comparing his work in this task to his conception of function and function notation within situation tasks and the Ferris wheel tasks with attributes switched, I can say that he conceived of function notation at a *function notation as a relationship between variables* level. He conceived of function as one input mapping to one output and different inputs mapping to the same output. He checked both notations to show that he conceived of both quantities to see which quantity will satisfy his conception of function. In this task, Jack switched axes labels to show that any variable could represent the independent variable. He chose function notation $s=h(t)$, so he did not match the variable s on the horizontal axis to the variable in the parentheses which was t.

Within the Pre interview situation tasks and throughout other interviews, Jack engaged in quantitative reasoning, variational reasoning, and covariational reasoning and employed a correspondence approach to function, so he conceived of function notation at a *function notation as a relationship between variables* level. He was not only thinking about how quantities changed together, but also thinking about interchanging the variables along the axes as long as it satisfied the definition of a function. Using Thompson and Carlson's (2017) definition of a function, Jack engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. He employed a correspondence approach to attend to the part of Thompson and Carlson's (2017) definition that the value of one quantity determined the value of the other quantity.

How Might Covariational Reasoning Related to Function Impact Students' Conceptions of Function Notation?

Jack consistently engaged in covariational reasoning within and across all interviews. For example, in situation tasks in the Pre interview, Ferris wheel interviews, and the Post interview, he provided evidence that he conceived of quantities changing together. He conceived of quantities changing together even when attributes were represented on different axes.

Jack engaged in quantitative reasoning, variational reasoning, and covariational reasoning and employed a correspondence approach which impacted his conception of function notation. He could think flexibly about graphs, where quantity representing the independent variable could be represented by either the horizontal axis or the vertical axis. In other words, he conceived of function notation at a *function notation as a relationship between variables* level.

How Do Students Conceive of a General Function Notation?

Jack preferred $y=f(x)$ over $g=r(m)$ because he was used to this function notation. He first showed that the independent variable could only be represented by the horizontal axis, but he provided evidence of a shift in his conception within the Pre interview situation tasks. Within the Pre interview plane situation task, he accepted that the independent variable could be represented by the vertical axis. He provided evidence of a shift in his conception of function notation from *function notation as convention* level to *function notation as a relationship between variables* level. His conception of function notation remained consistent for the rest of the Pre interview and across the Pre interview to Ferris wheel interviews to the Post interview.

Jack's conception of function notation shifted across the Pre interview to the Post interview for a particular task in tasks involving functions, graphs, tables and function rules. For example, in four sets of graphs and matching rules task in the Pre interview, he selected one function notation and conceived of function notation at a *function notation as convention* level such that the independent variable could only be represented by the horizontal axis. But, in the Post interview, he checked two function notations, but nothing more than that based on what he said in this particular task. Comparing his work in this task to his conception of function and function notation within situation tasks and the Ferris wheel tasks with attributes switched, where the independent variable was represented along the vertical axis, Jack demonstrated that he engaged in quantitative reasoning, variational reasoning, and covariational reasoning. So, I can say that he conceived of function notation at a *function notation as a relationship between variables* level in this task. He conceived of function as one input mapping to one output and different inputs mapping to the same output. He checked both notations to show that he conceived of both quantities to see which quantity will satisfy his conception of function.

Engaging in covariational reasoning and employing a correspondence approach to a function impacted Jack's conception of function notation. He conceived of an invariant relationship between quantities first and then employed a correspondence approach to interpret function notation. In other words, he conceived of function notation using Thompson and

Carlson's (2017) definition of function, which I refer to as a combination of covariational reasoning and correspondence approach. Jack engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. He employed a correspondence approach to attend to the part that the value of one quantity determined the value of the other quantity.

CHAPTER VI

CASE STUDY OF DAVE

In this chapter, I present a case study of Dave who conceived of function notation at a *function notation as label* and *function notation as convention* level within the Pre interview and the Post interview. His conception of the definition of function mitigated his conception of function notation. He operated with a different conception of the definition of function within and across the Pre interview and the Post interview than Ferris wheel interviews. Within Ferris wheel interviews, he conceived of function notation at a *function notation as a relationship between variables* level. I identify Dave's conceptions of function and function notation and his engagement in quantitative reasoning, variational reasoning, and covariational reasoning within and across interviews. Within Ferris wheel interview 1, he first conceived of distance from both distance and height measuring the same thing (length from the ground) and then conceived of distance increasing and only the height increasing and then decreasing. Within and across Ferris wheel interviews, he conceived of function notation at a *function notation as a relationship between variables* level, where he engaged in variational reasoning and covariational reasoning and then employed a correspondence approach, but not across the Post interview. Across the Pre interview to the Post interview, he conceived of function notation at a *function notation as label* and *function notation as convention* level.

I interviewed Dave last. Midway through data collection, I met with my advisor, Dr. Johnson, to discuss how tasks were providing opportunities for me to gather evidence of students' conceptions of function notation. Dr. Johnson provided me an idea to have students respond to others' claims about a graph (see also Johnson et al., 2018, August). As a result, we modified a few tasks (see Table 8 in Chapter 4). Because I interviewed Dave last, he worked on all the modified tasks in all interviews.

I included a selection of tasks that Dave worked on during the set of interviews. I selected five tasks from the set of interviews. There is one task from Ferris wheel interview 1 and one task from Ferris wheel interview 2. In the Pre interview and the Post interview, I include one task from tasks involving functions, graphs, tables, and function rules and two tasks from the situation tasks. I selected these tasks because they provided strongest evidence of Dave's individual forms of reasoning and his conceptions of function notation. Excerpts are representative of Dave's broader work across tasks. I have merged Wolcott's (1994) constructs of *description*, *analysis* (my interpretation), and *interpretation* (connections to literature) in the results. I use the term *interpret* to refer to Wolcott's (1994) analysis and interpretation levels. When I make connections to extant literature, I move from analysis to interpretation.

I organized this chapter in such a way as to make it easy to see Dave's growth in understanding function and function notation during and after intervention. Ferris wheel tasks are presented in chronological order. The Pre interview and the Post interview tasks are not presented in chronological order, because I present a task from the Pre interview and then a similar task from the Post interview. I present Ferris wheel tasks first, and then tasks from the Pre interview and the Post interview, to provide readers an opportunity to follow Dave's conceptions of function and function notation during the intervention and his conceptions of function and function notation from the Pre interview to the Post interview.

At the conclusion of this chapter, I present a summary that addresses my research questions. I include each research question and describe how Dave's work answered each of my research questions. Within and across Ferris wheel interviews, he conceived of function notation at a *function notation as a relationship between variables* level because he engaged in variational reasoning and covariational reasoning and then employed a correspondence approach. Within Ferris wheel interviews, he conceived of onto graphs representing a function such that two different inputs could map to the same output. Within the Pre interview and the Post interview and across the Pre interview to the Post interview, he conceived of function notation at a *function notation as label* level, where his conception of function notation was intertwined with physical characteristics such as the shape of a graph. Across the Pre interview to the Post interview, he applied the definition of a function such that there must be a one-to-one-correspondence for a graph to represent a function and conceived of function notation using this conception as well.

Ferris Wheel Interviews

In this section, I present two tasks to demonstrate Dave's reasoning with function and function notation in Ferris wheel interviews. I present one task from Ferris wheel interview 1 and one task from Ferris wheel interview 2 to demonstrate that he conceived of function notation at a *function notation as a relationship between variables* level. His conception of function notation remained consistent within Ferris wheel interview 2 and across Ferris wheel interview 1 to Ferris wheel interview 2.

Ferris Wheel Interview 1

Within Ferris wheel interview 1, Dave first conceived of distance measuring the same thing as height- a length from the ground. Later, he conceived of distance as increasing to the right, zero at the maximum, and decreasing to the left, while the height increased and then decreased. After watching the dynamic trace and the dynamic Ferris wheel together, he shifted in his conception of distance such that distance was the total distance traveled around the Ferris wheel and only the height increased and then decreased. Then he demonstrated that he engaged in covariational reasoning and conceived of function notation at a *function notation as a relationship between variables* level.

Ferris Wheel Interview 1 Task 1: Function Notation as a relationship between variables. Right after Dave provided evidence that he conceived of distance and height changing together, I asked him to interpret a response from a student named Pat who said that the graph below (see Figure 19) could be written as either *d=f(h)* or *h=f(d)*. In the excerpt below, Dave provided evidence that he conceived of an invariant relationship between quantities such that the distance continued to increase, and the height varied with it.

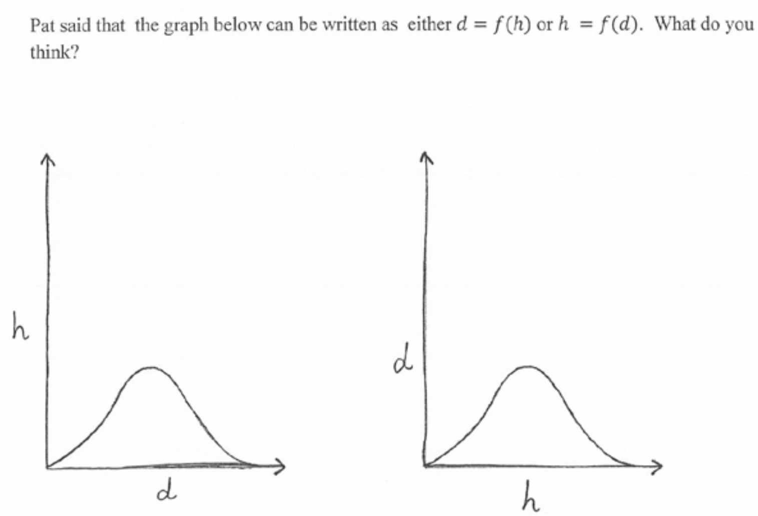


Figure 19: Graphs given to Dave with distance along the horizontal axis (left) and distance along the vertical axis (right).

Excerpt 16: Dave Ferris wheel interview 1 task 1

Azeem: Alright, so my next question is again about the notation. So, Pat said that this graph could be written as either *d* equals *f* of *h* (*d=f(h)*) or *h* equals *f* of *d* (*h=f(d)*) What do you think?

Dave: I do not think that they (axes) could be freely interchanged because distance keeps progressing I guess indefinitely as they would keep going around the circle and the height reaches a certain point and then decreases. So, this graph [see Figure

19 left], assuming *h* stands for height (pointed to *h* along the vertical axis, see Figure 19 left) and *d* stands for distance (pointed to *d* along the horizontal axis, see Figure 19 left) is true, but this one (pointed to Figure 19 right) cannot be because *d* does not reach a peak in value or *x*, distance, sorry. So, the two cannot be interchanged.

Azeem: Ok. So, which one of these could it be written as?

Dave: I think it would be *h* equals *f* of *d* ($h = f(d)$).

Azeem: Why?

Dave: Because I don't think that distance can be derived from a function of height if height reaches a specific height (a peak) and I do not think that distance can be derived from that [pointed to horizontal axis d] if there is a peak [moved pen to the max, see Figure 19 left] and distance continues to progress pass that [moved pen from the middle of the horizontal axis all the way to the end].

Azeem: What would happen in that case [pointing to $h = f(d)$]?

Dave: In the case of *h* equals *f* of *d* ($h=f(d)$), then height is equal to a function of distance. So, it is hard to express that I guess. Because distance is continuously increasing (moved pen along the horizontal axis, see Figure 19 left), it. I guess, uh, it's hard to put. I guess just because distance has more value to it or because it has more values in general, then I guess more values could be input into the equation or in the function to create *h* (pointed to the peak), I am really sorry.

Azeem: Yeah there are no right or wrong answers so don't be panicked. I just want to know how you are thinking about it.

Dave: I had it a second ago but

Azeem: So, you said no, you can't write it this way or that way earlier and I just want to know why you said that.

Dave: Ok, so I think that height can be the product of a function of distance because well with distance continuously increasing. I am sorry I already explained why distance can't equal a function of height.

Azeem: Umhum.

Dave: Um [pause 15 secs]. I guess it is difficult to put into words.

Azeem: So just maybe point to the graphs and just tell me what you are thinking.

Dave: Ok, well I guess, because distance is continuously increasing this way [moving pen along the horizontal axis of the left graph, see Fig.19] the increasing values allows for height to also increase[moved pen along the increasing part of graph on the left, see Fig.19] but reaches certain point [stopped pen at max] and then start to decrease [moved pen from the max to the decreasing part of the graph] while distance can continue to increase [moved pen along the horizontal axis], but if you were to switch the two [pointing to labels *d* and *h* in Fig.19 right graph], then that won't work out because the height is not continuously increasing [moved pen along the horizontal axis right graph] but instead reaches a certain high point.

In this excerpt, Dave provided evidence that he conceived of distance and height as quantities. He said that axes could not "be freely interchanged." He also pointed to the axes of a graph shown in Figure 19 on the left to show that distance increased along the horizontal axis and the height varied along the vertical axis. He pointed to a graph shown in Figure 19 on the right and said, "*d* does not reach a peak in value." I interpret that he conceived of distance as a quantity that increased and therefore, the vertical axis could not be labeled '*d*'. In other words, he provided evidence that Pat could not just interchange variables along the axes. Because to Dave, the variables along the axes represented how quantities changed, he engaged in *emergent shape thinking* (Moore & Thompson, 2015) when interpreting graphs.

Dave provided evidence of conceiving of two changing quantities and then employing a correspondence approach when interpreting function notation. When I asked him about function notations, he said, "it would be *h* equals *f* of *d* ($h=f(d)$)". I asked him, "why?" He explained why he could not say $d=f(h)$ by saying, "if there is a peak and distance continues to progress pass that," so I interpret that he meant that one *h* had different distances and $d=f(h)$ was not possible. Then I asked him about $h=f(d)$. He said, "more values could be input into the equation or in the function to create *h*," while pointing to the peak when he said *h*. I interpret that he meant that different distance inputs mapped to the same height, therefore, he could express a graph as $h=f(d)$. I interpret that he employed a correspondence approach. Then I asked him to point to the graphs to tell me what he was thinking. Dave explained how the distance and height changed, where distance kept going and the height increased up to a point and then decreased. He moved his pen along the horizontal axis (see Figure 19 right) and said, "height is not continuously increasing," so I interpret that he conceived of how quantities changed rather than paying attention to how the axes were labeled. I interpret that Dave engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level, because he explained that as distance increased, the height increased and then decreased. Dave engaged at a *function notation as a relationship between variables* level, because to him, function notation $h=f(d)$ expressed a relationship between two quantities such that as the distance increased, the height increased and then decreased. His reasoning with function notation is different from what Musgrave and Thompson (2014) term idiomatic expression.

Ferris Wheel Interview 2

In this section, I report on Ferris wheel interview 2. Within Ferris wheel interview 2 and across Ferris wheel interview 1 to Ferris wheel interview 2, Dave's reasoning with function and function notation remained consistent. I only present one task from the end of Ferris wheel interview 2 because this task was representative of Dave's reasoning throughout Ferris wheel interview 2 and across Ferris wheel interview 1 to Ferris wheel interview 2.

Ferris wheel interview 2 Task 2: Function notation as a relationship between variables. I asked Dave to interpret a response from a student named Nat who said that both graphs could be expressed as $h=f(d)$. The figure below shows his annotated graphs (see Figure 20). In the excerpt below, he demonstrated that he conceived of both graphs representing the changing distance just represented on different axes and then employed a correspondence approach to justify function notation, so he conceived of function notation at a *function notation as a relationship between variables* level.

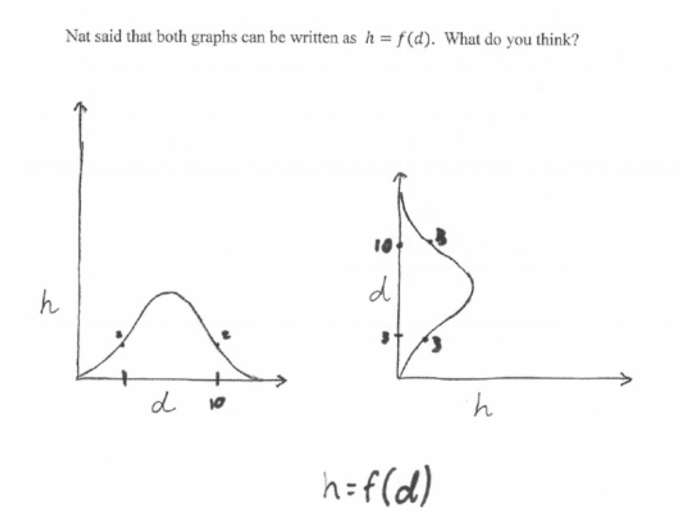


Figure 20: Dave's annotation related to function notation with distance on the horizontal axis (left) and distance on the vertical axis (right)

Excerpt 17: Dave Ferris wheel interview 2 task 2

Azeem: Ok. Alright. Let me stop that animation for a second and what I'll ask you next is this question. Nat said that both graphs that we talked about in Ferris wheel case, they can both be written as *h* equals *f* of *d* (*h=f(d)*). What do you think about that?

Dave: Theoretically I think that it could be applied to both. Well, sorry give me a second to look at it.

Azeem: Ok

Dave: Pause [26 sec]. Alright, they should both work.

Azeem: And why? And you can write on this paper.

Dave: So based on what I was thinking before, um, I guess the overall thing that needs to be considered is whether or not *d* has a repeated value with a different value of *h*, and since *d* is constantly increasing here [moving his pen over the horizontal axis left to right, see Figure 20 left] and you do not encounter the same value here multiple times, then this one [see Figure 20 left] still works with *h* equals *f* of *d* (*h=f(d)*), and over here [pointed to Figure 20 right] it works the same way, it's just on a different axis [referring to *d* and *h* on different axes].

Azeem: So, yeah, could you just maybe explain like in terms of the points like why it makes sense to write it this way [both graphs as *h=f(d)*]?

Dave: Yeah, sure. So if *h* equals *f* of *d* (*h=f(d)*) then *d* [wrote *h=f(d)*, see Figure 20] can't have well, *d* is the input has to equal a certain output and because of the graph, it keeps moving forward on this axis, *x*-axis [pointing to Figure 20 left] if we are talking about there, then say the points say *d=3*. If the point *d=3* equals say *h* of 2 or something like that then eventually when you reach the same height of *h*= 2 again, it would be something like *d=10*. Because of that it still works because you get multiple inputs that do not equal the same output.

Azeem: Ok, what about the second case?

Dave: In the second case, it works the same way if *d=3*, for example here and *h*=3, then *d=10*, *h*=3 again [wrote numbers (see Figure 20 right)]. So, at that rate, well you do not have the same input here for *d* that equals the same output. If you flipped it around so that *d=f(h)* then you'd have a problem where *h* could equal 2 in two places [pointed his pen to *h*=2, *h*=2, one then the other in Figure 20 left] and it will equal the same thing and have the different values for *d*.

To interpret function notation *h=f(d)*, Dave conceived of both graphs representing the same information and then employed a correspondence approach. He moved his pen over the horizontal axis left to right (see Figure 20 left) and said, "since *d* is constantly increasing here and you do not encounter the same value here multiple times, then this one still works with *h* equals *f* of *d* (*h=f(d)*)." I interpret that he engaged in variational reasoning at a *gross variation* level (Thompson & Carlson, 2017) because he conceived of distance as a quantity that increased. He also pointed to a graph shown in Figure 20 on the right and said, "it works the same way," it's just on a different axis," so I interpret that he meant that both graphs represented distance increasing; one graph had *d* on the horizontal axis and the other graph had *d* on the vertical axis. I also interpret that he meant that both graphs could be written as *h=f(d)*.

Within Ferris wheel interviews and across the Pre interview to Ferris wheel interviews, Dave provided evidence of a shift in his definition of a function when employing a correspondence approach to interpret function notation. I conjectured that he was using function notation to represent only one-to-one correspondence, and I wanted to learn more about this. So, I asked him to show me in terms of points what he meant. He showed with numbers (see Figure 20) that *h=f(d)* worked for both graphs. He said that *h* equals *f* of *d* (*h=f(d)*) worked for a graph shown in Figure 20 on the left. He pointed to that graph and used numbers and said, "… it still works because you get multiple inputs that do **not** [emphasis added] equal the same output." Because he used two inputs of 3 and 10 for *d* and used one output of 2 for *h*, so I interpret that he wanted to say that multiple inputs equaled the same output. When I asked him if a graph shown in Figure 20 on the right could be written as *h=f(d)*, he used 3 and 10 for *d*, and 3 for *h*. Then he said, "we do not have the same input here for *d* that equals the same output," so I interpret that he conceived of a definition of a function such that different inputs mapped to the same output and used this definition to state that a graph could be written as *h=f(d)*. Here Dave employed a correspondence approach to explain why *h=f(d)* worked.

Dave employed a correspondence approach to interpret function notation *d=f(h)*. He used numbers to explain why *d=f(h)* did not work for both graphs. He pointed to a graph shown in Figure 20 on the left and said, "if you flipped it around so that *d* equals *f* of *h* (*d=f(h)*), then you'd have a problem where *h* could equal 2 in two places (pointed his pen to[*h*=2, *h*=2], one then the other) and it will equal the same thing and have different values for *d*," I interpret that he conceived of *d*-values as outputs and *h*-value as an input because he pointed to *h*=2 twice, although he never used the words input/output here. He meant that same input *h* had different *d* outputs, so it violated the definition of a function and therefore, *d=f(h)* did not work. Here Dave employed a correspondence approach again to explain why *d=f(h)* did not work. He conceived of function notation at a *function notation as a relationship between variables* level, because he engaged in variational reasoning and employed a correspondence approach. Dave interpreted function notation as something more than what Musgrave and Thompson (2014) term idiomatic expression.

Tasks Involving Functions, Graphs, Tables, and Function Rules

In this section, I present one task to demonstrate Dave's conceptions of function notation. Dave was the only student who worked on this type of task in both the Pre interview and the Post interview, because all tasks were modified by the time I had scheduled his interviews. Across the Pre interview to the Post interview, he provided evidence of relating function notation to the shape of a graph and convention of matching the axes labels to the variables in function notation, so he conceived of function notation at *function notation as label* and *function notation as convention* levels.

General Function Notation Task: Function Notation as Label and Function Notation as Convention

Pre interview. I asked Dave to interpret a response from a student named Sam who said that both *m=t(p)* and *p=t(m)* could be used to describe a graph I provided, which Dave annotated (see Figure 21 below). I include an excerpt below to show that Dave's conceptions of function notation were intertwined with the shape of a graph and convention.

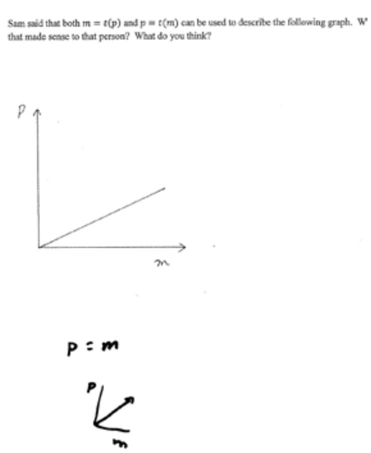


Figure 21: Dave's annotation of general function notation task in the Pre interview

Excerpt 18: Dave Pre interview

Azeem:	Ok. So there was Sam. Sam actually said that both *m* equals *t* of *p* ($m=t(p)$) and *p* equals *t* of *m* ($p=t(m)$) can be used to describe this graph. Why do you think that made sense to that person? What do you think about it?
Dave:	Well, it is a linear function so I would assume that the two could be equateable, but I don't think this was entirely right because it is not exactly proportionate one-to-one unless it's off scale, but yeah at that rate, it seems like *m* would have greater values than *p*, so I don't think the two were equateable.
Azeem:	So which one do you think is the right one?
Dave:	Um, let's see. I'd probably say *p* equals *t* of *m* ($p=t(m)$) actually.
Azeem:	Why?
Dave:	Just because if *m* is apparently a greater value or if it looks like a greater value, then I'd assume the function will modify the greater value to be lesser in here. Sorry that wasn't very good.
Azeem:	If *m* is a greater value?
Dave:	Yeah. Just from looking at the graph, *m* is a greater value so actually
Azeem:	What does this line represent?
Dave:	Oh, the line. I'd assume that it represents the inputs of *m* and *p* or the input of *m* resulting in *p*.
Azeem:	Ok.
Dave:	So actually, yeah at that rate, I would still think it was *p* equals *t* of *m* ($p=t(m)$) if *m* is the *x*-value which is supposed to be an input and *p* is a *y*-value.
Azeem:	And why do you think it's the input and why you think *p* is the output?
Dave:	Well, I guess I just think that because it's on the *x*-axis, then it would be the input and *p* on the *y*-axis would be the output.
Azeem:	So that's why you are picking this one? [pointed to $p=t(m)$]
Dave:	Yes.
Azeem:	Ok, what about the other one?
Dave:	The other one.
Azeem:	Why do you think the other person said that oh you could also express it like that [pointed to $m=t(p)$]?
Dave:	I guess just thinking that you could switch the values around if this was a perfect one-to-one, well if it was a perfect ratio of *p* equals *m*, then the two would be reversible, but it's not and I would just assume that this person never looked that
Azeem:	Oh so you are saying that. You said something about the ratio like could you explain that a little bit?
Dave:	Well, I'd say that if the value of *p* equaled the value of *m*(wrote $p=m$, see Figure 13), then the two would be reversible and if the *p*-value did not equal the *m*-value, then this being *m* and this being *p*, then it would be perfectly increasing graph (sketched a graphed, see Figure 21) I guess.

Dave conceived of function notation in conjunction with conceiving of physical characteristics of a graph as well as the convention that the variable in the parentheses should be along the horizontal axis. He said that *m* seemed to have greater values than *p*, so *p* equals *t* of *m* ($p=t(m)$) was the correct notation. I interpret that he conceived of function notation at a *function notation as label* level because he attended to the physical slant of the line. To Dave, *m* values seemed to increase faster than *p* values, so he decided to choose the variable with greater values to represent the independent variable in function notation. He said that both function notations $p=t(m)$ and $m=t(p)$ "could be equateable", but he said that the line was not "exactly proportionate one-to-one," so I interpret that he wanted to see the shape of a graph of $f(x)=x$ to be able to express a graph using both function notations. He conceived of the possibility of varying the intensity of the change (Johnson, 2012b) so that the shape of a graph would not have a slant that it had and then both function notations $p=t(m)$ and $m=t(p)$ could be used to express a linear graph. He wrote $p=m$ and sketched his own graph (see Figure 13) later when I asked him to explain what he meant by perfect ratio. Dave also conceived of function notation at a *function notation as convention* level because he matched the axes of labels to function notation. He provided evidence by saying that he chose "*p* equals *t* of *m*" because *m* was the input and on the *x*-axis, *p* was on the *y*-axis and was the output.

Post interview. I asked Dave to interpret a response from a student named Max who said that both *m* equals *s* of *r* and *r* equals *s* of *m* could be used to describe the graph. I provided a graph to Dave which he annotated (see Figure 22 below). I include this excerpt because here again, Dave's conceptions of function notation were intertwined with the shape of a graph, so he conceived of function notation at a *function notation as label* level.

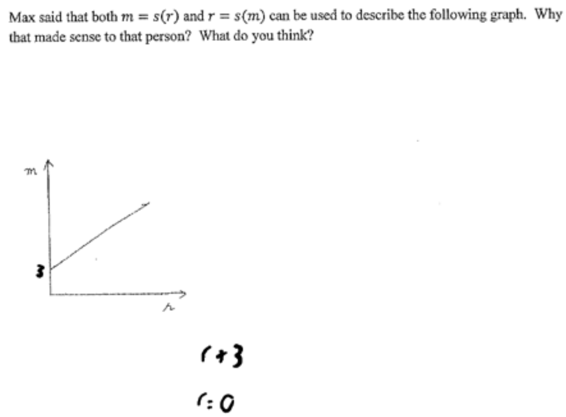


Figure 22: Dave's annotation of general function notation task in Post interview

Excerpt 19: Dave Post interview

Azeem:	Ok, alright. So, there was a student named Max and Max said that both *m* equals *s* of *r* (*m=s(r)*) and *r* equals *s* of *m* (*r=s(m)*) can be used to describe this graph. Why that made sense to that person? What do you think about that?
Dave:	I can see the *r* equals *s* of *m* (*r=s(m)*) that would make sense to me. *m* equals *s* of *r* (*m=s(r)*), I guess I could see that working if they thought that *s* as a function involves some kind of addition.
Azeem:	What kind of addition?
Dave:	Well, just because the *r* starts at zero but *m* starts a few points up if *s* was something of *r* plus 3 or something like that then *r* could theoretically equal zero as it does here but you could still end up with an output of 3 up here say this is a value of 3 [put '3' as the starting point]. But, at that rate I am not sure if it would produce a consistently increasing graph like this.
Azeem:	Hum, and why does *r* equal *s* of *m* work?
Dave:	*r* equal *s* of *m* (*r=s(m)*) yeah, *r* equal *s* of *m* (*r=s(m)*), because *m* has a starting value and *r* does not so because of that *r* could be an output derived from the input of a pre-existing value.

In the Post interview, the shape of a graph was intertwined with Dave's conception of function notation. In this excerpt, Dave said that "*m* equals *s* of *r* (*m=s(r)*)" would work if *s* was something of *r* plus 3. I interpret that he meant if *m=s(r+3)*. When he said, "I am not sure if it would produce a consistently increasing graph like this," I interpret that by the consistently increasing graph he meant that both *r* and *m* should start at the origin. Then he said "*r* equals *s* of *m* (*r=s(m)*)" worked because *m* had a starting value, but *r* did not, so *r* as an output could be derived "from the input of a pre-existing value," so I interpret that to Dave, the variable along the vertical axis had a higher starting value than the other variable, so to him, the same variable with a greater starting value had to represent the independent variable in function notation. He had similar reasoning for graph 3 in four sets of graphs and matching rules task as well. Because he matched the variable with higher starting value to represent the independent variable in function notation, he conceived of function notation at a *function notation as label* level where his conception of function notation was intertwined with the shape of a graph.

Situation Tasks

In this section, first I describe how Dave reasoned with function notation in a task related to function notation in the Pre interview and the Post interview. Then I describe his reasoning with function and function notation given a swing situation task in both the Pre interview and the Post interview. In both tasks, he conceived of function notation at a *function notation as label* level across the Pre interview to the Post interview. Dave's engagement with swing situation task provided evidence of how he conceived of quantities and how his conception of general function notation ($y=f(x)$) was separate from his covariational reasoning.

Situation Task 1: Function Notation as Label across the Pre Interview and the Post Interview

Pre interview: Function notation as label. To learn more about how Dave conceived of a function notation, I asked Dave to interpret a response from a student named Nat who said that for the first situation, the graph could be written as both $a=f(d)$ and $d=f(a)$. I gave him a graph which he annotated (see Figure 23). In the excerpt below, Dave provided evidence that he could write function notations for a graph that did not satisfy his definition of a function.

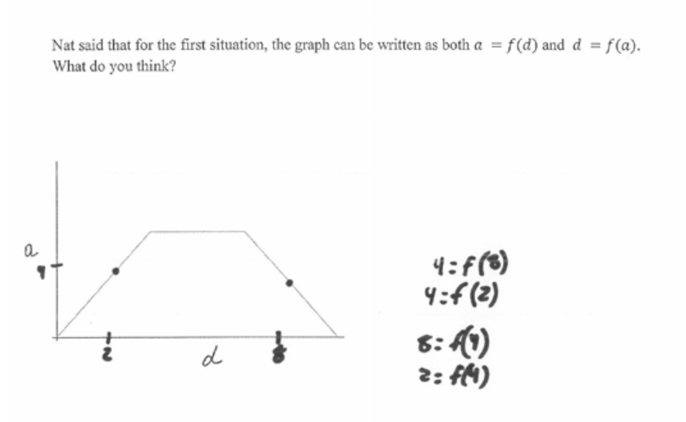


Figure 23: Dave's annotation of the task in Pre interview

Excerpt 20: Dave Pre interview

Azeem:	Nat said that for the first situation the graph can be written as both *a* equals *f* of *d* (*a=f(d))* and *d* equals *f* of *a* (*d=f(a)*). What do you think?
Dave:	I agree. I think that it could. Just because from what I saw in the previous one even if there is a problem with a function even if you switch the values for *d* and *a*, it still seems to encounter the same points so, yes, I'd say that the graph can be written as both *a* equals *f* of *d* (*a=f(d))*and *d* equals *f* of *a* (*d=f(a)*).
Azeem:	And why again? Can you write down something over here?
Dave:	Sure. So, at that rate like another example where *d* is 2 and 8 over here and *a* were to just equal 4 and hit the point of (2, 4) and (8, 4) and. Sorry trying to think and move at the same time. [wrote *4=f(8)* and *4=f(2)*]
Azeem:	So, what's happening?
Dave:	Umm kind of hitting a moment where I would think that they could both be written as a function, but I am running into them not working as a function.
Azeem:	Humm.
Dave;	So, I guess they could both be written as the same non-real function.
Azeem:	[pause] ok.
Dave:	So, let me try *d* equals *f* of *a* (*d=f(a)*) if I could.
Azeem:	Ok.
Dave:	So, 8 equals *f* of 4 (*8=f(4)*) and 2 equals *f* of 4 (*2=f(4)*), alright, yes they could be written as the same non-real function.
Azeem:	[pause]. Non-real function. Why did you say that?
Dave:	Umm, I say non-real because just based on the whole two different inputs equaling or the same input equaling two different outputs that it is not a functional function.
Azeem:	Ok, and what about a (referring to *a=f(d)*)

Dave; For *a* , in that case it would be 4 would equal *f* of 2 (*4=f(2)*) or 4 would equal *f* of 8 (*4=f(8)*), so it is a problem with the same output equaling two different inputs.
Azeem: So, they are both not functions.
Dave: Yes.

Dave conceived of function notations as variables with different letters that could be used to represent graphs that represented functions or graphs that did not represent functions. He first said that switching the values for *d* and *a* still gave the same points, so the graph could be written as both "*a* equals *f* of *d* (*a=f(d))* and *d* equals *f* of *a* (*d=f(a)*)." He put numbers later (see Figure 23) and said that the graph did not represent a function, although he said that he could write both function notations as the same *non-real function*. When I asked him what he meant by the term *non-real function*, he explained that by the term *non-real function*, he meant that one input had two different outputs and the same output had two different inputs which according to Dave did not define a "functional function." I interpret that he used numbers to show that a graph did not represent a function, but he could still use either function notation to represent this *non-real function*. He engaged in variational reasoning at a *Variable as symbol* (Thompson & Carlson, 2017) level because he conceived of function notation variables as symbols. Based on my levels of function notation, he conceived of function notation at a *function notation as label* level, where variables were just different letters because he used function notation for graphs that represented a function or for graphs that did not represent a function. A graph that represented an onto function did not satisfy Dave's definition of function, and his conception of the definition of a function impacted his conception of function notation, so he expressed the graph as both *a=f(d)* and *d=f(a).*

Post interview: Function Notation as Label. To learn more about how Dave conceived of a function notation, I asked him to interpret a response from a student named Chris who said that for the first situation, the graph could be written as both *a=f(d)* and *d=f(a)*. I gave him a graph which he annotated (see Figure 24). In the excerpt below, Dave provided evidence that his conception of the definition of a function impacted his conception of function notation and in this task, he did not write a function notation for a graph that did not define a function to Dave.

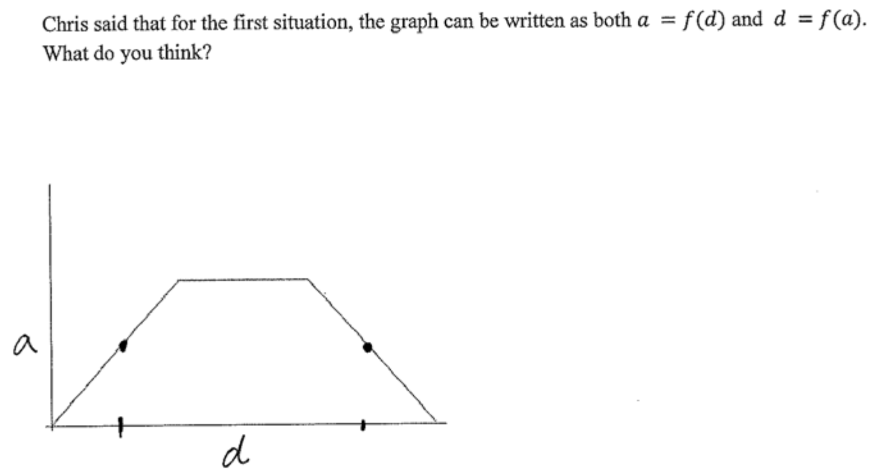


Figure 24: Dave's annotation of the task in Post interview

Excerpt 21: Dave Post interview

Azeem:	Ok, alright. So, there was a student named Chris and Chris said that for the first situation, the first one over here [pointing to graph 1], the graph could be written as both *a* equals *f* of *d* (*a=f(d))* and *d* equals *f* of *a* (*d=f(a)*). What do you think?
Dave:	[pause, 8 secs]. So *a* equals *f* of *d* (*a=f(d))* is more equatable to standard function format where the *y*-value is equatable yes the *y*-value equals the function of the *x*-value. But if *d* is the input then there is a problem where different inputs can well eventually hit the same output if *y* is the output, so you cannot have the same output for different inputs otherwise the function does not work or yes, my function does not work. If we flip it where distance is equal to a function of altitude (*d=f(a)*), then I think the same problem is still there where yes the same altitude will have, for the same altitude will hit different *x*-values [pause, 3 sec] so I don't think *a* equals *f* of *d* (*a=f(d))* or *d* equals *f* of *a* (*d=f(a)*) will work.

Dave provided evidence that a graph that did not satisfy his definition of a function could not be written as a function notation either. For *a=f(d)*, he said, "if *d* is the input then there is a problem" because different inputs could not have the same output. I interpret that he was consistent with his definition that a function must satisfy a one-to-one correspondence. For the case $d=f(a)$, he said, "the same altitude will hit different x-values," so I interpret that he was applying the same definition that graphs representing an onto function could not define a function. I interpret that he related his definition of a function to function notation and said that neither function notation worked and conceived of function notation at a *function notation as label* level.

Situation Task 2: Function Notation as Label across the Pre Interview to the Post Interview

In this section, I include a swing situation task from both the Pre interview and the Post interview. Dave's engagement with this task provided evidence of how he conceived of quantities and how his conception of general function notation ($y=f(x)$) was separate from his covariational reasoning.

Swing Situation Task in Pre interview: Covariational Reasoning

I gave Dave a task with the description such that a child had been swinging on a swing for some time. The graph represented the total distance traveled and the height of the swing (see Figure 25). He read the situation out loud. I asked him to label the axes and interpret the graph.

Excerpt 22: Dave Pre interview

Dave:	So, height hits peaks while x just keeps progressing.
Azeem:	Ok, so what does this point mean then?
Dave:	That point I would assume so if x is the distance then [pause 8 secs] sorry
Azeem:	No, that's ok.
Dave:	Yeah, I guess x should not be continuously increasing as the distance unless it is collective should stay about the same.
Azeem:	So, what is this point representing what is that point, what is this point, like what is it doing?
Dave:	Alright. Well this point should just be starting point maybe the swing is not swinging.
Azeem:	Ok.
Dave:	So, then distance equals zero and then y is the height this would be at the top of the swing and at the bottom I would say they would be at the bottom of the swing. But as it keeps going say maybe so then I'd assume it will go to here, this would

	also be at the bottom of the swing, but it will be also some sort of distance traveled. I might have mislabeled this. So [pause 11 secs]
Azeem:	Ok.
Dave:	Then I guess just the midpoint over here; actually, I am not quite sure if I can detach the graph to that situation.
Azeem:	Hum. What does not seem to work?
Dave:	Um, sorry just I am having trouble attributing an *x*-value on here to distance swung but that's probably just me overthinking it, so yeah at that rate, I guess distance would be being collected on every swing so then just these points would just be the varying heights of the swing while *x* is just increasing collective distance.
Azeem:	Umhum.
Dave:	Alright, so I guess under that it makes sense.

Reasoning with quantities. Dave conceived of both distance and height changing together. He first labeled the vertical axis 'height' (see Figure 25 left) and said, "*y* would equal height." Then he said, "*x* in this case would just equal." I gave him a hint by asking what the two things he was looking at were. He said the height and the distance traveled. He also said, "*x* should also be negative." Then he said, "*y* should equal height, but *y* should equal maybe equal forward distance (pointing to vertical axis) and *x* should equal backwards distance (pointing to horizontal axis). I interpret that he conceived of distance as forward distance, but he conceived of the vertical axis representing the forward movement of the swing. By pointing to the horizontal axis, he conceived of backward distance which represented the backward movement of the swing. I asked him if he could interpret the total distance and the height just the way it was. Then he said, "*x* would just equal the distance swung and *y* is just the height of the swing." Then he said, "height hits peaks while *x* just keeps progressing." I interpret that he now conceived of *x* as the total distance traveled and the height varied as distance increased. Then I asked him about the low point and the high points of the graph. Later he also said, "I am having trouble attributing an *x*-value on here to distance swung….these points would just be the varying heights of the swing while *x* is just increasing collective distance," so I interpret that he was having difficulty relating a graph to the swing situation, but still conceived of both height and distance changing together where distance increased, and the height varied with it. Dave engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level because he said that as the distance increased, the height varied with it.

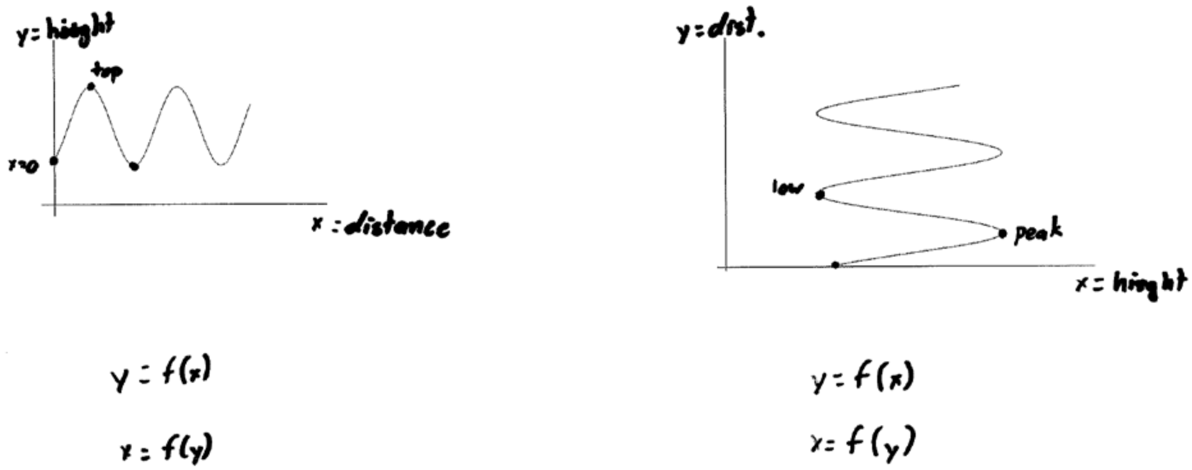


Figure 25: Dave's annotation of the swing situation with distance on the horizontal axis (left) and distance on the vertical axis (right)

Reasoning with quantities with attributes on different axes. I asked Dave to switch the attributes and interpret the graph. I include an excerpt below because he engaged in covariational reasoning.

Excerpt 23: Dave Pre interview

Azeem:	Ok, now we have the same situation but now I would like you to switch the attributes. The attributes are now on different axes so what would it look like.
Dave:	So, in this case *x* now equals height I still have spelled that wrong and now *y* is equal to distance
Azeem:	Ok.
Dave:	So then at that rate I would assume that *x* would start all the way over here (pointed to the starting point of a graph along the horizontal axis) in terms of height I guess because the swing is 10 feet off the ground.
Azeem:	Ok.
Dave:	And *y* is increasing continuously (moved pen in an upward motion along the vertical axis) as the collective distance is swung for every high *x*-value that is a

	peak [wrote peak, see fig.25 right] on the swing as a high point [darkened the max on graph] and for every value of where it just goes further zero that is well I guess a low point [darkened low point] on the swing and it is just progressively increasing.
Azeem:	Ok, alright. So, is it any different from the first situation?
Dave:	No, not necessarily no.

Dave conceived of height and distance changing together. In this excerpt, I asked Dave to switch the attributes and interpret the graph. He said, "x now equals height," and "y is equal to distance." He also labeled the axes (see Figure 25 right). He pointed to the starting point of the graph along the horizontal axis and said, "x would start all the way over here in terms of height I guess." Then he moved pen in an upward motion along the horizontal axis and said, "y is increasing continuously as the collective distance is swung for every high x-value that is a peak." He wrote the word peak and darkened the max on the graph to show that height was at its highest point. He darkened the low point, wrote the word low, and said, "that is, well I guess a low point on the swing and it is just progressively increasing." I interpret that he conceived of height varying from a high point to a low point as distance increased, which provided evidence that he was engaging in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level.

Interpretation of notation: Function notation as label. I asked Dave if it was possible to write the swing situations as $h=f(d)$ or $d=f(h)$. I include an excerpt below because his conception of function notation included the shape of a graph.

Excerpt 24: Dave Pre interview

Dave:	So, for the first one I don't think it would work for h equals f of d ($h=f(d)$) or d equals f of h ($d=f(h)$) because we have the same problem of multiple inputs equaling the same output just because y crosses the x-value at multiple points.
Azeem:	Ok, but when you have those (pointed to x in $y=f(x)$and y in $x=f(y)$, see figure 25 left) what happens to those inputs?
Dave:	Uh, well if you are putting x in for the input then it would be constantly increasing
Azeem:	Ok

Dave: but at that rate say if over here *x* is 2 and way over here *x* is 4 (pointed to two different points along the horizontal axis), right over here *x* is 4 then it would still have the same output (pointed to the maximum).

Azeem: Ok, and what about the second case?

Dave: in the second case, if you input *y*-value let's say your *y* is 4 (pointed to the first maximum) and your *y* is still 4 over here (pointed to the other maximum), then it would equal different *x*-values because over here (pointed to the horizontal axis) *x* might be 2 and you know over here (moved pen along the horizontal axis to the left of the previous point) *x* might be 4 but at the same time *y* equals 4 can produce different outputs.

Azeem: Ok. What about the second?

Dave: The second one I think would run into similar problems with *h* for *y* equaling *f* of *d* for *x* (wrote $y=f(x)$ see Figure 25 right), then so for every *x*-value say like I don't know between here of maybe 5 over here 7, you have the same problem of hitting the *y*-value of like 1 and then 2, and then 4, 5 (moved pen vertically over the graph) and it would just keep going. So that one does not work because *x*-values would be encountering the same *y*-values repeatedly and if I were to reverse that with *x* for distance equaling *f* of *y* (wrote $x=f(y)$, see Figure 25 right), then we have the same problem where something like I say or different *y*-values of like 0 and 2 and then 4 (moved pen vertically) would equal the same *x*-value (moved pen horizontally and then up) of like 3.

Azeem: Hum

Dave: Yeah so, I am constantly seeing inputs and outputs violating rule of one-to-one I guess

Azeem: Do they have to be one to one?

Dave: I am not so sure.

Azeem: You are not too sure.

Dave: Yeah, I am not too sure. I remember something like that, but I also remember there being exceptions for certain things and based on that I mean saying that most of these are not functions, I am starting to think I got it wrong.

The shape of a graph was intertwined with Dave's conception of function notation and he preferred letters *x* and *y* when writing a function notation (see Figure 25). I interpret that he selected *x* and *y* because he was used to seeing an *x* and *y* in function notation. He said that neither notation $h=f(d)$ or $d=f(h)$ worked for swing situation (Fig.25 left) because there was, "the same problem of multiple inputs equaling the same output." To clarify what he said, I pointed to *x* in $y=f(x)$and *y* in $x=f(y)$, see figure 25 left) and asked him what would happen if he used *x* and *y* as inputs. He said, "if you are putting *x* in for the input then it would be constantly increasing." I interpret that he conceived of *x* as a quantity that increased. Then he employed a correspondence approach to show that both $y=f(x)$ and $x=f(y)$ did not work for the first graph (see Figure 25 left). He picked values to show that in the case of $y=f(x)$, where x was the input, 2 different inputs had the same output. He also showed that if y was the input in $x=f(y)$, then the same input produced 2 different outputs. Then I asked him about the second graph, where attributes were represented on different axes (fig.25 right). He used numbers to explain that for function notation $y=f(x)$, "x-values would be encountering the same y-values repeatedly" and that was a problem. I interpret that he conceived of the variable x representing distance and the variable y representing height and he meant that different distance inputs (x) could not have the same height output (y). I interpret that it should be same input x giving different outputs y for figure 25 right, because in Figure 25 right, x represented height and y represented distance. From my perspective, he was using variables x and y imprecisely. It was fine with Dave to be what I saw as imprecise because he attended to the shape of a graph. If a graph violated the one-to-one property, it was enough to say that function notation did not work.

Dave provided evidence that he used the same definition to interpret function notation $x=f(y)$. For the notation $x=f(y)$, Dave said that different y-values had the same x-value. He used numbers again to explain that $x=f(y)$ did not work. Then Dave said, "I am constantly seeing inputs and outputs violating rule of one-to-one I guess." When I asked him if they had to be one-to-one, he said he was not sure. He said, "saying that most of these are not functions, I am starting to think I got it wrong," so I interpret that he was skeptical of his application of definitions of a function.

Because Dave was skeptical of his application of definitions of a function, so to learn more about how he conceived of onto functions and a possibility of a shift in his reasoning with function, I asked him to graph a function $f(x)=x^2$. He said, "it started off at x equals zero right there and then from there, as it went on, it would just keep increasing and if it was negative then it would do the same," while sketching a graph (see Figure 26). I asked him if it was considered a function. He said, "I recognize it as a function, but I can't explain it." He also said, "I think it is a function just because I have seen it in other cases used as a function." He also said he was not sure. I interpret that he remembered that the graph of a function $f(x) = x^2$ represented a function but did not explain why it represented a function. My conjecture was that he might have reasoned that different x-values had the same y-value or different inputs had the same output, so this graph represented a function. But, he did not want to say that, because he had been using this definition to state that a graph did not represent a function. My conjecture was that he did not want to be wrong. He conceived of a function as one (different) input mapping to a different output and held this definition of a function to interpret function notations as well. Dave conceived of function notation at a *function notation as label* level, where he employed a correspondence approach and his conception of function notation was intertwined with the shape of a graph.

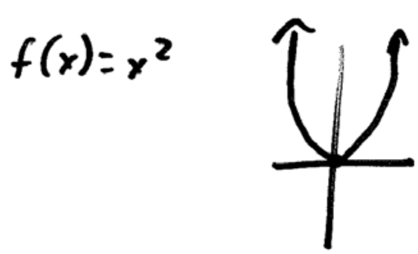


Figure 26: Dave's sketch of $f(x) = x^2$

Swing Situation in Post interview: Covariational Reasoning

I gave Dave a task with the same description as in the Pre interview, such that a child had been swinging on a swing for some time. The graph represented the total distance traveled and the height of the swing. I asked him to read the situation out loud and interpret the graph. He annotated the graphs (see Figure 27 left). He labeled the horizontal axis '*x*=total distance' and said, "*x* is the total distance being traveled by the swing." He labeled the vertical axis '*y*=height' and said, "*y* is the height." I include an excerpt below because he provided evidence that he conceived of distance and height as quantities that changed together.

Excerpt 25: Dave Post interview

Azeem:	Ok and we have this situation here. Please read it out loud and interpret the graph.
Dave:	Given the situation below interpret the graph. Suppose that a child has been swinging on a swing for some time. Here is a graph representing the total distance traveled and the height of the swing. Please interpret the graph [reading out loud].
Azeem:	Umhum.
Dave:	So, I am going to put the *x*-axis as, alright so *x* is the total distance being traveled by the swing [labeled horizontal axis, '*x*=total distance'] *y* is the yeah, so *y* is the height [labeled vertical axis '*y*=height] and we don't really have to worry about 0 because the swing is a few feet off the ground apparently.
Azeem:	Umhum.
Dave:	So as the child travels more or as the child swings more collective distance (moved pen left to right over the graph), the height increases and then decreases (traced the first curve with a closed pen), so yeah just as they swing forward and backward there (moved pen up and down), collected more distance (moved pen left to right along the horizontal axis) and there (traced entire graph with a closed pen) the height fluctuates accordingly.
Azeem:	So, what does this point represent (pointed to the first minimum)?
Dave:	This point (darkened the first minimum) is a low point probably when they were just about at ground level as they were swinging backwards or forwards.
Azeem:	Ok.
Dave:	And these peaks (darkened the maximums) are the heights of the swing going either way forward or backward.

Reasoning with quantities. Dave conceived of distance and height changing together because he said, "as the child swings more collective distance, the height increases and then decreases." He moved his pen left to right along the horizontal axis to show that the distance increased. He traced the entire graph with a closed pen to show that the height fluctuated. I interpret that Dave conceived of both height and distance changing together such that distance increased, and the height varied with it, which provided evidence that he engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level.

Reasoning with quantities with attributes on different axes. When I asked Dave to switch the attributes and interpret a graph, he annotated the graph (see Figure 27 right). I include an excerpt below because he provided evidence that he engaged in covariational reasoning.

Excerpt 26: Dave Post interview

Azeem:	Ok, alright. So, now the same situation now have your axes on different [pointing to the axes on 2nd graph], your attributes are on different axes.
Dave:	Well, so now the *y*-value is the total distance [labeled vertical axis '*y*=total distance'] and *x* is the altitude or height [labeled horizontal axis '*x*=height'].
Azeem:	Uhum.
Dave:	So, because of that the total distance is constantly increasing now along the y-axis and height is still hitting the same peaks of values as the swing moves forward and backward along its path and it is constantly moving upward because it is constantly accumulating distance as it swings forward and backward.
Azeem:	Umhum.
Dave:	So, yeah same as before.

Dave provided evidence that he conceived of a graph representing the same information and engaged in covariational reasoning. He labeled the axes and said, "the total distance is constantly increasing now along the *y*-axis and height is still hitting the same peaks," so I interpret that he still conceived of the same relationship between distance and height when the axes labels were switched. He moved pen vertically from the origin going up to show that the distance increased along the vertical axis. He also darkened the peaks to show that the height attained maximums and minimums. I interpret that he conceived of distance increasing and height increasing and decreasing based on distance, which provided evidence that he engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level.

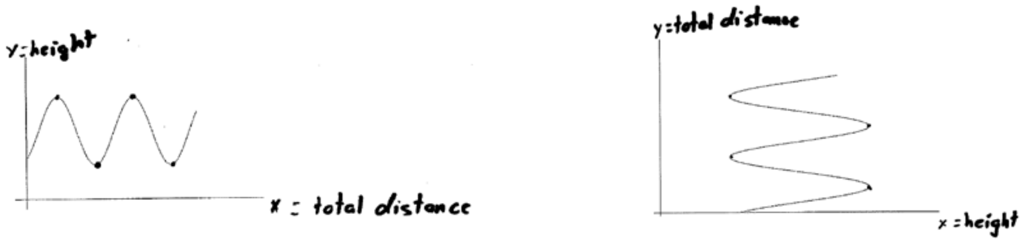
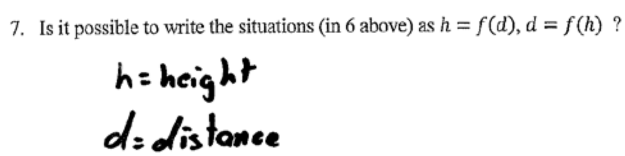


Figure 27: Dave's annotation of the swing situation with distance on the horizontal axis (left) and distance on the vertical axis (right)

Function notation as label. I asked Dave if it was possible to write the swing situations as *h=f(d)* or *d=f(h).* His work related to function notation is shown in Figure 28. I include an excerpt below to demonstrate how the shape of a graph impacted his conception of function notation.


7. Is it possible to write the situations (in 6 above) as $h = f(d)$, $d = f(h)$?

h= height
d= distance

Figure 28: Dave's work related to function notation in the Post interview

Excerpt 27: Dave Post interview

Azeem: Ok, what about notation? How can we write those [put paper with notation *h=f(d), d=f(h)*]?

Dave: Ok height equals *f* of *d* (*h=f(d)*), so I am gonna write *h* as height and *d* as distance and put that down [writing *h*=height, *d* =distance]. Ok, so because of that I'll try and apply both of those formats to the first one. Let's see so height is a function of distance (*h=f(d)*), well just looking at it again I don't know if I can write it as either one because um as you move along the *x*-axis or distance (moved pen left to right along the horizontal axis) you'll hit the same *y*-value repeatedly (pointed to peaks) depending on where you are. So, if I were to put that in as *h* equals *f* of *d* (*h=f(d)*) then we'd have a problem where multiple inputs can produce the same output but if I were to flip that where *d* equals *f* of *h* (*d=f(h)*), then we'd have another problem where, yeah, if we are using height as an input then there is a problem where, what is the way to put this, there are multiple heights (pointed to

	peaks) for different *x*-values still and because of that I don't think it would work for either.
Azeem:	Ok so you can't write either one of them (pointing to notations)
Dave:	Yeah and I think the same would apply for this (Figure 27 right) because if you are using *d* as an input then you hit the same, well the same *x*-value can hit multiple different *y*-values (moved pen from a point on the horizontal axis to the graph vertically)
Azeem:	Ok.
Dave:	Yeah, so that means that neither will apply to that (right graph) either.

Dave provided evidence that his reasoning with function notation was intertwined with the shape of a graph. In this excerpt, Dave mixed words with variables and wrote *h* = height and *d* = *distance* on the paper (see Figure 28). Dave said that neither function notation *h=f(d)* or *d=f(h)* worked for a graph shown in Figure 27 on the left. In the case of *h=f(d)* he said, "we'd have a problem where multiple inputs can produce the same output." For the case *d=f(h),* he said, "if we are using height as an input," then this was not possible because "there are multiple heights for different *x*-values." From my perspective, multiple heights meant different heights, but from Dave's perspective, height as an input had different outputs, which did not define a function, so *d=f(h)* did not work. It was fine with Dave to be what I considered imprecise because he conceived of function notation in conjunction with the shape of a graph. If a graph did not satisfy a one-to-one property, he decided that neither notation worked without carefully considering the input and output variables, because his reasoning with function notation was intertwined with the shape of a graph.

Dave conceived of function notation in conjunction with conceiving of the shape of a graph. He said that *h=f(d)* or *d=f(h)* could not be used to express a graph shown in Figure 27 on the right either. He said, "using *d* as an input", but he did not use *d* as an input. He moved his pen from a point on the horizontal axis to the graph vertically (see Figure 27 right) and said, "the same *x*-value can hit multiple different *y*-values." Then he said, "neither will apply to that (right graph) either." From my perspective, he only showed that $d=f(h)$ did not work and was using h as an input, but from Dave's perspective, he showed that both $h=f(d)$ and $d=f(h)$ did not work. I interpret that what I considered imprecise was fine with him, because he attended to the shape of a graph. If a graph violated the one-to-one condition, he decided that neither function notation worked without carefully considering which variable should be used as an input variable and which variable should be the used as an output variable. This meant that his conception of function notation was intertwined with the shape of a graph, so he conceived of function notation at a *function notation as label* level.

Summary

Dave is a case of a student who conceived of function notation at a *function notation as label* and *function notation as convention* level within the Pre interview and the Post interview. Within Ferris wheel interviews, he conceived of function notation at a *function notation as a relationship between variables* level, where he engaged in quantitative reasoning and covariational reasoning and also employed a correspondence approach to justify function notation. Within Ferris wheel interviews, he conceived of the definition of a function such that different inputs could map to the same output. Across Ferris wheel interviews to the Post interview, he applied a different definition of a function such that different inputs could not map to the same output. Within the Pre interview and the Post interview and across the Pre interview to the Post interview, he did not demonstrate quantitative reasoning, variational reasoning, or covariational reasoning when conceiving of function notation. Next, I briefly summarize how Dave's case answers my research questions.

How Might Students' Conceptions of Function Impact Their Conceptions of Function Notation?

Dave's conceptions of the definition of a function shifted from the Pre interview to Ferris wheel interviews and within the Post interview, he shifted back to how he defined it in the Pre interview. Within the Pre interview and the Post interview and across the Pre interview and the Post interview, Dave operated with the definition of a function such that it had to satisfy a one-to-one property (only a different input could map to a different output) and operated with the same definition to interpret function notation as well. In other words, in the Pre interview and the Post interview context, to Dave same input mapping to different outputs and different inputs mapping to the same output did not represent a function. Within Ferris wheel interviews and across Ferris wheel interviews, Dave operated with a definition of a function (different from the Pre interview and the Post interview) such that different inputs could map to the same output and operated with the same definition to interpret function notation as well. In Ferris wheel interviews 1 and 2, he conceived of an invariant relationship between quantities and then employed a correspondence approach to justify function notation. In other words, he engaged in covariational reasoning and employed a correspondence approach to function. I interpret that he conceived of function notation using Thompson and Carlson's (2017) definition of function such that there was an invariant relationship between quantities and one value of a quantity determined one value of the other quantity. A possible reason for a shift in his conceptions of function in Ferris wheel interviews was because he engaged in quantitative reasoning and covariational reasoning. Within Ferris wheel interviews, he also had a different definition of a function such that different inputs could map to the same output. Across the Pre interview to the Post interview, he used the same definition of function such that a function satisfied the one-to-one condition which impacted his function notation. For example, in a swing situation task, his reasoning with function notation from my perspective was imprecise, but from his perspective was just fine. If a graph violated the one-to-one condition, he decided that neither function notation could be used to express a graph without carefully deciding the input and output variables. In other words, the shape of a graph was intertwined with his conception of function notation.

How Might Covariational Reasoning Related to Function Impact Students' Conceptions of Function Notation?

Within the Pre interview and across the Pre interview to the Post interview, Dave engaged in covariational reasoning, but his conception of function notation was separate from his covariational reasoning. His conception of function notation was intertwined with the shape of a graph. For example, in a swing situation task, when reasoning with function notation, it was fine with Dave to be what I saw as imprecise because he attended to the shape of a graph. If a graph violated a one-to-one definition of a function, he did not carefully consider which variable should be used as an input or output and concluded that he could not write a function notation either.

Within Ferris wheel interviews, Dave engaged in covariational reasoning and employed a correspondence approach which impacted his conception of function notation. He was not only thinking about how quantities changed together, but also thinking about interchanging the variables along the axes as long as it satisfied the definition of a function. For example, in the Ferris wheel tasks when he chose function notation, he conceived of an invariant relationship between quantities and then employed a correspondence approach to justify function notation. In other words, he conceived of function notation using Thompson and Carlson's (2017) definition of function which states: "A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (p.444). Thompson and Carlson's (2017) definition of function is important and I refer to it as a combination of covariational reasoning and a correspondence approach, because Dave engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. He employed a correspondence approach to attend to the part that the value of one quantity determined the value of the other quantity.

How Do Students Conceive of a General Function Notation?

Dave preferred the letters *x*, *y*, and *f* when interpreting function and function notation. If variables other than *x* and *y* were used, he converted different variables to *y* equals *f* of *x* and then decided how to label the axes. He used variable *x* along the horizontal axis, variable *y* along the vertical axis, and preferred *f* as the name of a function. One possible reason of why he converted letters to *x*, *y*, and *f* is because textbooks usually represent a function notation as $y=f(x)$. It is a convention (see also Moore, Paoletti, and Musgrave, 2013; Thompson et al., 2014) to represent the independent variable *x* in parentheses along the horizontal axis and the dependent variable *y* on the other side of an equal sign along the vertical axis.

Across the Pre interview to the Post interview, Dave conceived of function notation at *function notation as label* and *function notation as convention* levels. For example, in the Pre interview, both shape of a graph and convention of matching axes labels with variables in function notation were intertwined with his conception of function notation. He considered the variable along the horizontal axis to have greater values and used that variable to represent the independent variable in function notation. He also matched the labels of axes with function notation such that the independent variable within the parentheses was along the horizontal axis and the variable on the left-hand side was along the vertical axis. In the Post interview, he did not provide evidence of matching the label of axes to function notation, but the shape of a graph alone was intertwined with his conceptions of function notation. To Dave, the variable along the vertical axis had a higher starting value than the other variable, so the same variable with a greater starting value had to represent the independent variable in function notation.

Across the Pre interview to the Post interview, Dave conceived of function notation at a *function notation as label* level in a task of interpreting a response from a student named Nat (Pre interview) and a student named Chris (Post interview). In the Pre interview, Dave expressed function notation to describe all graphs whether they represented a function to him or not. Whereas, in the Post interview, he used function notation only for graphs that satisfied how he conceived of a function. Within the Post interview, if the same input had different outputs and different inputs had the same output, that did not define a function to Dave and he used the same measure to interpret function notation such that neither function notation could work. A viable explanation for this shift is his engagement with Ferris wheel tasks that I presented earlier in this chapter.

Within Ferris wheel interview 1, Dave shifted in his conception of distance from both distance and height measuring the same thing (length from the ground) to conceiving of distance increasing and only the height increasing and then decreasing. After a shift in his conception of distance, he had a shift in his reasoning with function notation within Ferris wheel interview1 that stayed consistent across Ferris wheel interview 1 to Ferris wheel interview 2. Within Ferris wheel interviews 1 and 2 and across Ferris wheel interviews 1 and 2, Dave conceived of function notation at a *function notation as a relationship between variables* level. In other words, he conceived of function notation using Thompson and Carlson's (2017) definition of function, which I refer to as a combination of covariational reasoning and a correspondence approach. He engaged in quantitative reasoning, variational reasoning, and covariational reasoning and employed a correspondence approach to attend to the part that the value of one quantity determined the value of the other quantity.

CHAPTER VII

CASE STUDY OF LISA

In this chapter, I present a case study of Lisa who demonstrated that within the Pre interview, she first conceived of function notation at *function notation as convention* level and then conceived of function notation at *function notation as a relationship between variables* level. Within Ferris wheel interview 1, she had difficulty conceiving of function notation. Within Ferris wheel interview 2, she demonstrated that she first conceived of function notation at *function notation as convention* level and then conceived of function notation at *function notation as a relationship between variables* level. After she conceived of function notation at a *function notation as a relationship between variables* level within Ferris wheel interview 2, her conception of function notation remained consistent across the Post interview. I provide evidence to illustrate how her engagement with the Ferris wheel tasks impacted her conception of function notation.

Lisa was one of the first students I interviewed. Midway through data collection, I met with my advisor, Dr. Johnson, to discuss how tasks were providing opportunities for me to gather evidence of students' conceptions of function notation. Dr. Johnson provided me an idea to have students respond to others' claims about a graph (see also Johnson et al., 2018, August). As a result, we modified a few tasks (see Table 8 in Chapter 4). Because Lisa was one of the first students I interviewed, she worked on the modified tasks only in the Post interview.

I included a selection of tasks that Lisa worked on during the set of interviews. I selected nine tasks from the set of interviews. I present three tasks from Ferris wheel interview 1 to show that her reasoning with quantities and function notation stayed consistent within Ferris wheel interview 1. I present three tasks from Ferris wheel interview 2 to show how her conceptions of graph and quantities impacted her conception of function notation. I include a plane situation task from the Pre interview and the Post interview. Within the Pre interview plane situation task, Lisa first conceived of function notation at a *function notation as convention* level and then conceived of function notation at a *function notation as a relationship between variables* level. Within the Post interview plane situation task, she conceived of function notation at a *function notation as a relationship between variables* level. I include one task from tasks involving functions, graphs, tables, and function rules to show that Lisa conceived of function notation at a *function notation as a relationship between variables* level. Because the tables, functions, and graphs task was one of the tasks I modified, she did not work on this task in the Pre interview. I include another task from the Post interview to show that Lisa conceived of function notation at a *function notation as a relationship between variables* level. Again, this was one of the tasks that I modified, and as a result Lisa did not work on this task in the Pre interview. I selected these tasks because they provided strongest evidence of Lisa's individual forms of reasoning and her conceptions of function notation. Excerpts are representative of Lisa's broader work across tasks. I have merged Wolcott's (1994) constructs of *description*, *analysis* (my interpretation), and *interpretation* (connections to literature) in the results. I use the term *interpret* to refer to Wolcott's (1994) analysis and interpretation levels. When I make connections to extant literature, I move from analysis to interpretation.

I organized this chapter in such a way as to make it easy to see Lisa's growth in understanding function and function notation after intervention. The Ferris wheel tasks are presented in chronological order. The Pre interview and the Post interview tasks are not presented in chronological order, because I present a task from the Pre interview and then a similar task from the Post interview. I present Ferris wheel tasks first, and then tasks from the

Pre interview and the Post interview or just from the Post interview, to provide readers an opportunity to follow the impact of intervention on Lisa's reasoning with function and function notation from the Pre interview to the Post interview.

At the conclusion of this chapter, I present a summary that addresses my research questions. I include each research question and describe how Lisa's work answered each of my research questions. In the Pre interview plane situation task and Ferris wheel interview 2, she shifted from conceiving of function notation at a *function notation as convention* level to conceiving of function notation at a *function notation as a relationship between variables* level.

Ferris Wheel Interviews

In this section, I present Lisa's work from Ferris wheel interviews to show how she progressed in her conception of function and function notation. I present three tasks from Ferris wheel interview 1 in chronological order to show that Lisa's conception of distance and height and her conception of function notation stayed consistent throughout Ferris wheel interview 1. The first task shows that Lisa's conception of distance and height stayed consistent within Ferris wheel interview 1. Within Ferris wheel interview 1, she conceived of distance and height both increasing up to the top of the Ferris wheel and then decreasing. The second and third tasks show that Lisa had difficulty conceiving of function notation because she conceived of distance and height that represented the same thing- a length from the ground. I present three tasks from Ferris wheel interview 2, in chronological order, to demonstrate changes in Lisa's conception of distance and her conception of function notation within Ferris wheel interview 2 and across Ferris wheel interview 1 to Ferris wheel interview 2. Within Ferris wheel interview 2, she first conceived of distance and height changing just like in Ferris wheel interview 1, but then shifted in her conception of distance. After she shifted in her conception of distance within Ferris wheel interview 2, she engaged in quantitative reasoning and covariational reasoning and conceived of function notation at a *function notation as a relationship between variables* level within Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview.

Ferris Wheel Interview 1

In this section, I include three tasks from Ferris wheel interview 1 in chronological order. The first task shows that Lisa's conception of distance and height stayed consistent within Ferris wheel interview 1. The other two tasks show that her conceptions of distance and height impacted her conception of function notation and she had difficulty conceiving of function notation throughout Ferris wheel interview 1.

Ferris wheel Interview 1, task 1: Lisa's conception of distance and height. Lisa conceived of distance and height as quantities such that both increased up to the top of the Ferris wheel and then decreased. I first asked Lisa how distance and height changed without the animation. Then I showed her the animation and asked again how distance and height changed. I provide an excerpt below to describe what happened after she watched the Ferris wheel animation. In this excerpt, she demonstrated that she conceived of distance and height as measuring the same kind of thing – a length from the ground.

Excerpt 28: Lisa Ferris wheel Interview 1 task 1

Azeem: Ok, alright. So, what I would like you to do is to click on animate point over there.

Lisa: Ah, look at that.

Azeem: So, I'll ask you the same question again like how is the distance changing?

Lisa: [pause/watching animation]. Ok [still watching animation]

Azeem: So, as you move from the starting point, how is the distance changing around the circle?

Lisa: So, as you move from the starting point it increases and it seems to increase [moving cursor counterclockwise up to the max], and then it stays the same here [moved cursor back from the max to half way from the start] and then it decreases [cursor on the left side of the max], seems like pi.

Azeem: Humm.

Lisa:	And then it decreases, decreases distance, distance, until you get to zero till you are at the ground.
Azeem:	Ok, could you point with your finger like what is happening to the distance?
Lisa:	So, the distance from the ground is increasing [moved finger along the FW start to the top] so you are getting further and further away from the ground and then you are getting further until you are at the diameter away [pointing finger at the max] and then it starts decreasing as you are going back towards the ground [moving finger from max to the left].
Azeem:	Ok, so then what is the height doing? Height from the ground.
Lisa:	distance. I don't know I guess I am relating them the same.
Azeem:	Ok.
Lisa:	Distance from the ground as height from the ground.
Azeem:	Ahumm
Lisa:	That's not the same thing?
Azeem:	[pause]. So, distance is the distance traveled around the Ferris wheel and the height is the height from the ground.
Lisa:	Oh, ok. So the distance is not changing from the center.
Azeem:	So what is happening to it?
Lisa:	Um your, so the height is
Azeem:	Is it going up, is it. What is happening to it or is it increasing, decreasing?
Lisa:	Ok, so the distance from your seat on the Ferris wheel um in direct relation to the height. So, the distance from the center of the Ferris wheel is not going to change but it is going to be in direct relation to the height that your distance from the ground like from your seat on the Ferris wheel is in direct relation to your height off the ground.

Lisa talked about distance and height as being "from the ground". In this excerpt, when I asked her how the distance changed, she said, "the distance from the ground is increasing." She also moved her finger along the Ferris wheel from start to the top. She pointed to the maximum with her finger and said, "then you are getting further until you are at the diameter away." She moved her finger from the top of the Ferris wheel to the left and said, "then it starts decreasing as you are going back towards the ground." I interpret that she conceived of distance as distance from the ground that increased and then decreased. When I asked her about height, she said, "I guess, I am relating them the same." Because Lisa said, "distance from the ground as height from the ground" she provided evidence that she conceived of both distance and height as a length from the ground.

Ferris wheel Interview1, task 2: Lisa's difficulty conceiving of function notation. After Lisa predicted how distance and height changed, I asked her to graph the relationship between distance and height. Her sketched graph is shown below (see Figure 29). Then I asked her about function notation. I include an excerpt below to show that she had difficulty expressing distance and height as a function notation.

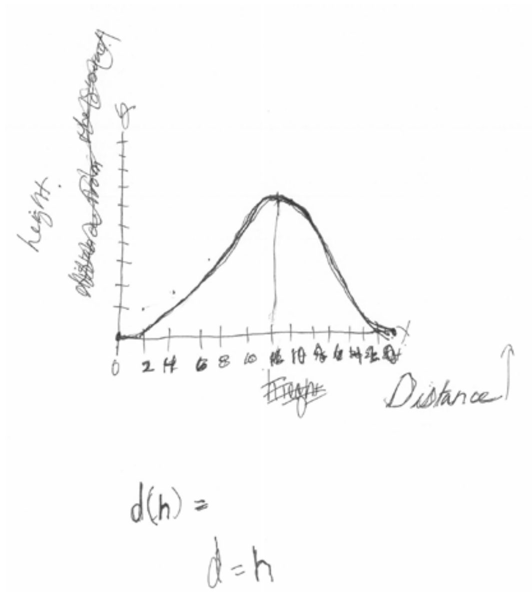


Figure 29: Lisa's graph

Excerpt 29: Lisa Ferris wheel Interview 1 task 2

Azeem:	Ok, Alright. So, could you express, could you use symbols or letters to write a rule for this situation?
Lisa:	Umm [pause]. I guess I am just confusing the distance with the height like they seem synonymous to me.
Azeem:	Ok.
Lisa:	Umm [pause], so distance is a function of (wrote ($d(\ ) =$) under her graph.

Lisa provided evidence that she conceived of distance and height as the same thing again which impacted how she conceived of function notation. She said that distance and height seemed "synonymous" to her. She only wrote *d ( ) =*, so I interpret that Lisa was trying to fit *d* and *h* in a function notation. She had difficulty writing a symbol expression, because to her, *d* and h represented the same thing. Lisa said, "distance is a function of", so I interpret that she conceived of function notation, but did not write f anywhere in her symbol expression. Lisa's conception of distance and height impacted her conception of function notation and she had difficulty expressing distance and height in a function notation.

Ferris wheel Interview1, task 3: Lisa's difficulty conceiving of function notation. Right after Lisa worked on function notation task, she worked on a task related to dynamic segments. She first saw each dynamic segment separately and then watched the dynamic segments together. After watching the dynamic segments together, she said, "it looks like the distance continues to increase while the height is going back down." This was the first time Lisa said that distance continued to increase but then said, "that does not make sense to me though. I guess it is because I am confusing the distance and the height." I interpret that even after watching the dynamic segments, she did not accept that distance could increase, because what she watched conflicted with her conception of distance.

Then I asked Lisa about function notation again. In the excerpt below, she provided evidence that she still conceived of function notation expressing a relationship between distance and height which meant the same thing to Lisa - a length from the ground.

Excerpt 30: Lisa Ferris wheel Interview 1 task 3

Azeem:	Ok, so my next question to you would be again about the notation part. If I ask you to use symbols or letters to write a rule could you write one?
Lisa:	[pause]. But, that does not really work maybe no then it would just keep going up and up and up and then it would not go down. Umm, I don't know so 10, 8, 6, 4, 2, 0 [changed the numbers on the x-axis from the middle point to 10, 8,…,0]. Umm.
Azeem:	Let me give you another paper [pause] so write on this one.
Lisa:	[sketching a graph]
Lisa:	So, (whispered distance or height is a function of distance and wrote $(h(d))$.
Azeem:	So, when I say I want you to express it as symbols what I really mean is if you could write it as h equals f of d ($h=f(d)$).
Lisa:	Oh, ok.
Azeem:	How about that? Could we do that?

Lisa: [pause] (whispers height is a function of distance and wrote (*f(d)=h*)).
Azeem: So I really want to know what you are thinking about it. Is there something missing or?
Lisa: Umm. Is there something missing?
Azeem: Like how are you making sense of the notation?
Lisa: Ok. So, the further that you get from the ground, so distance from the ground the higher off the ground you will be. So, as a function of distance as the independent because as the independent in this one, in this equation or situation then height is dependent upon how far you are from the ground. So, height would be a function of or yeah, height would be the dependent on how far you are from the ground as a function of like.

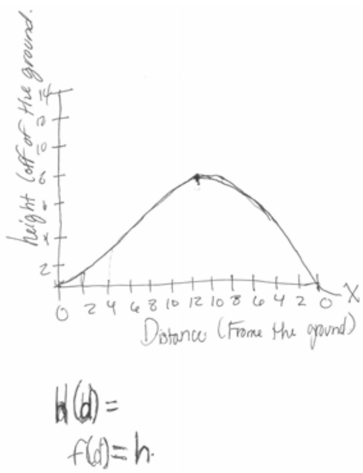


Figure 30: Lisa's graphical representation

Lisa had difficulty conceiving of function notation *h=f(d)* because to her, both distance and height measured the same thing- a length from the ground. When I asked her to write a function rule, she started changing numbers on her graph (see Figure 29). Then I gave her another paper and she sketched another graph (see Figure 30). She said, "distance or height is a function of distance" and wrote *h(d)*. I interpret that it was difficult for Lisa to write height as a function of distance in symbols. When I asked Lisa if she could "write *h* equals *f* of *d*" (*h=f(d)*), then she said, "height is a function of distance" and wrote *f(d)=h*. I interpret that she preferred to write function notation as *f(d)=h* instead of *h=f(d).* I also interpret that she only wrote function notation *f(d)=h* because I asked her to. I wanted to know how she conceived of *d* and *h*, so I asked her what function notation meant to her. She said, "height would be the dependent on how far you are from the ground," so I interpret that there was no shift in her conception of distance and height, because she conceived of both *d* and *h* representing a length from the ground. I interpret that function notation $h=f(d)$ meant something more than what Musgrave and Thompson (2014) term idiomatic expression. To Lisa, both *d* and *h* represented something and that was a length from the ground. Lisa provided evidence that her conceptions of distance and function notation remained consistent within Ferris wheel interview 1.

Ferris Wheel Interview 2

In this section, I present three tasks from Ferris wheel interview 2 in chronological order. In the first task, Lisa's conception of distance changed from conceiving of both distance and height as a length from the ground to conceiving of distance increasing and height increasing up to a point and then decreasing and she engaged in covariational reasoning. I present a second task of dynamic trace/dynamic Ferris wheel to provide evidence that Lisa engaged in covariational reasoning and that impacted her conceptions of graph and her conceptions of function notation. I present a third task to demonstrate that she shifted in her conception of function notation from conceiving of function notation at a *function notation as convention* level to conceiving of function notation at a *function as a relationship between variables* level.

Ferris wheel Interview 2, task 1: Lisa's conception of distance. I include an excerpt below to provide evidence of a shift in Lisa's conception of distance. In this excerpt, I also mention Peter who is a graduate student (see chapter 4). I asked Peter to rephrase a question, because it looked like he wanted to say something, but he did not say anything.

Excerpt 31: Lisa Ferris wheel interview 2 task 1

Lisa: I don't know because the last time that I was here I had the same problem. I am having trouble understanding the distance relative to what, like the distance from the ground, the distance from the center of the circle, the distance from what because distance is a measurement of far and height is a measurement of tall.

Azeem:	Peter you might wanna add something to that. So, it's not the distance from the ground, it's just the total distance traveled like what is another way to
Lisa:	Oh, the distance along the circumference? [moved her finger along the Ferris wheel pic]
Azeem:	Yes.
Lisa:	Oh, okay, that's like pi.
Azeem:	What is it [the distance] doing?
Lisa:	Umm
Azeem:	Is it decreasing, is it increasing?
Lisa:	Distance along the circumference ---- that helps me in my understanding. Ok, so
Azeem:	[Handed a new paper]
Lisa:	Ok. Man, I am not good at drawing circles (sketching a circle). So, if your car starts right here, the distance that you are traveling along the circumference is increasing the further you are going. So, it's a measurement of length really, I guess, a lack of a better term so. You are going further and further, further so even as the height decreases you are still going further. So, you are traveling the circumference of the circle (tracing along the circumference with her pen, see Figure 31)

Lisa provided evidence of a shift in her conception of distance and then provided evidence that she engaged in covariational reasoning. When Lisa said, "like the distance from the ground, the distance from the center of the circle, the distance from what," so the shift in her conception of distance resulted when she was reflecting on her own difficulty. I wanted to clarify that distance was the total distance traveled and not the distance from the ground. I did not even finish my sentence and she was making sense for herself. She said, "oh, the distance along the circumference?" I said, "yes." This was the first time Lisa was clear on what distance meant in this situation. Then she sketched a circle and traced along the circumference to show that the distance increased (see Figure 31). When she said, "You are going further and further, further so even as the height decreases you are still going further," I interpret that at this point, she made a distinction between distance and height, where distance kept increasing and the height increased and then decreased. She engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson & Carlson, 2017) because she said that as the height increased and decreased, distance kept increasing.

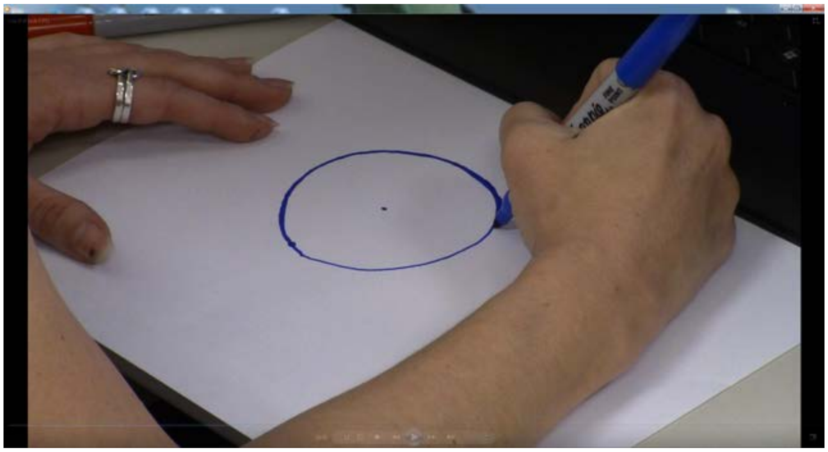


Figure 31: Lisa tracing along the circumference to show that distance increases.

Ferris wheel Interview 2, task 2: Covariational reasoning impacting Lisa's conception of graph and function notation. I include an excerpt below that came after what I described earlier. While I was showing Lisa the dynamic trace and the dynamic Ferris wheel together, she provided evidence that she engaged in covariational reasoning and provided evidence of a shift in her conception of what a graph represented.

Excerpt 32: Lisa Ferris wheel Interview 2 task 2

Azeem:	Ok. So, if we compare this to what you graphed before, do you want to make changes to what you have or.
Lisa:	Um (pause), well see (pause 22 secs)
Azeem:	So, if you compared your graph (pointed to her sketched graph) to that graph (pointed to the dynamic trace), what do you notice?
Lisa:	That it is one line; there is not two linear functions, it is one line and they can relate.
Azeem:	So, you want to make changes or (Lisa started sketching the axes) yes just do another one (graph)
Lisa:	Ok (sketching the coordinate plane and labeling axes). This is a weird thing for me (sketched a graph, see Figure 32)
Lisa:	Ok, so (labeled the vertical axis '*y*').
Azeem:	So, can you explain what is going on
Lisa:	Ok, so this is independent variable [underlined label '*y*' with her pen], so this would be function of height equals (wrote *f(h)*= and labeled the horizontal axis '*x*'). So, you are going (put a mark at the max), so (putting ticks along the horizontal axis and writing numbers and putting ticks along the vertical axis and put 50 at the top of the vertical axis). So, this forces me to think differently. Ok, so this, you are going up and up and up along the distance traveled (putting ticks over a graph), because your distance traveled is increasing no matter what

because you are going around the circumference of the Ferris wheel and then your height off the ground will increase (put dots along her graph). This is crazy because I have not been taught this way before. So, your height off the ground is increasing along that *x*-axis (putting dots along the increasing curve and moved her pen along with the dots) and then when you reach the top of the Ferris wheel (darkened the maximum with her pen) and my numbers are wrong but (whispered this needs to be fixed) and then you are going back down (moving her pen along the decreasing part of her graph) along the *x*-axis, because you are going closer to the ground so you are less feet off the ground.

Azeem: So as far as the quantities distance and height go, are they changing any differently than we had at the first task last week? (darkened the *h* in *f(h)=* she had written before)

Lisa: Umm, No. Last week, I was relating height and distance as equal they meant the same thing, but I was not understanding it. I think it was just my misunderstanding was of what the distance was meaning, so last week I am sure it meant the same thing, but I just was not understanding it the same way.

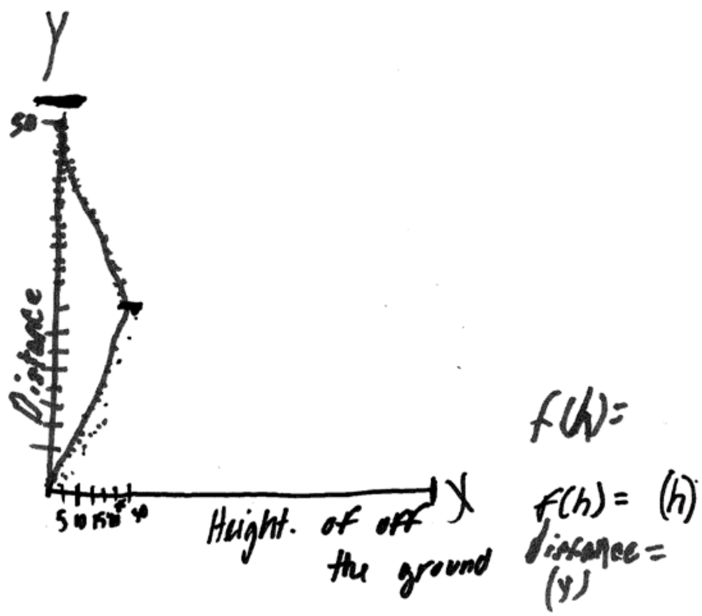


Figure 32: Lisa's sketch of a graph after watching the animated trace

Lisa provided evidence that she shifted in her conception of what a graph represented after watching a dynamic trace. The dynamic trace did not work until she was conceiving of distance and height in ways consistent with what the animation was representing. I asked Lisa to compare her sketched graph to the dynamic trace. She said, "it's not two linear functions. It is one line and they can relate," so I interpret that she conceived of a single "line" representing two quantities changing together. Because Lisa said that "one line can relate" both distance and height, I interpret that she conceived of a graph as what Thompson and Carlson (2017) called a *multiplicative object*.

Lisa provided more evidence that she conceived of both distance and height as quantities that changed together. She sketched a graph (see Figure 32) to further explain what the trace meant. She put numbers along the axes to demonstrate that she conceived of distance and height as possible to measure. She moved her pen along different parts of a graph to show that distance increased, and height increased and then decreased. I interpret that she engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson & Carlson, 2017), because she conceived of distance and height as quantities that changed together such that as the distance increased, the height increased and decreased.

Lisa provided evidence that to conceive of function notation, she was beginning to shift from *function notation as convention* level. I did not ask her about function notation, but she wrote *f(h)=* while she was labeling axes of her sketched graph (see Figure 32). When she underlined the label '*y*' along the vertical axis and said, "this is independent variable, so this would be function of height equals," I interpret that Lisa provided evidence that she was beginning to shift her conception that the horizontal axis always represented the independent variable. Because Lisa did not write a complete function notation, I interpret that she did not just want to switch *d* an *h* in function notation. Later, when I asked her if distance and height changed differently than what she had last week, she said that it may have "meant the same thing", but she misunderstood it last week. At the same time, she darkened the *h* in *f(h)=*, which meant that she conceived of function notation, and my conjecture was that she was not sure if she could put *d* in a function notation yet. Lisa also said, "this forces me to think differently," so my interpretation is that she was beginning to conceive of function notation at a *function notation as a relationship between variables* level.

Ferris wheel Interview 2, task 3: Function notation as convention to function notation as a relationship between variables. In this task, Lisa provided evidence that she first conceived of function notation by associating numbers and conceived of function notation at a *function notation as convention* level. Later, she conceived of function notation at a *function notation as a relationship between variables* level.

Lisa associating numbers and conceiving of function notation as convention to interpret h=f(d). In the excerpt given below, Lisa provided evidence that she could translate *h=f(d)* into words and associated numbers to interpret *h=f(d)*.

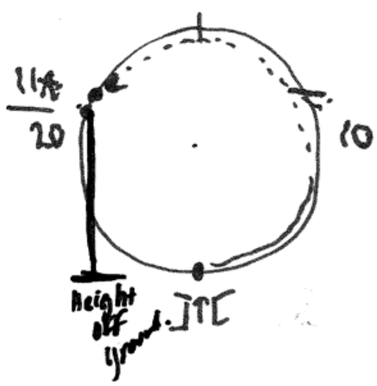


Figure 33: Lisa's work related to function notation

Excerpt 33: Lisa Ferris wheel Interview 2 task 3

Azeem: Now, I'll ask you to write a letter or a symbol for this relationship between distance and height.

Lisa: (wrote *f(h)*= and wrote the word 'distance' and wrote (*y*) below the word 'distance'). I don't know how to relate them symbolically or algebraically speaking, I don't know how to relate them.

Azeem: Ok. How about *h* equals *f* of *d* (*h=f(d)*)? Write here (handed a new sheet of paper) *h* equal *f* of *d* (*h=f(d)*).

Lisa: (wrote *h=f(d)*).

Azeem: What does that mean *h* equals *f* of *d* (*h=f(d)*)?

Lisa: Height off the ground equals the function of the distance traveled around the circumference.

Azeem: Ok.

Lisa: Is that true though? That does not seem true.

Azeem: Why it doesn't seem true?

Lisa: Because height off the ground equals a function of the distance. Well, yes it does.

Azeem: Why?

Lisa: From this same thing (looking at Figure 33), so the height off the ground if you are able to determine you are 11ft off the ground here, then you can determine you traveled either 10ft around circumference or 20 feet around the circumference. I mean I don't know the numbers.

Azeem: Yes, ok.

Lisa: So, say you are 11ft off the ground and you have traveled 20ft around the circumference and then you came over here and you are 11 feet off the ground then you can determine. Then you can also say you have traveled 30ft around the circumference.

Azeem: So, then we can say, or we cannot say.

Lisa: You can say that. I just do not know how.

Azeem: You can't say what? Would you explain what you can say?

Lisa: You can say that you can determine your height off the ground (started writing a statement above *h=f(d)*). But I don't know how to say it algebraically. I guess that is it (pointed to *h=f(d)* in figure 34].

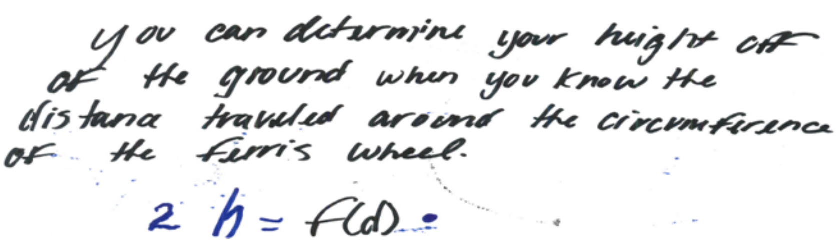


Figure 34: Lisa expressing *h=f(d)* in words

Lisa provided evidence of translating symbols (see Figure 34) to words and also provided evidence that she associated numbers to interpret function notation *h=f(d)*. I conjectured that she conceived of function notation at a *function notation as convention* level. When I asked her what *h=f(d)* meant, she explained in words that distance equaled the function of distance, but she also said, "that does not seem true." My conjecture was that she might be conceiving of *h=f(d)* where the variable *d* did not match with the variable *h* along the horizontal axis, so she said that *h=f(d)* was not true. When I asked her to explain why *h=f(d)* was not true, she showed with numbers (see Figure 33) that given the same height, she could determine two different distances. My conjecture was that she might be letting *h* as an input and *d* as an output without using the words input and output, but I did not have enough evidence at this point. So, I conjecture that she conceived of function notation at a *function notation as convention* level. Lisa also said that she did not know how to relate distance and height "algebraically," so I interpret that she wanted to see a formula. She wrote a statement (see Figure 34) to show that she could translate function notation $h=f(d)$ to words and also demonstrated that it was difficult for her to interpret a functional relationship between *d* and *h* and she wanted to see a formula instead of $h=f(d)$.

Lisa associating numbers and conceiving of function notation d=f(h) as convention. The excerpt given below came right after what I presented above. In the excerpt below, Lisa provided evidence that she conceived of function notation $d=f(h)$ at a *function notation as convention* level. She also associated numbers and provided evidence that it was difficult for her to decide if she could say $d=f(h)$ "algebraically" just like she said in the previous excerpt.

Excerpt 34: Lisa Ferris wheel Interview 2 task 3

Azeem:	Ok, alright. So, what if I asked you *d* equals *f* of *h* ($d=f(h)$).
Lisa:	Silent.
Azeem:	Just write *d* equals *f* of *h* ($d=f(h)$) and explain it to me what that means to you. I mean can we say that, we cannot say that, explain it.
Lisa:	(Wrote $d=f(h)$). (looking at her sketched graph (sketch after watching trace)). Yes, you can say both.
Azeem:	So, again maybe write down like with the picture show me what that means. Just do a picture over here and then try to explain.
Lisa:	Ok. [drew a circle next to $d=f(h)$]. So further you go around the circumference (putting marks along the circumference of a circle), so say you have traveled 5 feet around the circumference of the Ferris wheel, you can say that you are, I don't know 6 feet or however many feet off the ground. So, you can say, because you have traveled 11 feet along the circumference of the Ferris wheel, you are *x* amount of feet off of the ground, because if you had traveled 0 feet along the circumference of the Ferris wheel, you would be 0 ft off the ground. So, if you had traveled 6 feet along the circumference of the Ferris wheel, then you can say that you are *x* amount of feet off ground. If you had traveled *x* amount of distance around the circumference then you could say that you are *x* feet off ground. You could even I just don't know how to express it algebraically correctly. So, then

you could say that you traveled over here (put *x* to the right side of max) *x* amount of feet around the circumference of the circle, you are *x* amount of feet off the ground. Yes, so you can say both, I think. I believe, it's making sense to me to say both. I just don't know how to say it algebraically unless that's exactly it (pointing to *d=f(h)*).

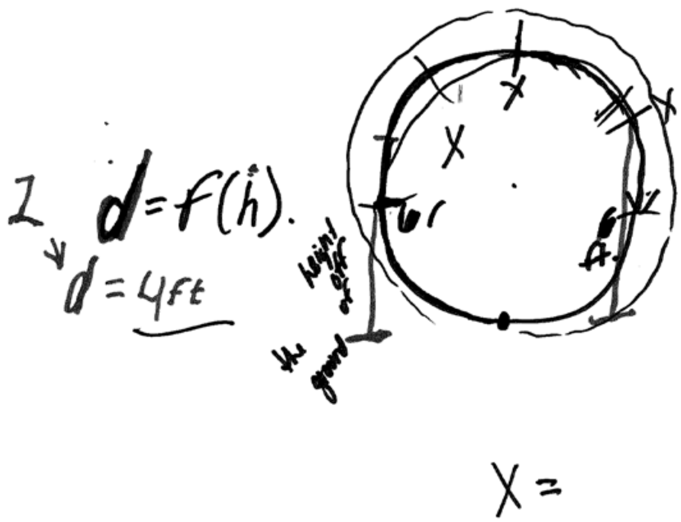


Figure 35: Lisa exploring relationships between *d* and *h*

Lisa provided evidence that she conceived of function notation *d=f(h)* at a *function notation as convention* level. She wrote *d=f(h)* and looked at her sketched graph (see Figure 32), so I interpret that she matched the horizontal axis label *h* to variable *h* in *d=f(h),* because it is a convention to express the independent variable in function notation along the horizontal axis. When I asked her to explain, Lisa showed with numbers and letters (see Figure 35) that if she knew the height, she could determine two different distances without using the words inputs or outputs, so my conjecture was that she might be conceiving of *h* as an input giving two outputs *d*, but I did not have enough evidence at this point. Just like in the previous excerpt, she again said that she did not know how to relate distance and height "algebraically," so I interpret that she wanted to see a formula. Lisa demonstrated that she conceived of function notation *d=f(h)* at a *function notation as convention* level and wrote *d=f(h)*. It was difficult for her to interpret relationships between *d* and *h* and she wanted to see a formula again instead of function notation *d=f(h)*.

Lisa beginning to relate definition of a function (inputs/outputs) to d=f(h). In the excerpt below that came after what I presented above, Lisa provided evidence that she conceived of a graph failing the vertical line test and conceived of inputs/outputs, but it was not clear what *h* meant to Lisa when reasoning with function notation *d=f(h)*.

Excerpt 35: Lisa Ferris wheel Interview 2 task 3

Azeem:	So, you are saying if you know the distance, or if you know the height you can also (pause) know the distance.
Lisa:	What is confusing me is, because you are going to have two heights that are the same. You are not going to have two distances that are the same. Does that make sense.
Azeem:	Umhumm.
Lisa:	Like your distances are all going to be different (put a circle around her sketched circle) That's a linear thing, it's all (while sketching the circle around her sketched graph) and then your heights, you are going to have two heights that are the same because it's a circle.
Azeem:	Then could we express it like that *d* equals *f* of *h* (*d=f(h)*)?
Lisa:	(Silent).
Azeem:	So, what does *f* mean?
Lisa :	As a function of.
Azeem:	So, can we say that then?
Lisa:	Umm (whispered distance as a function of). No, not the way that it is written because there are two heights. Numerically speaking, if you were to plug it in to the variable, to the symbol that is representing set number, it can be represented twice and so. I know the vertical line test thing but that is where it would fail it, because it is running in to the, you cannot have two outputs (pointing to *h* in *d=f(h)*)

Lisa provided evidence that she was beginning to relate a function to function notation and was beginning to use the words inputs/outputs when reasoning with *d=f(h)*. First, she said that she could not say *d=f(h)*, "because there are two heights," so my conjecture was that she was beginning to conceive of inputs and outputs. Then she said that, "it would fail the vertical line test", so I interpret that she conceived of a graph. Then she pointed to *h* in *d=f(h)* and said, "you cannot have two outputs", so I interpret that she was trying to justify that a graph would fail the vertical line test. My conjecture was that she was trying to relate the graph, the vertical line test, and inputs/outputs. She provided evidence that she was trying to relate different conceptions of function and it was difficult for her to interpret *d=f(h)*. Because it was not clear how she conceived of *h* in *d=f(h)*, I investigated further as given in the next excerpt.

Lisa conceiving of d=f(h) as a relationship between variables. In the excerpt below that came right after the excerpt I presented above, I asked Lisa about *h* in *d=f(h)*. Because she pointed to *h* in *d=f(h)* when she said "you cannot have two outputs" in the previous excerpt, so I wanted to learn more about how Lisa conceived of inputs and outputs in function notation. She provided evidence of what *h* meant. She also provided evidence of a shift in her conception of function notation *d=f(h)* from saying that she could say *d=f(h)* to saying that she could not write *d=f(h)*.

Excerpt 36: Lisa Ferris wheel Interview 2 task 3

Azeem:	But is *h* your output?
Lisa:	Well, no [shaking her head as a no]. No, on this one (pointing to *h* in *d=f(h)*) it is your input, but I mean, if you put in then you would have two outputs; distance will be your output (wrote *d* below *d=f(h)*). So, if you had height here 6 feet, then you would have distance traveled 4 feet (wrote = 4 next to *d* below *d=f(h)*) and then if you put same input 6 feet, you would have to say 20 feet or whatever however many feet. So, you would have to have a different output here (drew an arrow pointing to *d*) and you will have two outputs (wrote '2' above the arrow, see Figure 35) for the same input (underlined 4 ft), because of height.
Azeem:	Oh, ok.
Lisa:	Because the distance only, the distance continues because you are continuing to go further and further around, so you are going to have two outputs for the same height, because it is a circle.
Azeem:	Hum, uhumm.
Lisa:	So, maybe you cannot say that *d* equals *f* of *h* (*d=f(h)*).

Lisa clarified to me what *h* meant and later provided evidence that she conceived of function notation *d=f(h)* at a *function notation as a relationship between variables* level. I interpret that she employed a correspondence approach because she used numbers to show that same height had two different distances. When I asked Lisa if *h* was the output, she pointed to *h* in *d=f(h)* again and said, "no." When Lisa said, "the distance continues because you are continuing to go further and further around, so you are going to have two outputs for the same height", I interpret that Lisa engaged in quantitative reasoning and employed a correspondence approach. After engaging in quantitative reasoning and employing a correspondence approach, she provided evidence of a shift in her reasoning with function notation *d=f(h)* from saying that she could say *d=f(h)* to saying that she could not write *d=f(h).*

Lisa conceiving of h=f(d) and d=f(h) as a relationship between variables. In the excerpt below that came right after the excerpt I presented above, Lisa conceived of function notations *h=f(d)* and *d=f(h)* at a *function notation as a relationship between variables* level.

Excerpt 37: Lisa Ferris wheel Interview 2 task 3

Lisa:	So, the *h* one (looking for the other paper)
Azeem:	So, you want to go back to the other one.
Lisa:	The *h* one.
Azeem:	So, the *h* one you said earlier, so you want to look at that (handed her the paper with *h=f(d)*)
Lisa:	Oh, yeah that's the one. This you could say, because you are not gonna have [umm]well maybe
Azeem:	(Handed another paper)
Lisa:	Let's see.
Azeem:	And you also said that there right (pointed to paper with *h=f(d)*). You said you can determine the height when you know the distance.
Lisa:	Uh, well, you can determine it, but it's like you are gonna have two of these heights (put 2 before *h* in *h=f(d)*) they are gonna be the same. No, this is the other one [looking at *h=f(d)*]
Azeem:	If you want to explore that a little bit right here you can write here [pointed to the blank sheet of paper].
Lisa:	Ok. Uhh. (wrote what's given in Figure 36)
Lisa:	So, the distance increases while the height will decrease because you are going back around or the further along you go. So, I don't know they seem different to me because if you put the distance around then you can put. This one is correct (pointing to *f(d)=h*). This one is not (pointing to *d=f(h)*).

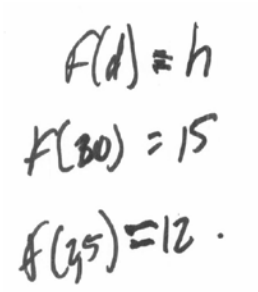


Figure 36: Lisa's work related to function notation *h=f(d)*

Lisa provided evidence that she engaged in covariational reasoning and employed a correspondence approach when interpreting function notations *h=f(d)* and *d=f(h)*. She said, "the distance increases while the height will decrease because you are going back around," so I interpret that she engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson & Carlson, 2017). She pointed to *f(d)=h* in Figure 36 and said that *f(d)=h* was the correct function notation. She also pointed to *d=f(h)* and said that *d=f(h)* was not correct. I interpret that she employed a correspondence approach because she substituted numbers to show that one input had one output and two inputs had one output which satisfied the definition of a function (see Figure 36). So, Lisa conceived of function notation at a *function notation as a relationship between variables* level and said that she could only write *h=f(d),* because *h=f(d)* represented a functional relationship between *d* and *h*.

Situation Tasks: Function Notation as Convention to a Relationship between Variables from the Pre Interview to the Post Interview

In this section, I present two tasks to show how Lisa conceived of function and function notation. I present a plane situation task to show that within the Pre interview, Lisa first conceived of function notation at a *function notation as convention* level and then conceived of function notation at a *function notation as a relationship between variables* level. Within the Post interview plane situation task she conceived of function notation at a *function notation as a* *relationship between variables* level. In the second task, she also provided evidence that she conceived of function notation at a *function notation as a relationship between variables* level. She worked on the second task only in the Post interview because it was one of the modified tasks. Lisa's engagement with these tasks provides evidence of how she conceived of quantities and how her quantitative reasoning and covariational reasoning impacted her conception of general function notation ($y=f(x)$).

Plane Situation in Pre interview: Covariational Reasoning

I gave Lisa a plane situation task such that as the plane covered distance along the ground, its altitude changed. When I asked her to interpret a graph, she drew on it (see Figure 37, left). She labeled the horizontal axis *x* and wrote "distance along the ground" below the horizontal axis (see Figure 37 left). She labeled the vertical axis *y* and wrote "altitude" along the vertical axis (see Figure 37 left). In the excerpt below, Lisa provided evidence that she engaged in covariational reasoning.

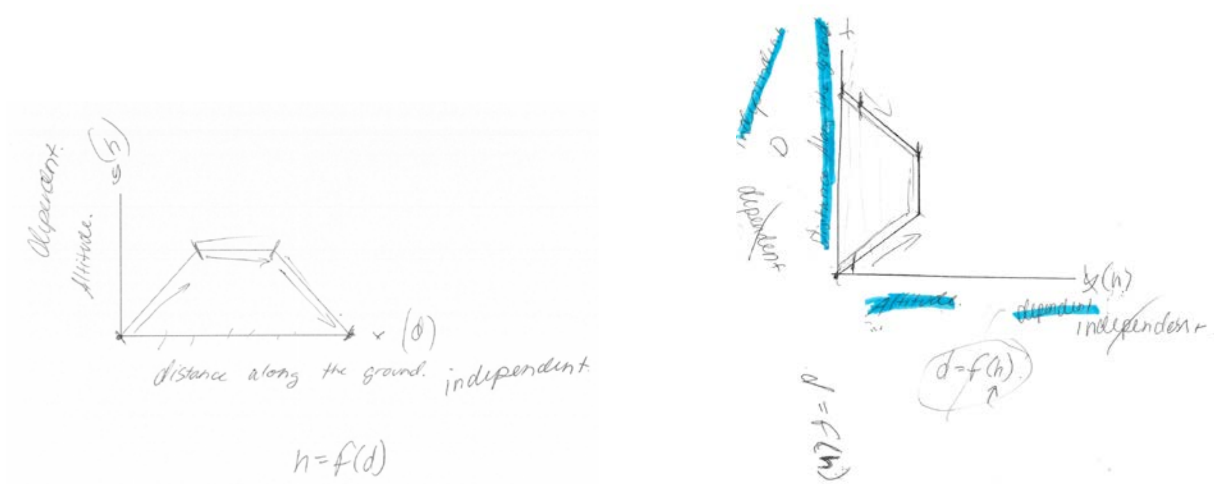


Figure 37: Lisa's annotation of the situation with distance on the horizontal axis (left) and distance on the vertical axis (right)

Excerpt 38: Lisa Pre interview

Lisa: Ok [labeled the horizontal axis *x* and wrote distance along the ground, labeled *y* along the vertical and wrote altitude]. So, what this graph is saying to me is this is, its altitude is increasing [drew a line along the graph that showed increase in altitude [see Figure 37, left] as its distance along the ground [drew tick marks along the *x*-axis] as it is essentially taking off [the plane], and this it would be in the air [sketched a line along the horizontal portion of Figure 37 left] and this would be when it lands [drew an arrow pointing towards the *x*-axis showing decrease in altitude, see Figure 37 left]. That's what this graph says to me because then it is distance along the ground is decreasing at these intervals, at this interval [to the right of the horizontal part] and increasing at this interval [left part up to where it is horizontal] as it takes off. And then this [drew a line above the horizontal part, Figure 37 left] it's going to in theory, remain at the same altitude while it is in flight depending on the turbulence or what not, birds perhaps.

Reasoning with quantities. Lisa provided evidence that she conceived of distance and altitude as quantities that changed together. She drew tick marks along the horizontal axis and drew arrows along different parts of a graph (see Figure 37 left) to show that she conceived of distance and altitude as possible to measure. She said that "as it's distance along the ground," the altitude increased, remained the same, and then decreased. I interpret that she engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level because she said that as distance increased, the altitude increased, stayed the same, and then decreased.

Reasoning with quantities with attributes on different axes: Quantitative reasoning and covariational reasoning. Then I asked Lisa to have the same attributes represented on different axes. I asked her to interpret a graph and she drew on it (see Figure 37 right). In the excerpt below, Lisa demonstrated that she conceived of a graph (see Figure 37 right) representing the same relationship between distance and altitude. She also demonstrated that she engaged in covariational reasoning.

Excerpt 39: Lisa Pre interview

Lisa: [Pause]. It's making me view functions differently that's for sure. Well, it's weird because if you turn it this way [turned the paper with Figure 37 right counterclockwise], it does not fail the vertical line test [moved her pen over the graph vertically], but if you turn it this way [Figure 37 right and moving the pen over the graph vertically] it does. But, then what's weird is that essentially what the graph is doing it's describing a relationship between two different bits of information and it is describing the same amount of information. It is describing the exact same information really. It's just a different perspective. So, algebraically speaking how would you handle this? But, that would be my thought process in that.

Azeem: But, if you were to explain what was going on between distance and altitude, could you explain that relationship like you did in the previous case.

Lisa: Yes, it just would look a little bit different. So, I would measure it probably differently, but it would be the same numbers. So, maybe this is on the flight back [darkens the starting point or the origin in Figure 37, right, then went to the top showing an increase in altitude]. It is taking off from I don't know San Diego. So, then you are going and the distance off the ground, the altitude is increasing, then you are in flight and then you are descending. But, because you have to interpret the graph from left to right, so if you are interpreting these from left to right, this is decreasing [started from the top point up to the horizontal part] and then this is increasing [drew an arrow going from Figure 37, right origin up to the horizontal part]. It's the same information really.

Lisa provided evidence of a shift in her reasoning with function such that the vertical axis could represent the independent variable and conceived of a graph (see Figure 37 right) describing the same information. She said that turning a graph shown in Figure 37 on the right counterclockwise passed the vertical line test and she also said, "it is describing the exact same information," so I interpret that she engaged in quantitative reasoning and demonstrated flexibility in her reasoning that the vertical axis could also represent the independent variable.

Lisa provided evidence that she conceived of distance and altitude as quantities that changed together. She drew arrows to show that she conceived of distance and altitude as quantities. She also said, "then you are going and the distance off the ground, the altitude is increasing, then you are in flight and then you are descending," so I interpret that she engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level.

Function notation as convention to a relationship between variables. I asked Lisa about function notation right after she provided evidence that she conceived of distance and altitude as quantities that changed together. In the excerpt below, she provided evidence that to her, function notation *h=f(d)* meant more than just letters. She provided evidence of a shift in her conception of function notation *d=f(h)* from *function notation as convention* level to *function notation as a relationship between variables* level.

Excerpt 40: Lisa Pre interview

Azeem: Okay, Alright. So, now I will ask you this question: Is it possible to express these two situations in either both of those notations (pointed to *h=f(d)*, *d=f(h)* written on a paper), or one, or not at all.

Lisa: Both of these notations or not at all. So, height versus distance so [labeled axes]. So, height off the ground [pause, 9 secs] this would be height is a function of distance, [wrote *h=f(d)* under Figure 37 left]. This would be distance is a function of height [wrote *d=f(h)* under Figure 37 shown on the right]. But, this is super confusing to me because it is the same information and it is not a function if you turn it that way [pointed Figure 37 right with her pen], because there are multiple outputs but it's also, it's so weird. It's blowing my mind right now.

Azeem: Ok, so what does the variable *h* mean in here?

Lisa: Height or whatever altitude you are at. Well, and if you are describing this in like a life situation in terms of plane taking off in DIA, it also deals with space time, because you can have the same exact distance but at a different time and you can have the same altitude but at different times so it's describing a relationship that I don't even think mathematically I know how to do yet. So, basically what it is saying is when you know what your altitude is you know what your distance off the ground is going to be.

Azeem: So, do you think it's possible?

Lisa: Not in this instance, I don't think so [pointing to *d=f(h)* she wrote under fig.37 shown on the right]

Azeem: So, can we express it like *d=f(h)*?

Lisa: Well, no. No, I don't think so. No. It's (*d=f(h)*) the only one that makes sense in terms of the letters.

Azeem: Do the letters matter? Could they be anything or?

Lisa: They matter in terms of what they represent in terms of numbers. They matter in terms of symbols, but they don't matter in terms of what letter they are. It just matters in terms of like. To me relating it to information requires knowing language and this is unfamiliar, so it takes me a second so no. In the long run it doesn't. You can put whatever symbol there to represent a number.

Azeem: So, you said first graph can be represented as height as a function of distance (*h=f(d)*) Why?

Lisa: The distance off the ground is going to tell you what altitude you are.

Azeem: So, can Fig.37 left be expressed like *d* equals *f* of *h* (*d=f(h)*) or cannot?

Lisa: It's just so confusing to me. Well, yes [moved Fig. 37 right counterclockwise, so it looks like Fig.37 left. Then changed the axes labels Fig 37 right, wrote 'dependent' along the vertical axis and labeled the horizontal axis 'independent']. So, if that's the dependent then it is a function of *h*. So, if that's the dependent then it is a function of *h* [wrote down *d=f(h)* by the side, Fig.37 right] It is a matter of turning it sideways. It can be, because your distance along the ground like the same here [now pointing to Fig.1], like your distance along the ground like that's you are gonna be driving. Your altitude is determined by your distance along the ground. So, [now pointing to Fig.37 right], your altitude is determined by the distance along the ground. It's just weird because it is distance and you can have the same distances, so it has to be a function of time.

Azeem: Why do you think it has to be a function of time?

Lisa: I mean for altitude, you are going to be at the same altitude like if you are talking linear [put 2 marks along a graph, fig. 37 right] it is going to be linear, you are going to be the same altitude but it is going to be at two different times and it is going to be the same here [pointed to fig.37 left] but like this is how it is showing up for me in my brain. So, it seems that because *d* is the dependent as distance as a function of height, your altitude, or your distance along the ground like that is the dependent of your altitude, or your altitude is a dependent upon that. Let's see, I think so that's where I am getting confused. So, the altitude is the dependent [wrote 'dependent' along the vertical axis, fig 37 left] so these [crossed the word 'dependent' along the vertical axis, Fig.37 right and wrote 'independent'. Then crossed out the word 'independent' and wrote 'dependent' along the horizontal axis, Fig 37 right. Then wrote 'independent' along the horizontal axis, Fig.37 left]

Azeem: So, now can you briefly talk to me what you just did? [gave a highlighter] so mark it or something.

Lisa: Ok, so your distance along the ground [highlighted the words 'distance along the ground' along the vertical axis, fig.37 right] is the independent variable [highlighted the word 'independent' along the vertical axis, see fig.37 right], so that information is going to be what it is no matter what. The altitude [highlighted the word 'altitude' along the horizontal axis, Fig.37 right] is dependent upon your distance along the ground, so that's why I changed it to the dependent [highlighting the word "dependent" along horizontal axis, Fig.37 right], because your altitude will change depending on where you are along the ground.

Lisa's Interpretation of function notation *h=f(d)*. Lisa provided evidence that function notation *h=f(d)* meant more than letters to her. She first stated that Figure 37 left could be written as "height is a function of distance" and then wrote *h=f(d).* Later in the interview, I asked her why she chose *h=f(d)* for figure 37 left. She said that distance determined the altitude. Because Lisa said that distance determined the altitude, I interpret that she explained what symbols meant to her in words.

Lisa's conception of function notation *d=f(h)* from convention to a relationship between variables. Lisa provided evidence that she conceived of function notation at a *function* *notation as convention* level. Lisa first wrote *d=f(h)* under Figure 37 shown on the right and then said that she could not write *d=f(h)*, "because there are multiple outputs" and she also said, " its (*d=f(h)*) is the only one that makes sense in term of letters," so I interpret that she followed convention of matching the variable within the parentheses in function notation (*h*) to the variable along the horizontal axis (*h*), and that's why she said that *d=f(h)* was not possible because *h* as an input had two outputs. In other words, she conceived of function notation at a *function notation as convention* level.

Lisa provided evidence of a shift from *function notation as convention* level to *function notation as a relationship between variables* level. When I asked her if figure 37 on the left could be written as *d=f(h)*, she said that it was confusing. She said, "it is a matter of turning it sideways" and then pointed to both graphs and said, "your altitude is determined by the distance along the ground." I interpret that she conceived of both graphs representing the same information, because she pointed to both graphs and said that distance determined the altitude. She crossed out old labels and wrote new labels (see Fig 37 left and right) and when I asked her what she did, she highlighted words along the axes and said, "your distance along the ground is the independent variable so that information is going to be what it is no matter what." I interpret that she changed labels of axes and then highlighted new ones to show that distance was the independent variable in both graphs just represented on different axes. Because of what Lisa said and because she highlighted the words "distance along the ground" and "independent" along the vertical axis (see figure 37 on the right), Lisa engaged in quantitative reasoning and had opportunities to *break convention* (Moore et al., 2013; Moore et al., 2014) such that quantity representing the independent variable should always be on the horizontal axis. Lisa provided evidence of a shift in her conception of function that independent variable could be represented along the vertical axis.

Lisa did not explicitly write figure 37 on the right as *h=f(d)* in symbols, however, she provided evidence of what function notation *h=f(d)* meant to her in words. Looking retrospectively on the data, I should have asked Lisa if she could write a graph shown in Figure 37 on the right as *h=f(d)* to have a stronger evidence. After highlighting the labels, Lisa said, "your altitude will change depending on where you are along the ground," so I interpret that with this statement, Lisa provided evidence that she conceived of both graphs representing altitude as a function of distance. Therefore, Lisa conceived of function notation at a *function notation as a relationship between variables* level.

Plane Situation in Post interview: Covariational Reasoning

I gave Lisa a plane situation task such that as the plane covered the distance along the ground, its altitude changed. This was the same task that Lisa worked on in the Pre interview. I asked her to label the axes and interpret a graph. She drew on a graph (see Fig.38 left). I include an excerpt below because she provided evidence of engaging in covariational reasoning.

Figure 38: Lisa's annotation of the plane situation with distance on the horizontal axis (left) and distance on the vertical axis (right)

Excerpt 41: Lisa Post interview

Lisa: Okay, so as the plane covers the distance along the ground, its altitude changes so it is starting at 0 feet off the ground it looks like and then it is going up. So, it is moving forward is what is happening, distance along the ground (drew tick marks and wrote numbers along the horizontal axis). So, this is the altitude that's increasing right here (drew ticks along the vertical axis). Um, it is actually kind of an incorrect statement because you are not going along the ground when you are up in the air. So, you are having one input (put dots over different parts of a graph). Because you have one input as distance along the ground, you are going to have two different outputs. Because when you are—you land in it does not tell you, in San Diego, then you are going to decrease your altitude (drew arrow along the decreasing part of a graph and wrote 'Altitude') but you are going to be a completely different area, so you are going to have a completely different input.

Azeem: So, like in terms of the plane and the distance along the ground, and the height or the altitude, from here to here (pointed to the increasing part of graph), what's happening?

Lisa: It's taking off (wrote 'taking off'). So, it's increasing in altitude (drew an arrow along the increasing part of the graph), has to get up in the air, it is flying to San Diego in a high (drew a horizontal line under the horizontal part of the graph), and then they get to a specific point, where they are like, oh we got to land, and then so you decrease.

Lisa conceived of distance and altitude as quantities that changed together. She first labeled the horizontal axis as "distance along the ground" and the vertical axis as "altitude". Then she put numbers on the horizontal axis starting from 0 and going all the way up to 13. I interpret that by putting numbers along the horizontal axis, she showed that she conceived of distance as possible to measure. She said, "as the plane covers the distance along the ground, its altitude changes." I interpret that Lisa's reasoning was consistent with a level of covariation framework offered by Thompson and Carlson (2017) called *Gross Coordination of Values* because she said that as distance increased, the altitude increased, leveled off, and then decreased.

Reasoning with quantities on different axes. Next, I asked Lisa to have the same attributes on different axes and again she drew on the graph (see Figure 38 right). In the excerpt below, Lisa provided evidence that she conceived of label of axes related to quantities describing them.

Excerpt 42: Lisa Post interview

Azeem:	So, if you have the same attributes but on different axes what would you say?
Lisa:	This is what happened before. So, if it is put along the *y*-axis [labeled the vertical axis '*y*' Altitude and labeled the horizontal axis '*x*' and wrote distance along the ground] So, if you keep the axes the same it does not work, because it is not describing the distance along the ground.
Lisa:	Ok
Azeem:	No, so I did not want you to change them. I didn't say that it has to be on different axes now.
Lisa:	Ok. [wrote '*x*' over label *y* along the vertical axis, drew arrows to show altitude should be on the horizontal axis and distance along the ground should be on the vertical axis. Wrote '*y*' over label *x* along the horizontal axis]. So, if this is the *x*-axis. So, this
Azeem:	And also notice that, so you are changing the labels right
Lisa:	yes.
Azeem:	So, you are also changing the yes, this is now the altitude and that is the distance. So, you change that with a different marker if you want to rewrite it (referring to the words altitude and distance)

Lisa provided evidence that she conceived of the variable and the quantity representing the independent variable to be represented along the vertical axis. When Lisa said, "if you keep the axes the same it does not work, because it is not describing the distance along the ground," I thought that when I said, "same attributes but on different axes," Lisa interpreted my statement as if she could not represent distance along the ground on the vertical axis. That is why I said, "I did not want you to change them. I didn't say that it has to be on different axes now." Lisa then interpreted what she wanted to interpret. She denoted the vertical axis by letter '*x*' and the horizontal axis by letter '*y*'. She also drew arrows to show that distance should be along the vertical axis and altitude should be along the horizontal axis, so I interpret that she shifted from following convention that the independent variable and the quantity representing it are always along the horizontal axis to a shift that the horizontal axis could represent the dependent variable and the quantity representing the dependent variable.

Variational and covariational reasoning. Right after Lisa labeled her axes, I asked her how distance and height changed. In the excerpt below, Lisa provided evidence that she engaged in variational reasoning and covariational reasoning.

Excerpt 43: Lisa Post interview

Azeem:	So, what is happening to distance now?
Lisa:	So, now you are flying back.
Azeem:	No, just the distance. What is happening to the distance?
Lisa:	It is increasing.
Azeem:	So, what is happening to the altitude?
Lisa:	It's increasing, staying the same and then decreasing.
Azeem:	Okay.
Lisa:	You are going back but you can't go negative feet because you are traveling so say you are flying back. You have to take off the plane or the plane has to take off and then you are in the air and then you are landing back at DIA. So, your distance has decreased. That won't ever decrease because you can't go negative feet. But your altitude can equal zero and your distance traveled can only equal zero when you first take off because after that. That's why it will continue to travel up and this would flip back from like this (pointed to Figure 38 left).

Lisa provided evidence of variational reasoning. When I asked Lisa how distance changed, she said that distance increased. When I asked her about altitude, she said, "it's increasing, staying the same and then decreasing." I interpret that she engaged in variational reasoning because she talked about distance first, then altitude. Her reasoning was consistent with a lower level of variation framework offered by Thompson and Carlson (2017) called *gross variation*, because she said that distance increased and then altitude increased, leveled off, and then decreased independently.

Lisa demonstrated that she conceived of distance and altitude as quantities that changed together. She said, "you are in the air and then you are landing back at DIA. So, your distance has decreased. That won't ever decrease because you can't go negative feet." I interpret that she engaged in covariational reasoning at a level called *Gross Coordination of Values* (Thompson &

Carlson, 2017) because she said that as the plane took off, stayed "in the air", and landed (changes in altitude), then the distance continued to increase.

Function notation as a relationship between variables. I asked Lisa if it was possible to express both graphs as $a=f(d)$ or $d=f(a)$ (see figure 38 left and right). She mixed variables and function notation when interpreting function notation (see Figure 39). I include an excerpt below to provide evidence that Lisa engaged in covariational reasoning and employed a correspondence approach to interpret function notation.

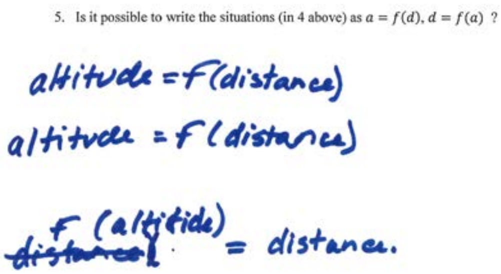


Figure 39: Variables mean more than letters to Lisa

Excerpt 44: Lisa Post interview

Azeem: Is it possible to write those situations as *a* equals *f* of *d* ($a=f(d)$), or *d* equals *f* of *a* ($d=f(a)$)?

Lisa: Um this one, altitude is a function of distance [writes altitude= *f*(distance), see Fig 39]

Lisa: Um, altitude is a function of distance, yes, because you can put exactly one distance, you are just like I said you cannot go negative distance so you are continuing to accumulate feet further and further you go, so altitude can absolutely be written as a function of distance [wrote altitude =*f*(distance) again, see fig. 39]. If you have gone 2300 ft., like you can probably find the altitude in which you are or whatever. Um, Distance as a function of altitude [wrote *f*(altitude)=distance, see fig.39]. No, especially if you are talking about-if your altitude is right here, and so you have a continued constant right here, you cannot determine from this number how far you have traveled based on the altitude. As a function of altitude, you would not be able to determine your distance.

Azeem: Reason why?

Lisa: Because if you put in let's say you were 4000ft or 40000 ft., I don't know how far they travel, but if you put in 40,000 ft., you would not be able to determine your distance based on 40,000ft. The fact that you were 40,000 ft. in the air will not tell you the distance. It would give you different numbers because you are traveling along but you have traveled 40,000ft, you are still 40,000 ft. It's the same number that will determine the different outputs. It just wouldn't make any sense. It does not make any sense.

Azeem: Is it for both situations or like which ones were you talking about right here? Like for situation a or situation b?

Lisa: For as a function of altitude, you could determine the distance along the ground.

Azeem: In both cases.

Lisa: No. Well I don't (shook her head as a no). I mean your distance along the ground will tell you what your altitude is, because if you are 2300ft in the air or I mean along the ground, then it's going to tell you where you are even if you left San Diego and your distance before leaving was 1600 because that's how far it is to San Diego, then it would tell how far you are off the ground you could be or altitude wise you could be 40,000ft in the air determining if you have traveled 2300 miles, you should be 40,000 ft. in the air.

Lisa first interpreted function notations *a=f(d)* and *d=f(a)* by engaging in covariational reasoning and then employed a correspondence approach to justify them. She wrote altitude =*f*(distance), (see fig.39) and said, "um altitude is a function of distance, yes, because you can put exactly one distance….you are continuing to accumulate feet further and further you go," I interpret that she conceived of distance and height as quantities and she meant that one distance input could give one output. She wrote *f*(altitude)=distance (see fig.39) and said, "Um, Distance as a function of altitude. No." She further stated, "you cannot determine from this number how far you have traveled based on the altitude. As a function of altitude, you would not be able to determine your distance." I was not sure why she said it was not possible to write distance equals function of altitude, so I asked her why. Then she said, "The fact that you were 40,000 ft. in the air will not tell you the distance… It's the same number that will determine the different outputs," I interpret that Lisa meant that one altitude input (40,000 ft) could give different distance outputs which was not consistent with the definition of function (same input cannot have different outputs) and therefore distance as a function of altitude (*d=f(a)*) was not possible. To Lisa, function notation *h=f(d)* was something more than what Musgrave and Thompson (2014) term idiomatic expression. For Lisa, function notation expressed a relationship between two quantities such that the distance determined the altitude, so she conceived of function notation at a *function notation as a relationship between variables* level.

Lisa did not explicitly write which graph (one or both) could be written as *a=f(d)*, however, she provided evidence of what function notation *a=f(d)* meant to her. Earlier in the Post interview (see Excerpt 42), Lisa provided evidence that an independent variable could be along the vertical axis and in this excerpt, she engaged in covariational reasoning and employed a correspondence approach to show that *a=f(d)* worked, so combining the two forms of reasoning, I can say that Lisa conceived of both graphs expressed as *a=f(d).*

Task from the Post Interview: Function Notation as a Relationship between Variables

To learn more about how Lisa conceived of a function notation, I asked her to interpret a response from a student named Chris, who said that a graph given below which Lisa annotated (see Figure 40) could be written as both *a=f(d)* and *d=f(a).* This is one of the modified tasks that Lisa worked on only in the Post interview, so I could not compare her reasoning from the Pre interview to the Post interview. However, in the excerpt below, she provided clear evidence that a graph (see Figure 40) could only be expressed as *a=f(d).*

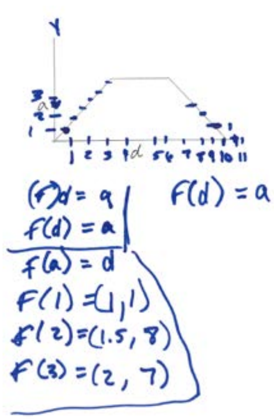


Figure 40: Lisa providing evidence of employing a correspondence approach

Excerpt 45: Lisa Post interview

Azeem: Okay, alright. So here Chris said that for the first situation, it could be written as both *a* equals *f* of *d* ($a=f(d)$) and *d* equals *f* of *a* ($d=f(a)$). So, what would you say?

Lisa: I don't know. It's like the same as what the other guy said but umm, because if you put one output for distance [drew ticks along the axes and wrote numbers], you will have two different instances where the altitude is 1 or 100 or 1000 whatever. It does not make sense because you are trying to determine your distance along the ground like the altitude of it. So, timing wise, if you are like 10,000 miles along the ground or far wise, then you should be able to say they are landing in San Diego their altitude is one, but that is the same output. So, if you put in you will get two different outputs for that altitude [substituted numbers in $f(a)=d$]. I don't know. So, if you put in as a function of altitude, the distance along the ground you are going to get two different *y*-values.

Azeem: So, what did you say for the other one? (pointed to $a=f(d)$)

Lisa: So, when you put in as a function of distance, you are only going to get one *y*-value.

Azeem: Okay, so you are saying which one is okay then?

Lisa: The distance.

Azeem: So, can't say that (pointed to $d=f(a)$).

Lisa: No, because of this (pointed to Figure 40)

Lisa provided evidence that she conceived of function notations $a=f(d)$ and $d=f(a)$ at a *function notation as a relationship between variables* level, because she conceived of distance and height as quantities and then employed a correspondence approach. She put numbers along the horizontal axis and the vertical axis. With numbers, Lisa demonstrated that she conceived of distance and height as possible to measure. In other words, she conceived of distance and height as quantities. Then she substituted numbers in function notation $f(a)=d$ (see Figure 40) and said, "if you put in as a function of altitude, the distance along the ground you are going to get two different *y*-values," so I interpret that she employed a correspondence approach to show that a graph could not be expressed as $f(a)=d$. When I asked her about function notation $a=f(d)$, she said, "when you put in as a function of distance, you are only going to get one *y*-value," so I interpret that she employed a correspondence approach again to show that function notation $a=f(d)$ worked. Lisa conceived of function notation at a *function notation as a relationship* *between variables* level, where she engaged in quantitative reasoning and then employed a correspondence approach.

Function and Notation in Tasks Involving Functions, Graphs, Tables, and Function Rules

In this section, I present another modified task from the Post interview where Lisa interpreted a response from another student regarding general function notation (*y=f(x)*) for a linear graph. Lisa worked on this task only in the Post interview because she did not work on modified versions in other interviews (see Table 8, Chapter 4). In this task, Lisa engaged in quantitative reasoning and conceived of function notation at a *function notation as a relationship between variables* level.

Task from the Post interview: Function notation as a relationship between variables. To learn more about how Lisa conceived of function notation, I asked her to interpret a response from another student named Max who said that both *m=s(r)* and *r=s(m)* could be used to describe a linear graph. I gave her a graph which she drew on (see Figure 41). I include an excerpt below to provide evidence that for a linear graph, Lisa conceived of function notation at a *function notation as a relationship between variables* level. She also provided evidence that she conceived of function notation at a *function notation as convention* level for graphs representing onto functions.

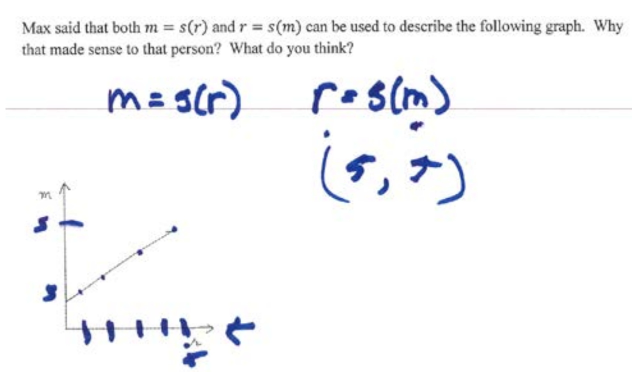


Figure 41: Lisa's annotation of the linear graph

Excerpt 46: Lisa Post interview

Azeem:	Ok. So, somebody, a student named Max said that you could express this graph [see. Fig.41] as both *m* equals *s* of *r* (*m=s(r)*) and *r* equals *s* of *m* (*r=s(m)*). Why do you think that made sense to that person and what do you think about it?
Lisa:	Because it's true.
Azeem:	Why?
Lisa:	Because it is a function. If it wasn't a function, I don't think that would be true but if you, so *m* equals *s* of *r* (wrote *m=s(r)*) so if *r* is your input, and you are going to put in 1, 2, 3, 4, 5 (sketching a graph) and you are going to get a specific output matching these like 3 and a half, you are going to get a specific number. And, then the same goes here. So, *r* equals as a function of *m* (wrote *r=s(m)*), so if you are going to put in a specific number for *m* (put a dot under *m* in *r=s(m)*), you are going to get a specific output (put a dot under *r* in *r=s(m)*) along the *r*-axis. That's why he said that. Um so, it makes complete sense to me. If it wasn't a function, if it was this or that [pointing to graphs 2 and 4], these would be incorrect. But they are not. That is a correct statement I believe because if you put *m* as a number so if you are matching let's say 5 (wrote *x*-coordinate 5), you are going to get this specific 5 (wrote *y*-coordinate 5) number along the *r*-axis as well and then so any number you pick on this line (moved her pen along the vertical axis), you are going to be able to determine what the *r*-axis is. Any number that you pick along the *r*-axis (pointed to *r* in *m=s(r)* she had written) like just say you pick -3 or 3, you are gonna be able to determine along the *y* or the *m* axis (moved her pen along the vertical axis) what that number is. That's why that is a true statement.
Azeem:	So that is true in both cases?
Lisa:	Yeah.

Lisa provided evidence that she could think flexibly about function notation for a linear graph because she conceived of the graph representing a function. She used numbers to explain that letting any value of *m* as an input gave an output and letting *r* be any input value also gave one output, therefore, to her both function notations could describe a linear graph. In other words, either axis could represent the independent variable. She also moved her pen along the vertical axis and then pointed to the variable in function notation, which I take as evidence that Lisa conceived of *r* and *m* as quantities. I interpret that she engaged in quantitative reasoning and conceived of function notations *m=s(r)* and *r=s(m)* at a *function notation as a relationship between variables* level.

Lisa demonstrated that for a graph representing an onto function, she conceived of function notation at a *function notation as convention* level. She pointed to graphs 2 and 4 from the previous task (four set of graphs task) and said, “if it wasn’t a function, if it was this or that, these would be incorrect.” I interpret that she conceived of a graph representing an onto function such that the horizontal axis represented the independent variable, and therefore, she could not interchange the variables in function notation either. In other words, she provided evidence that for a graph representing an onto function, she followed the convention such that the horizontal axis represented the independent variable.

Summary

Lisa provided evidence that she engaged in variational reasoning, quantitative reasoning, and covariational reasoning. Within the Pre interview, she first conceived of function notation at a *function notation as convention* level and then conceived of function notation at a *function notation as a relationship between variables* level. Within Ferris wheel interview 1, she conceived of distance and height measuring the same thing-a length from the ground, which impacted her conception of function notation. She demonstrated that she conceived of function notation at a *function notation as a relationship between variables* level within Ferris wheel interview 2 and the Post interview. Next, I briefly summarize how Lisa’s case answers my research questions.

How Might Students’ Conceptions of Function Impact Their Conceptions of Function Notation?

Lisa’s conceptions of function remained consistent across the Pre interview to the Post interview. She conceived of a function as every input mapping to an output. She was consistent throughout the tasks in the Pre interview, Ferris wheel tasks, and the Post interview, that same input could not map to different outputs, but different inputs could map to the same output.

Within the Pre interview, Lisa demonstrated a shift in her conception of function notation from *function notation as convention* level to conceiving of function notation at a *function notation as a relationship between variables* level. For example, in a plane situation task, she demonstrated that she could think flexibly about graphs such that quantity representing the independent variable could be represented by either the horizontal axis or the vertical axis.

Within Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview, Lisa conceived of function notation at a *function notation as a relationship between variables* level. She conceived of function notation using Thompson and Carlson's (2017) definition of function, which I refer to as a combination of quantitative reasoning and covariational reasoning and a correspondence approach. In other words, Lisa engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. She employed a correspondence approach to attend to the other part of the definition that the value of one quantity determined the value of the other quantity.

How Might Covariational Reasoning Related to Function Impact Students' Conceptions of Function Notation?

Lisa engaged in covariational reasoning in the Pre interview and provided evidence of a shift in her conception of function notation from *function notation as convention* level to conceiving of function notation at a *function notation as a relationship between variables* level. Within Ferris wheel interview 2, Lisa had a shift in her conception of distance from distance increasing and decreasing to increasing. After that shift, she provided evidence that she engaged in covariational reasoning throughout Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview. Within and across Ferris wheel interview 2 to the Post interview, she conceived of function notation at a *function notation as a relationship between variables* level.

How Do Students Conceive of a General Function Notation?

Lisa had some preferences regarding function notation. She always wrote $f(a)=d$ or $f(d)=a$ and mostly said function of altitude equals the distance across tasks, except when I said the other way like a equals f of d ($a=f(d)$). She also mixed words and function notation to show that both variables a and d meant more than letters to her. In other words, she provided evidence that function notation was more than what Musgrave and Thompson (2014) termed idiomatic expression.

Within the Pre interview plane situation task, Lisa first conceived of function notation at a *function notation as convention* level. Later, she provided evidence that she engaged in quantitative reasoning and covariational reasoning. In this task, Lisa did not clearly state if a graph shown in Figure 10 on the right could be expressed as $h=f(d)$, however, engaging in quantitative reasoning and covariational reasoning, she demonstrated that she conceived of an invariant relationship between quantities. So, she conceived of function notation at a *function notation as a relationship between variables* level.

In Ferris wheel interview 1, Lisa conceived of both distance and height as the same thing- a length from the ground and had difficulty conceiving of function notation. Within Ferris wheel interview 2, she had a shift in her conception of distance from distance increasing and decreasing to increasing. After that shift, she provided evidence that she engaged in covariational reasoning throughout Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview, which impacted her conception of function notation. Lisa conceived of function notation at a *function notation as a relationship between variables* level within Ferris wheel interview 2 and modified tasks in the Post interview. For example, in the Post interview, I asked her to interpret a response from a student named Chris, who said that a graph (see Figure 13) could be written as both *a=f(d)* and *d=f(a)*. She clearly stated that a graph could only be expressed as *a=f(d)*, because she conceived of function notation at a *function notation as a relationship between variables* level.

Within Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview, Lisa conceived of function notation at a *function notation as a relationship between variables* level. Using Thompson and Carlson's (2017) definition of function, Lisa engaged in quantitative reasoning and covariational reasoning and demonstrated conceptions of an invariant relationship between quantities. She employed a correspondence approach to attend to the other part of the definition that the value of one quantity determined the value of the other quantity.

CHAPTER VIII

CROSS CASE ANALYSIS

In this chapter, I summarize Jack, Dave, and Lisa's reasoning and conceptions of function and function notation within and across interviews and also discuss contrasts across cases within and across interviews. Next, I present each research question and summarize how each case answers my research question as well as address similarities and differences in students' reasoning across cases.

Students' Reasoning and Conceptions of Function Notation Within Interviews

In this section, I elaborate on similarities and contrasts in students' reasoning and conceptions of function and function notation within the Pre interview, within Ferris wheel interview 1, within Ferris wheel interview 2 and within the Post interview.

Similarities across cases within the Pre interview. Within the Pre interview situation tasks, all three students provided evidence that they engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level. Jack, Dave, and Lisa conceived of function notation at a *function notation as convention* level in different tasks. For example, in a statement task, Jack engaged in covariational reasoning at *Gross Coordination of Values* (Thompson & Carlson, 2017) level and conceived of function notation as convention because he matched the variables in function notation to label the axes of his graph. He labeled the horizontal axis with the same variable that was given in parentheses in function notation and labeled the vertical axis with the variable that was on the other side of an equal sign. Similarly, in a graphs and rules task, he conceived of function notation at a *function notation as convention* level because he selected function notation based on the labels of axes. Lisa engaged in quantitative reasoning and covariational reasoning at *Gross Coordination of Values* (Thompson

& Carlson, 2017) level and first conceived of function notation as convention in the plane situation task. Dave also conceived of function notation at a *function notation as convention* level in a task of interpreting a response from a student named Sam regarding function notations for a linear graph.

Both Jack and Lisa had the same conception of function that one input could not have multiple outputs and multiple inputs could map to the same output. In a plane situation task, both Jack and Lisa provided evidence that they first conceived of function notation at a *function notation as convention* level and then shifted to *function notation as a relationship between variables* level.

Contrasts across cases within the Pre interview. There were contrasts across all three cases. Dave's conception of function was different than Jack and Lisa. To Dave, only one-to-one graphs represented a function. For example, in a task of interpreting a response from Nat regarding function notation for the plane situation, he conceived of function notation at a *function notation as label* level. Dave's conception of the definition of function impacted his conception of function notation, because he used function notation for graphs that represented functions or did not represent functions to him. Similarly, in contrast to Jack and Lisa who only provided evidence of conceiving of *function notation as convention* and *function notation as a relationship between variables* levels, Dave demonstrated that he conceived of function notation at a *function notation as label* and *function notation as convention* level in another task of interpreting a response from a student named Sam regarding function notations for a linear graph.

Within the Pre interview situation tasks, both Jack and Lisa conceived of function notation at a *function notation as a relationship between variables* level, yet there were some differences. Jack not only engaged in covariational reasoning, but also employed a correspondence approach to interpret function notation that remained consistent across Ferris wheel interview 1 to Ferris wheel interview 2. He also stated that a graph with attributes represented on different axes (see Fig. 9 on the right) could also be represented as *h=f(d)*. On the other hand, Lisa did not explicitly state if a graph with attributes represented on different axes could be expressed as *h=f(d)*. A possible reason for that is that I did not ask her if she could write a graph shown in Figure 9 on the right as *h=f(d)*. If I had asked her, I could have gathered stronger evidence regarding her reasoning with *h=f(d)* for a graph shown in Figure 9 on the right. In contrast to Jack and Lisa, in a swing situation task, Dave conceived of function notation at a *function notation as label* level because both shape of a graph and his conception of function impacted his conception of function notation. In addition, Dave demonstrated that he engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017), yet his conception of function notation was separate from his covariational reasoning.

Similarities across cases Within Ferris wheel interview 1. Both Dave and Lisa conceived of distance and height measuring the same thing- a length from the ground. Both Dave and Lisa's initial conception of distance as a length from a ground can be explained by what Bell and Janvier (1981) term situational distraction. Bell and Janvier (1981) explained that situational distractions occur when the student's experience of the situation interferes with his/her ability to attend to the meanings of the features of the graphs. I interpret that both Dave and Lisa experienced a situational distraction because they conceived of the Ferris wheel as a circle and to them both distance and height increased and then decreased. In particular, Lisa provided evidence that she interpreted her sketched graph as a representation of both distance and height increasing up to a point and then both distance and height decreasing.

Within Ferris wheel interview 1, in a modified task, both Jack and Dave conceived of function notation at a *function notation as a relationship between variables* level, where Jack engaged in quantitative reasoning and variational reasoning and Dave engaged in covariational reasoning.

Contrasts across cases Within Ferris wheel interview 1. Lisa's case contrasts with Dave and Jack because she provided evidence that it was difficult for her to interpret function notation because she conceived of distance and height measuring the same thing- a length from the ground. On the other hand, Jack and Dave conceived of function notation as a relationship between variables. Dave shifted in his conception of distance first and then provided evidence of engaging in covariational reasoning and conceived of function notation at a *function notation as a relationship between variables* level.

Similarities across cases Within Ferris wheel interview 2. Jack, Dave, and Lisa conceived of function notation at a *function notation as a relationship between variables* level and engaged in different forms of reasoning. Both Jack and Lisa engaged in quantitative reasoning and covariational reasoning, and Dave engaged in variational reasoning, covariational reasoning and quantitative reasoning in different tasks within Ferris wheel interview 2.

Contrasts across cases Within Ferris wheel interview 2. There were contrasts in the cases of Dave and Lisa. Dave conceived of function notation at a *function notation as a relationship between variables* level throughout Ferris wheel interview 2. However, Lisa shifted in her conception of distance which impacted her conception of function notation. She had formed strong conceptions of distance and height that were difficult to change. Within Ferris wheel interview 2, she reflected on her own activity and as a result shifted in her conception of distance. A possible reason for adopting that distance was the total distance traveled and that the distance increased was because it was difficult for her to express the relationship between distance and height in a graph and function notation with her current conception. Within Ferris wheel interview 2, I learned more about how she conceived of symbols in *h=f(d)*. She first wanted to see a formula when interpreting function notations *h=f(d)* and *d=f(h)*. Lisa's initial interpretation was consistent with what other researchers found: that students can think a function must be defined by a single algebraic formula (e.g., Breidenbach et al., 1992; Carlson, 1998; Clement, 2001; Even, 1990; Even 1993; Sierpinska, 1992). Later, Lisa provided evidence that she engaged in covariational reasoning and employed a correspondence approach to interpret function notations *h=f(d)* and *d=f(h)*, so she conceived of function notation at a *function notation as a relationship between variables* level.

Similarities across cases Within the Post interview. In a plane situation task, both Jack and Lisa conceived of function as one input mapping to one output and different inputs mapping to the same output. In the plane situation task, both Jack and Lisa conceived of function notation at a *function notation as a relationship between variables* level where Lisa engaged in covariational reasoning and employed a correspondence approach and Jack engaged in quantitative reasoning and employed a correspondence approach.

Contrasts across cases Within the Post interview. Within the Post interview, in contrast to Dave and Lisa, Jack was the only student who shifted from conceiving of function notation as convention to *function notation as a relationship between variables* in a graphs and rules task. For graph 4 Jack provided evidence of a shift in his conception of function notation. He checked two function notations, but nothing more than that based on what he said in this particular task. Comparing his work in this task to his conception of function and function notation within situation tasks and the Ferris wheel tasks with attributes switched, I can say that he conceived of function notation at a *function notation as a relationship between variables* level. Jack's shift demonstrates the possibilities when students move beyond convention following, consistent with the recommendations of Moore and colleagues (Moore et al., 2013; Moore et al., 2014; Moore et al., in press).

There were contrasts among Dave's and Lisa's conceptions of function and function notation. Dave conceived of function notation at a *function notation as label* level throughout the Post interview, whereas Lisa conceived of function notation at a *function notation as a relationship between variables* level throughout the Post interview. For example, in a task of interpreting a response from a student named Max regarding function notations for a linear graph, Dave conceived of function notation at a *function notation as label* level because the shape of a graph was intertwined with his conception of function notation. On the other hand, Lisa worked on the same task and conceived of function notation at a *function notation as a relationship between* variables level. Lisa also demonstrated that she engaged in quantitative reasoning and employed a correspondence approach, whereas Dave only attended to the shape of a graph. In a situation task, there were differences in Dave's and Lisa's conceptions of function notation because they had different conceptions of function. Dave conceived of function such that different inputs could not map to the same output, whereas Lisa conceived of function such that different inputs could map to the same output. Within situation tasks, Lisa conceived of function notation at a *function notation as a relationship between variables* level where she engaged in covariational reasoning and employed a correspondence approach. However, Dave's conception of function notation was separate from his covariational reasoning, and he conceived of function notation at a *function notation as label* level where his conception of function and the shape of a graph impacted his conception of function notation.

Students' Reasoning and Conceptions of Function Notation Across Interviews

In this section, I elaborate on similarities and contrasts in students' reasoning and conceptions of function and function notation across the Pre interview to the Post interview and from Ferris wheel interviews to the Post interview.

Similarities across cases from the Pre interview to the Post interview. In a plane situation task, both Jack and Lisa conceived of function notation at a *function notation as a relationship between variables* level from the Pre interview to the Post interview.

Contrasts across cases from the Pre interview to the Post interview. Dave's case contrasts with Jack and Lisa because Dave's conception of function was different from Jack and Lisa. Both Jack and Lisa conceived of the definition of a function such that different inputs could map to the same output and same input could not have different outputs, whereas Dave only operated with the definition such that different inputs could not map to the same output. In other words, Dave conceived of a function satisfying a one-to-one characteristic.

In a graphs and rules task, Jack was the only student who provided evidence of a shift in his conception of function notation from conceiving of function notation as convention to conceiving of function notation at a *function notation as a relationship between variables* level. In the Pre interview, he selected function notation based on convention following so that function notation variables matched with the labels of a graph such that the variable on the left-hand side of an equal sign should be the dependent variable and the variable on the right-hand side within the parentheses should be the independent variable. But, in the Post interview, he checked both notations, for example, $s=h(t)$ and $t=h(s)$. Based on his previous work in situation tasks within the Pre interview and Ferris wheel tasks with attributes represented on different axes, I can say that within this task, he conceived of quantities that satisfied his conception of function such that one input could map to one output or 2 inputs could map to the same output.

Similarities across cases from Ferris wheel interviews to the Post interview. Within Ferris wheel interview 2, all three students conceived of function notation at a *function notation as a relationship between variables* level, where they provided evidence of engaging in quantitative reasoning and covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level and employed a correspondence approach. Both Jack and Lisa's conception of function notation remained consistent across Ferris wheel interview 2 to the Post interview.

Contrasts across cases from Ferris wheel interviews to the Post interview. Dave's conception of function notation contrasted with Jack and Lisa from Ferris wheel interviews to the Post interview. Both Jack and Lisa conceived of function notation at a *function notation as a relationship between variables* level, whereas, Dave went back to how he conceived of function notation in the Pre interview. Dave conceived of function notation at a *function notation as label* level because he drew on the shape of a graph. As I explained before that he was operating with different definitions of a function within Ferris wheel interviews than the Pre interview and the Post interview, so that could be the reason why a shift within Ferris wheel interviews did not stay consistent all the way to the Post interview. In Ferris wheel interviews, he operated with the definition of a function such that different inputs could map to the same output. Whereas, in the Post interview, he conceived of a function satisfying a one-to-one condition which also impacted his function notation. Dave's conception that a graph must satisfy a one-to-one property was consistent with findings from other researchers such as Dubinsky and Harel (1992) and Dreyfus and Vinner (1982) who stated that students conceived of *univalence* as equivalent to satisfying a one-to-one property.

In a swing situation task, the shape of a graph was intertwined with Dave's conception of function notation as I explained earlier in this section. My findings are similar to what Bell and Janvier (1981), Carlson (1998), and Moore and Thompson (2015) found that students often reason about graphs based on physical characteristics such as the shape of a graph, but the only difference is that Dave demonstrated that he extended the shape of a graph to his reasoning with function notation.

Research Questions Cross Case Analysis

In this section, I describe how my cases answered each of my research questions. I describe similarities across all three cases and then across two cases. Then I contrast across cases where I highlight unique aspects of each individual.

How Might Students' Conceptions of Function Impact Their Conceptions of Function Notation?

Similarities across cases. Within Ferris wheel interviews, all three students conceived of function notation at a *function notation as a relationship between variables* level because they had the same conception of function. Using Thompson and Carlson's (2017) definition of a function such that there was an invariant relationship between quantities and the value of a quantity determined the value of the other quantity, students engaging in quantitative reasoning and covariational reasoning demonstrated that they conceived of an invariant relationship between quantities and employing a correspondence approach, they attended to the part that the value of one quantity determined the value of the other quantity.

Both Jack and Lisa operated with the same definition of a function such that same input could not map to different outputs, but different inputs could map to the same output across all interviews, whereas, Dave operated with different definitions in the Pre interview and the Post interview context. Dave's definition of a function matched with Jack and Lisa's definition of a function, but only in the Ferris wheel context.

Contrast across cases. Jack, Dave, and Lisa conceived of function notation at a *function notation as a relationship between variables* level, but in different interviews. Jack engaged in variational reasoning, quantitative reasoning, and covariational reasoning and employed a correspondence approach to interpret function notation across all interviews. Dave conceived of an invariant relationship between quantities and then employed a correspondence approach to justify function notation within and across Ferris wheel interviews 1 and 2. Lisa provided strong evidence that she conceived of function notation by engaging in quantitative reasoning and covariational reasoning and employing a correspondence approach to function within Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview.

How Might Covariational Reasoning Related to Function Impact Students' Conceptions of Function Notation?

Similarities across cases. Within Ferris wheel interviews, all three students were consistent in their conception of function and they conceived of function notation at a *function notation as a relationship between variables* level. They engaged in variational reasoning, quantitative reasoning, and covariational reasoning and employed a correspondence approach to function.

Contrast across cases. Jack, Dave, and Lisa engaged in different forms of reasoning and conceived of function notation at different levels across interviews. Some of those differences can be explained by the tasks that students engaged in, and also on their conceptions of function. Jack consistently engaged in covariational reasoning within and across all interviews, yet his conception of function notation was different in certain tasks. For example, across the Pre interview to the Post interview, in a statement task, he engaged in covariational reasoning at least a *Gross Coordination of Values* (Thompson & Carlson, 2017) level, but he conceived of function notation at a *function notation as convention* level. At this level, he matched the independent variable in the parentheses with the variable along the horizontal axis and the dependent variable on the other side of an equal sign with the variable along the vertical axis. Across the Pre interview to the Post interview, in a plane situation task, Jack engaged in covariational reasoning at a *Coordination of Values* (Thompson & Carlson, 2017) level, and he conceived of function notation at a *function notation as a relationship between variables* level. At this level, he conceived of distance and altitude as quantities that had an invariant relationship where one value of distance determined one value of altitude (Thompson & Carlson, 2017). In other words, he engaged in covariational reasoning and employed a correspondence approach to function. He could think flexibly about graphs, where quantity representing the independent variable could be represented by either the horizontal axis or the vertical axis.

Dave engaged in covariational reasoning in all interviews, but his conception of function notation was different in Ferris wheel interviews than in the Pre interview and the Post interview. Within the Pre interview and across the Pre interview to the Post interview, Dave engaged in covariational reasoning and conceived of function notation at a *function notation as label* level. For example, in a swing situation task, his conception of function notation was separate from his covariational reasoning. When conceiving of function notation, it was fine with Dave to be what I saw as imprecise because he attended to the shape of a graph. He conceived of function notation at a *function notation as label* level because he drew on the shape of a graph. Dave's conception of function also impacted his conception of function notation. He only conceived of one-to-one graphs representing a function. If a graph violated a one-to-one definition of a function, he did not carefully consider which variable should be used as an input or output and concluded that he could not write a function notation either. Within Ferris wheel interviews, he shifted in his conception of function such that one input could not have multiple outputs but multiple inputs could have the same output. Dave engaged in covariational reasoning and employed a correspondence approach, so he conceived of function notation at a *function notation as a relationship between variables* level. He was not only thinking about how quantities changed together, but also provided evidence of interchanging the variables along the axes as long as it satisfied his new conception of the definition of a function.

Lisa engaged in quantitative reasoning and covariational reasoning in the Pre interview plane situation task and provided evidence of a shift in her conception of function notation from *function notation as convention* level to *function notation as a relationship between variables* level. She said that distance determined the altitude, however, she did not explicitly write a function notation. Within Ferris wheel interview 1, she had difficulty conceiving of function notation because of her conceptions of distance and height representing the same thing - a length from the ground. Within Ferris wheel interview 2, she conceived of distance as the total distance traveled and then she provided evidence that she engaged in covariational reasoning throughout Ferris wheel interview 2 and across Ferris wheel interview 2 to the Post interview. Within and across Ferris wheel interview 2 to the Post interview, she engaged at a *function notation as a relationship between variables* level. By that I mean that she demonstrated that she engaged in quantitative reasoning and covariational reasoning and employed a correspondence approach.

How Do Students Conceive of a General Function Notation?

Similarities across cases. Jack, Dave, and Lisa preferred letters *x*, *y*, and *f* when interpreting function notation. Jack preferred $y=f(x)$ over $g=r(m)$ because he said that he was used to this notation. Dave preferred the letters *x*, *y*, and *f* when interpreting function and function notation. If variables other than *x* and *y* were used, he converted different variables to *y* equals *f* of *x* and then decided how to label the axes. He used the variable *x* along the horizontal axis, the variable *y* along the vertical axis, and preferred *f* as the name of a function. One possible reason of why he converted letters to *x*, *y*, and *f* is because textbooks usually represent a function notation as $y=f(x)$. Lisa also labeled her axes as *x* and *y* in certain tasks. For example, in Ferris wheel interview 2, she sketched a graph after watching the animated trace and labeled the vertical axis as *y* and the horizontal axis as *x*. Lisa wrote $f(a)=d$ or $f(d) =a$ and mostly said function of altitude equals the distance across tasks, except when I said the other way like *a* equals *f* of *d* ($a=f(d)$). All three students liked to have *x*, *y*, and *f* because they were used to those letters.

Jack, Dave, and Lisa conceived of function notation as something more than what Musgrave and Thompson (2014) termed idiomatic expression. Both left-hand side and the right-hand side of function notation meant something to all three students. In particular, Lisa mixed words and function notation to show that both variables *a* and *d* meant more than letters to her. Dave and Jack also wrote words instead of letters to show that the letter *h* meant height and the letter *d* meant the distance traveled. In other words, to all three students, function notation meant more than letters.

Within Ferris wheel interviews, all three students conceived of function notation at a *function notation as a relationship between variables* level. Using Thompson and Carlson's

(2017) definition of a function, students engaging in quantitative reasoning and covariational reasoning demonstrated that they conceived of an invariant relationship between quantities and employing a correspondence approach, they attended to the part that the value of one quantity determined the value of the other quantity.

Contrasts across Cases. Within the Pre interview, Jack and Lisa shifted in their conception of function notation from *function notation as convention* to *function notation as a relationship between variables*. After this shift in Jack's conception, he conceived of function notation at a *function notation as a relationship between variables* level throughout all interviews. However, Lisa's conception of distance impacted her conception of function notation. Once she conceived of distance as the total distance traveled within Ferris wheel interview 2, she conceived of function notation at a *function notation as a relationship between variables* level and her conception of function notation remained consistent from Ferris wheel interview 2 to the Post interview.

Dave's conception of function notation was different from Jack and Lisa and he drew on the shape of graphs, so he conceived of function notation at a *function notation as label* level. Dave's reasoning with function notation intertwined with the shape of a graph can be explained by Bell and Janvier (1981), Carlson (1998), and Moore and Thompson (2015) findings that students often reason about graphs based on physical characteristics such as the shape of a graph. Bell and Janvier (1981) used a term called *pictorial distractions*. They explained that pictorial distractions occurred when the student confused the aspects of the situation. My findings are similar to what Bell and Janvier (1981) term pictorial distraction (1998) and what Moore and Thompson (2015) term static shape thinking, but the only difference is that Dave demonstrated that he extended pictorial distraction and static shape thinking to his reasoning with function notation.

Main Finding of this study

Looking across all 3 cases, my results showed that the students could be engaged in lower levels of covariational reasoning but still conceived of function notation at a higher level called *function notation as a relationship between variables*. I initially created a table (see Table 3) to classify students' conception of function notation based on my conjectures that they may conceive of function notation at higher levels as they would engage in higher levels of covariational reasoning, but the results were quite different than what I conjectured.

Based on my results, I found that within Ferris wheel interviews, all three students provided evidence that they engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level, yet they could conceive of function notation at a *function notation as a relationship between variables* level. The reason for this consistency in their conception of function notation is that all three students had the same conception of function that one input could not map to different outputs and different inputs could map to the same output. In other words, students' conception of function impacted their conception of function notation. At a *function notation as a relationship between variables* level, students employed Thompson and Carlson's (2017) definition of function which I refer to as a combination of covariation and correspondence approaches. The students engaged in a quantitative reasoning and covariational reasoning to demonstrate conceptions of an invariant relationship between quantities. They employed a correspondence approach to attend to the part of a definition that the value of one quantity determined the value of the other quantity.

CHAPTER IX

DISCUSSION & IMPLICATIONS

In this chapter, I discuss the importance of my results including the limitations of this study. I wanted to learn if covariational reasoning impacted students' conceptions of function notation. In this study, I found that when conceiving of general function notation ($y=f(x)$), within Ferris wheel interviews, all students provided evidence of engaging in quantitative reasoning and covariational reasoning and employed a correspondence approach. I also include implications these findings have for teaching and describe goals for future research.

Students' Covariational Reasoning and their Conceptions of Function Notation ($y=f(x)$).

My analysis of the data provided evidence of students engaging in some levels of covariational reasoning more than the others. Students' engagement at levels of *Gross Coordination of Values* and *Coordination of Values* (Thompson & Carlson, 2017) were more evident. Not all students engaged in *Coordination of Values* (Thompson & Carlson, 2017), yet they conceived of function notation at a *function notation as a relationship between variables* level that I expected would be associated with higher levels of covariational reasoning. Within Ferris wheel interviews, it was interesting that when all three students engaged in *Gross Coordination of Values* (Thompson & Carlson, 2017) level, then they conceived of function notation *as a relationship between variables*. Students' conception of function also impacted their conception of function notation within Ferris wheel interviews. Within Ferris wheel interviews, all three students were consistent in their conception of function such that one input could map to one output and different inputs could map to the same output, so they all conceived of function notation at a *function notation as a relationship between variables* level.

Within the Pre interview and the Post interview, students engaged in the same kind of reasoning (e.g., Gross covariation) were at different levels of function notation. For example, in a statement task, Jack engaged in covariational reasoning, but he conceived of function notation at a *function notation as convention* level. Later, he engaged in covariational reasoning and conceived of function notation at a *function notation as a relationship between variables* level in the plane situation task. Dave engaged in covariational reasoning within the Pre interview and the Post interview, but his conception of the definition of function impacted his conception of function notation and he conceived of function notation at a *function notation as label* level or *function notation as convention* level. For example, in a swing situation task, Dave conceived of function notation as label because he attended to the shape of a graph and provided evidence of engaging in covariational reasoning, which was separate from his conception of function notation.

Conjectured levels of students' conceptions of function notation. In my literature review, I conjectured that students may conceive of function notation in ways that I expected would be associated with higher levels of covariational reasoning. In the table below (see Table 10), I have linked my levels of students' conceptions of general function notation ($y=f(x)$) from a lower level to a higher level to covariational reasoning. The left column categorizes different conceptions of function notation and the right column provides a description of how students may link conceptions of function notation to their variational or covariational reasoning in a Cartesian coordinate system. When I started the study, my conjecture was that as students engaged in higher levels of covariational reasoning, they would also conceive of function notation at a higher level. For example, an individual who conceives of *function notation as label* level may conceive of $y=f(x)$ just as different letters and may draw on the shape of a graph.

This will be the weakest level. At a *function notation as convention* level, students may engage in covariational reasoning or their conceptions of function notation may be separate from covariational reasoning. At a *function notation as a relationship between variables*, I adapted Thompson and Carlson (2017) definition of function to function notation and conjectured that students may engage in covariational reasoning and also employ a correspondence approach when conceiving of function notation. Students at this level may think that function notation is not just how quantities are changing together, but also a special way (function) in which they are related.

Table 10

Conjectured levels of function notation before conducting the study

Name	Description of what students do in a Cartesian coordinate system
Function notation as label	At this level, students match a label to a graph. Students associate function notation $y=f(x)$ or $x=f(y)$ with the shape of a graph. Students may employ a correspondence approach without engaging in covariational reasoning.
Function notation as convention	Student may engage in variational or covariational reasoning and may use convention of matching the axes labels to function notation. Function notation can be written either as $x=f(y)$ or $y=f(x)$. If the horizontal axis is labeled as x, then function notation should be written as $y=f(x)$, so that the variable in the parentheses should match the variable on the horizontal axis. If the horizontal axis is labeled as y, then function notation should be written as $x=f(y)$, so that the variable in the parentheses should match the variable on the horizontal axis.
Function notation as a relationship between variables	Thompson and Carlson (2017) definition of function: When students employ a correspondence approach and engage in covariational reasoning, function notation is not just how quantities change together, but also a special way (function) in which they are related. The independent variable can be represented along the horizontal axis or the vertical axis, and the function notation can be written as $y=f(x)$ or $x=f(y)$ as long as value of one quantity x satisfies the value of the other quantity y or the value of one quantity y satisfies one value of the other quantity x.

Connections to students' reasoning with their conceptions of function notation after conducting the study. I initially thought that students engaged in *Coordination of Values* (Thompson & Carlson, 2017) or higher may engage in covariational reasoning and may employ a correspondence approach together when conceiving of function notation (see Table 10). Based on the results of this study, I found that within Ferris wheel interviews and across Ferris wheel interviews to the Post interview, all students engaged in a lower level of covariational reasoning called *Gross Coordination of Values* (Thompson & Carlson, 2017) and they could conceive of function notation at a *function notation as a relationship between variables* level. I modified my table (see Table 11) to include students' conceptions of function notation connected to their reasoning. Students engaged in variational reasoning, quantitative reasoning, and covariational reasoning and also employed a correspondence approach to function when conceiving of function notation.

Table 11

Connections to students' reasoning with levels of function notation after conducting the study

Name	Description of what students do in a Cartesian coordinate system	Connections to variational reasoning, quantitative reasoning, and covariational reasoning
Function notation as label	At this level, students match a label to a graph. Students associate function notation $y=f(x)$ or $x=f(y)$ with the shape of a graph. Students may employ a correspondence approach without engaging in covariational reasoning.	Students' conceptions of function notation are separate from their variational reasoning, quantitative reasoning, and covariational reasoning.
Function notation as convention	Student may engage in variational or covariational reasoning and may use convention of matching the axes labels to function notation. Function notation can be written either as $x=f(y)$ or $y=f(x)$. If the horizontal axis is labeled as x, then function notation should be written as $y=f(x)$, so that the variable in the parentheses should match the variable on the horizontal axis. If the horizontal axis is labeled as y, then function notation should be written as $x=f(y)$, so that the variable in the parentheses should match the variable on the horizontal axis.	Students may engage in variational reasoning, quantitative reasoning, or covariational reasoning, and may conceive of function notation using convention of Cartesian coordinate system. Students may conceive of function notation separately from covariational reasoning or in conjunction with covariational reasoning.
Function notation as a relationship between variables	Thompson and Carlson (2017) definition of function: When students employ a correspondence approach and engage in covariational reasoning, function notation is not just how quantities change together, but also a special way (function) in which they are related. The independent variable can be represented along the horizontal axis or the vertical axis, and the function notation can be written as $y=f(x)$ or $x=f(y)$ as long as value of one quantity x satisfies the value of the other quantity y or the value of one quantity y satisfies one value of the other quantity x.	Students engaged in variational reasoning, quantitative reasoning, and covariational reasoning and employed a correspondence approach to function.

Explanation of higher level of function notation. Thompson and Carlson's (2017) definition of a function was very useful to interpret student's reasoning with function notation. Thompson and Carlson (2017) argued that if students engaged in covariational reasoning, then students could reason without considering which variable represented the dependent variable and which variable represented the independent variable. The results of my study showed that when interpreting function notation, some correspondence was useful. Quantitative reasoning, a correspondence approach, and covariational reasoning together seemed to make a difference in students' reasoning with function notation. I adapted Thompson and Carlson's (2017) definition of function to function notation and referred to it as a combination of covariational reasoning and a correspondence approach. Although Thompson and Carlson (2017) did not characterize their definition in terms of dependent or independent variables and did not imply a correspondence perspective, but I make that distinction when interpreting their definition of a function. In other words, when conceiving of function notation, students engaged in quantitative reasoning and covariational reasoning and demonstrated that they conceived of an invariant relationship between quantities and employed a correspondence approach to attend to the part that the value of one quantity determined a value of the other quantity.

Results Consistent with *Convention Acting in the Capacity of Convention*

Moore et al. (in press) reported that some pre-service and in-service teachers were able to reason in ways that were different than what's practiced by the community (or convention). Moore et al. (in press) found that some teachers could conceive of x as a function of y by reasoning that rotating a graph counterclockwise expressed the same relationship between distance and height. The authors called those viable ways of reasoning *convention qua convention*. My results were consistent with the findings of Moore et al. (in press). I found that in a plane situation task with distance along the vertical axis and the height along the horizontal axis, Jack rotated a graph counterclockwise and said that the graph expressed the same relationship between distance and height. Jack also wrote function notation $h=f(d)$, so he conceived of a graph and function notation using convention qua convention. Convention qua convention is important because it allows to learn how students conceive of invariant relationships between quantities among different representations.

Types of Tasks Employed

Moore's Tasks

In this study, I built on the work of Moore who also investigated students' shifts from convention following to quantitative reasoning (Moore et al., 2013; Moore & Thompson, 2015; Moore et al., 2014, Moore et al., in press) and emphasized interpreting graphs as relationships between two quantities. I expanded Moore's work to study shifts in students' reasoning with function notation. All students in my study said that the tasks related to graphs with switched attributes were different from what the students were used to, but still demonstrated that they could engage in quantitative reasoning regardless of which axis represented the independent variable. I found that students' ways of thinking necessary for interpreting graphs as relationships between quantities were also useful for them to interpret general function notation ($y=f(x)$). My work builds on Moore's work on quantitative reasoning and covariational reasoning and extends to general function notation ($y=f(x)$).

Johnson's tasks

I built on Johnson's Ferris wheel tasks and tasks asking students to interpret responses from others' claims about a graph (Johnson et al., 2018, August) by adding the function notation to it. The Ferris wheel tasks were dynamic in nature, whereas, the tasks asking students to interpret responses from others' claims about a graph did not involve dynamic graphs. I found that ways of thinking necessary for interpreting graphs as relationships between quantities were also useful to interpret general function notation ($y=f(x)$) in these tasks.

Students' reasoning on different task types. I found different results depending on the types of tasks employed to investigate students' reasoning with function and function notation. In tasks involving functions, graphs, tables, and function rules, my results were consistent with other researchers (e.g., Akkoc & Tall, 2005; Breidenbach et al., 1992; Even, 1993; Oehrtman, Carlson, & Thompson, 2008) who documented that given graphs, students relied on the vertical line test to interpret a function. In tasks involving functions, graphs, tables and function rules, students in my study also relied on the vertical line test and did not conceive of x as a function of y. In tasks involving functions, graphs, tables, and function rules, my results were consistent with Moore and Thompson (2015) *static shape thinking* because students looked at the shape of a graph and relied on the vertical line test. Their reasoning was also consistent with Dubinsky and Harel's (1992) action conception because students only conceived of y as a function of x, because of how the axes were labeled such that the horizontal axis represented the independent variable x and the vertical axis represented the dependent variable y.

In situation tasks and the Ferris wheel tasks, students' reasoning was quite different, and their reasoning was consistent with Dubinsky and Harel's (1992) process conception because students could conceive of several inputs mapping to a set of outputs without determining specific values and relying on the vertical line test. Situation tasks with attributes represented on different axes, Ferris wheel tasks with attributes represented on different axes, and tasks asking students to interpret responses from others' claims about a graph (Johnson et al., 2018, August) were the most useful to learn how students engaged in quantitative reasoning and covariational reasoning and how it impacted their interpretation of function notation. For example, Jack, Dave, and Lisa all combined quantitative reasoning and a correspondence approach in those tasks. With those tasks, I found that students could conceive of x as a function of y, which is consistent with Moore et al. (in press) findings and they use the term *convention qua convention* to classify students' reasoning that is different than what is practiced by the community. In situation tasks, Ferris wheel tasks, and tasks asking students to interpret responses from others' claims about a graph (Johnson et al., 2018, August), my data provided evidence of all students engaging in a construct called *emergent shape thinking*.

Because of the dynamic nature of the Ferris wheel tasks, they were useful despite the challenges. For example, when Dave and Lisa initially engaged in the Ferris wheel tasks, they conceived of both distance and height measuring the same thing- a length from the ground. The tasks asked students to interpret the total distance traveled, however, they conceived of distance as distance from the ground. Both Dave and Lisa's initial conception of distance as a length from a ground can be explained by what Bell and Janvier (1981) term situational distraction. In contrast, when interpreting function notation in Ferris wheel tasks with attributes switched on the axes, I found that students could conceive of height as a function of distance ($h=f(d)$), which is consistent with Moore et al. (in press) findings and they use the term *convention qua convention* to classify students' reasoning that is different than what is practiced by the community. All students provided evidence that they engaged in quantitative, variational, and covariational reasoning and employed a correspondence approach to interpret function notation. This explains the complexity and challenges when students initially engage in Ferris wheel tasks, yet the dynamic nature of these tasks provided students an opportunity to engage in variational and covariational reasoning later which also played an important role in their conceptions of function notation.

A study done by Fonger et al. (2016) focused on middle school students' conceptions of function notation of specific rules such as quadratic functions and had levels of correspondence that were fine- grained. Fonger et al. (2016) found that covariation alone was powerful, and correspondence emerged simultaneously with students'covariational reasoning. My focus was on students' reasoning with general function notation ($y=f(x)$) and I found something different. I found that students interpreted function notation by engaging in covariational and quantitative reasoning and employing a correspondence approach.

Students' reasoning can vary depending on the type of task. I found that Dave's reasoning with function and function notation was different across contexts. A possible explanation was that he operated with different definitions of a function in Ferris wheel interviews context than in the Pre interview/Post interview context and that can explain differences in his reasoning with function notation across contexts. I thought that Dave will shift in his reasoning with function notation from Ferris wheel tasks to the Post interview, but he went back to how he conceived of function notation in the Pre interview. Johnson et al. (2017b) presented a case of a student named Ana who demonstrated shifts in her reasoning within Ferris wheel tasks, but not on the bottle problem. The results of this study were consistent with the findings of Johnson et al. (2017b), and my findings extended to interpreting general function notation ($y=f(x)$).

Limitations of this study

One limitation of my study is that the participants in my study were from the same university. I only interviewed a small number of participants who agreed to be interviewed. People who volunteered to be interviewed were all white students; therefore, the results cannot be representative of the diverse university population. If I had interviewed students of color and students from different universities, the results may have been different.

A possible limitation of this study was how I posed a question or did not ask a question at times and may have affected students' responses. For example, in the Pre interview, Lisa engaged in quantitative reasoning and demonstrated that she shifted from following convention that the horizontal axis should always represent the independent variable. But, I did not ask her explicitly if the graph could be expressed as function notation $h=f(d)$. My finding may have been different if I had asked her about $h=f(d)$. The kinds of questions we ask impacts what is possible for us to learn as researchers.

Another limitation of this study is that not all students had a chance to work on the same tasks. For example, Lisa did not work on the same kinds of tasks (responses to other students' claims) in the Pre interview and the Ferris wheel tasks. It was evident that when Dave and Jack worked on tasks of responding to other students' claims in Ferris wheel interviews, they provided evidence of engaging in a combination of quantitative reasoning and covariational reasoning and employed a correspondence approach in a single task. Lisa provided evidence of engaging in quantitative reasoning and covariational reasoning in the Ferris wheel tasks and then employed a correspondence approach, but the evidence was not as strong as the other students. If Lisa had the chance to work on tasks asking to respond to other students' claims in all interviews, that may have impacted my finding. Despite the limitations, modifying the tasks helped me to achieve my goal, and the benefits outweighed the limitations to really being able to look across tasks.

Implications

I recommend that researchers/teachers incorporate technology-rich tasks, tasks with attributes switched (different from what they learn in a classroom), and tasks of making sense of others' claims to foster students' variational and covariational reasoning and provide students an opportunity to relate the changing quantities to their conceptions of function and function notation. When I asked students to work on tasks involving functions, graphs, tables, and function rules it was difficult to learn if students could conceive of quantities as varying and covarying. It was also difficult to learn if students could conceive of a function notation more than just different letters. There were two types of tasks from which I gathered the most evidence of links between students' covariational reasoning and the interpretation of function notation. The first set of tasks included situation tasks and the Ferris wheel tasks with attributes switched. The second type of tasks were the tasks with names and making sense of others' reasoning about graphs (Johnson et al., 2018, August). All students demonstrated that they could combine quantitative and covariational reasoning and a correspondence approach to interpret function notation. So, these tasks may allow teachers to learn about students' conceptions and how students connect different representations of the same thing. Similarly, because of the dynamic nature of the Ferris wheel tasks, teachers can learn in what other ways students can conceive of general function notation ($y=f(x)$) as expressing an invariant relationship between quantities.

College Algebra course is meant to prepare students for higher level mathematics courses, but the current design of College Algebra course only serves 5-10% of the students to be prepared for PreCalculus (Herriott & Dunbar, 2009). There is a need to change the curriculum to provide College Algebra students an opportunity to have a deeper understanding of concepts such as function and general function notation ($y=f(x)$). Our current College Algebra textbooks do not emphasize what $y=f(x)$ really means. Students are usually asked to substitute values, complete the tables, and graph function notation of the form $f(x) = 3x+5$, but when it comes to general function notation ($y=f(x)$), students find it very difficult to interpret what $y=f(x)$ is.

College Algebra students should be given an opportunity to work on tasks mentioned above, so that they can conceive of general function notation as an expression relating two quantities *x* and *y*.

In an effort to promote covariational reasoning in College Algebra classes, Implementing Techtivities to Promote Students' Covariational Reasoning in College Algebra (ITSCoRe) is working to develop instruments measuring students' covariational reasoning. Dr. Johnson is a principal investigator (PI) of ITSCoRe grant and its purpose is to increase and enhance College Algebra students' engagement in tasks so that they can be successful not only in their College Algebra class, but also be prepared for STEM professions. My work aligns with ITSCoRe mission as I focused on College Algebra students' covariational reasoning and links between student's covariational reasoning and their reasoning with general function notation ($y=f(x)$). Covariational reasoning is not emphasized in curricular materials, but it can make a difference in students'conceptual understanding of function and general function notation ($y=f(x)$).

Future research

Future research could include a critical theory lens with emphasis on a more representative sample of college student's population. Using a larger sample size is needed to substantiate the claim that students engaged in covariational reasoning at a *Gross Coordination of Values* (Thompson & Carlson, 2017) level could conceive of function notation using Thompson and Carlson's (2017) definition of function. Given that my sample was small, it seemed that those classification levels were developed in my study, but future research could look at if they hold up with a larger sample including students of color and first-generation college students.

To foster students' reasoning and conceptions of function and function notation, more opportunities must be provided to them. Shifts in students' conceptions occurred as students reflected on the tasks that I posed and I found those shifts interesting. For example, the situation tasks, the Ferris wheel tasks and tasks asking students to interpret responses from others' claims about a graph (Johnson, et al., 2018, August) seemed to make a difference in students' conceptions of general function notation ($y=f(x)$). While I identified shifts in students' conceptions and reasoning, explaining the mechanism of how those shifts happened is beyond the scope of this study. Future study should work to explain shifts in students' reasoning.

Concluding Remarks

Students' reasoning with quantities impacted their conceptions of function notation. I found that students' engagement in quantitative reasoning and covariational reasoning along with a correspondence approach to function allowed then to view function notation $y=f(x)$ representing a relationship between two quantities. In this study, I found that students viewed general function notation ($y=f(x)$) differently than what Musgrave and Thompson (2014) termed idiomatic expression. To all three students, general function notation ($y=f(x)$) meant more than letters and they conceived of function notation expressing a relationship between two quantities x and y at a *function notation as a relationship between variables* level.

REFERENCES

Akkoc, H., & Tall, D. (2005). A mismatch between curriculum design and student learning: The case of the function concept. In D. Hewitt, & A. Noyes (Eds.), *Proceedings of the sixth British Congress of mathematics education* University of Warwick, (pp. 1-8).

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34-42.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp.20-32). Reston, VA: NCTM.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*, 247-285.

Cajori, F. (1928). *A history of mathematical notations* (Vol. 1: Notations in elementary mathematics). La Salle, IL: Open Court.

Cajori, F. (1929). *A history of mathematical notations* (Vol. 2: Notations mainly in higher mathematics). La Salle, IL: Open Court.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A.H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (pp.114-162). Washington, DC: Mathematical Association of America.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Carlson, M., & Oehrtman, M. (2005). Key aspects of knowing and learning the concept of function. In A. Selden & J. Selden (Eds.), MAA Online Research Sampler. Retrieved from http://www.maa.org/t_and_l/sampler/rs_9.html.

Castillo-Garsow, C. (2010). *Teaching the Verhulst Model: A teaching experiment in covariational reasoning and exponential growth.* Unpublished doctoral dissertation, Arizona State University, Tempe, AZ.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 55-73). Laramie, WY: University of Wyoming College of Education

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31-37.

Chazan, D. (2000). *Beyond formulas in mathematics and teaching: Dynamics of the high school algebra classroom.* New York: Teachers College Press.

Chen, X. (2013). *STEM Attrition: College Students' Paths Into and Out of STEM Fields* (NCES 2014-001). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washignton, DC.

Clement, J. (1982). Algebra word problem situations: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13,* 16-30.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Clement, L. (2001). What do students really know about functions? *Mathematics Teacher, 94*(9), 745-748.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*, 135-164.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Cooney & Wilson (1996). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg & E. Fennema (Eds.), Integrating research on the graphical representation of functions (pp. 131-158). Hillside, NJ: Lawrence Erlbaum.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). London: Sage Publications.

Dogbey, J. & Kersaint, G. (2012). Treatment of variables in popular middle -grades mathematics textbooks in the USA: Trends from 1957 through 2009. *International Journal for Mathematics Teaching and Learning, 2*(1), 1-30.

Drew, D. E. (2011). *Stem the tide: Reforming Science, Technology, Engineering and Math Education in America*. Baltimore: Johns Hopkins University Press.

Dreyfus, T. & Vinner, S. (1982). Some aspects of the function concept in college students and junior high school teachers. *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education*, Antwerp, Belgium.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht, The Netherlands: Kluwer.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. In G. Harel, & E. Dubinsky (Eds.), *The concept of functions : Aspects of epistemology and pedagogy* (pp.85-106). United States: The Mathematical Association of America.

Dubinsky, E., & Wilson, R. T. (2013). High school students' understanding of the function concept. *The Journal of Mathematical Behavior,32*, 83-101.

Duval, R. (1987). The role of interpretation in the learning of mathematics. *Diastasi, 2,* 56-74.

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathemtaics Education, 38*(3), 194-229.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. J. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215-238). New York: Springer.

Ellis, A. B., Ozgur, Z., Kulow, T., Dogan, M. F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning, 18*(3), 151-181

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2013). An exponential growth learning trajectory. In A. M. Lindimeier & A. Heinze (Eds.), *Proceedings of the 37 th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 273-280). Kiel, Germany: PME

Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C.C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*(1), 135-155. doi: 10.1016/j.jmathb.2015.06.004

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, *21*(6), 521-544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24*(2), 91-116.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Fonger, N., Ellis, A. B., & Dogan, F.M. (2016). Students' conceptions supporting their symbolization symbolization and meaning of function rules. *Proceedings of the 38th annual meeting of the International Group for the Psychology of Mathematics Education North American chapter* (pp.155-163). Tucson, AZ.

Freudenthal, H. (1973). *Mathematics as an education task*.Dordrecht: Reidel

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Boston: D. Reidel Publishing Company.

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology, 24*(5), 645-657.

Goldin, G., & Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In. L Steffe & P. Nesher (Eds.), *Theories of mathematical learning.* Mahwah, New Jersey: LEA.

Goldin, G. A. (2000). A scientific perspective on structured, task- based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates.

Gordon, S. P. (2018). What's Wrong with College Algebra? *PRIMUS, 18*(6), 516-541.

Gray, E. M., & Tall, D. (1994). Duality, ambiguity, and flexibility: A "Proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116-140.

Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *Journal of Mathematical Behavior, 25*, 57-72.

Herriott, S. R., & Dunbar, S. R. (2009). Who Takes College Algebra? *PRIMUS, 19*(1), 74-87.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior, 17*(1), 123-134.

Hornsby, J., Lial, M. L., & Rockswold, G. (2019). *A graphical approach to Algebra & Trigonometry* (7th ed.). Boston: Pearson Education, Inc.

Hung, D.W. (2000). Some insights into the generalizing of mathematical meanings. *The Journal of Mathematical Behavior*, *19*, 63-82.

Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child* (E. A. Lunzer & D. Papert, Trans.). New York: W. W. Norton & Company, Inc.

Jackiw, N. (2009). *The Geometer's Sketchpad (Version 5.0) [Computer Software]*. Emerville, CA: Key Curriculum Technologies.

Janvier, C. (1987). Translation processes in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27-32).Hillsdale, NJ: Lawrence Erlbaum Associates.

Johnson, H. L. (2012a). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In R. Mayes & L.L. Hatfield (Eds.), *Quantitaitve reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol.2, pp. 39-53). Laramie, WY: University of Wyoming College of Education.

Johnson, H. L. (2012b). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313-330.

Johnson, H. L. (2013). Reasoning about quantities that change together. *Mathematics Teacher, 106*(9), 704-708.

Johnson, H. L. (2015a). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64-90.

Johnson, H. L. (2015b). Task design: Fostering secondary students' shifts from variational to covariational reasoning. In Beswick, K., Muir, T., & Wells, J. (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 129-136). Hobart, Australia.

Johnson, H. L., Coles, A., & Clarke, D. (2017a). Mathematical tasks and the student: Navigating "tensions of intentions" between designers, teachers, and students. *ZDM: The International Journal on Mathematics Education, 49(6), 813-822. 49*(6). doi: 10.1007/s11858-017-0894-0 (OPEN ACCESS)

Johnson, H. L., Hornbein, P., & Azeem, S. (2016). Investigating function with a Ferris wheel. *Mathematics Teacher. 110*(5), 345-351.

Johnson, H. L., McClintock, E., & Hornbein, P. (2017b). Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning across tasks with different backgrounds and features. *ZDM, 49*(5).

Johnson, H. L., McClintock, E., Kalir, R., & Olson, G. (2018). Networking theories to design dynamic covariation techtivities for college algebra students. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1224-1233), San Diego, CA: RUME.

Johnson, H. L., Olson, G., Gardner, A., & Smith, A. (2018, August): From soliciting answers to eliciting reasoning: questioning our questions in digital math tasks. Colorado Council of Teachers of Mathematics.

Kaput, J.J. (1994). Democratizing access to calculus: New routes to old roots. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp.77-156). Hillsdale, NJ: Erlbaum.

Kaput, J. J., & Roschelle, J. (1999). The mathematics of change and variation from a millennial perspective: New content, new context. In C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum* (pp. 155-170). London: Falmer Press.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Kieran, C. (1992). The learning and teaching of school algebra. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp.390-419). New York, NY: macmillan.

Kinzel, M.T. (1999). Understanding algebraic notation from the students' perspective. *Mathematics Teacher*, *92*(5), 436-442.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *The College Mathematics Journal, 20*(4), 282-300.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2011). Middle school students' understanding of core algebraic concepts: Equivalence & variable. In J. Cai & E. J. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 259-276). New York: Springer.

Leindhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1-64.

Lesh, R., Post, T., & Behr, M. (1987a). Rational number relations and proportions. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp.41-58). Hillsdale, NJ: Lawrence Erlbaum.

Lesh, R., Post, T., & Behr, M. (1987b). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp.33-40). Hillsdale, NJ: Lawrence Erlbaum.

Marsden, J. E. & Hoffman, M. J. (1993). *Elementary classical analysis* (2^{nd} ed.). New York: W.H. Freeman and Company

Monk, S. (1992). Students understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy, MAA Notes* (Vol. 25, pp. 175-193).

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld, & J.J. Kaput (Eds.), *Research in collegiate mathematics education*, 1, issues in mathematics education (Vol. 4, pp. 139-168). Providence, RI: American Mathematical Society.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102-138.

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*,*32*(3), 461-473.

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking Conventions to Support Quantitative Reasoning. *Mathematics Teacher Educator, 2*(2), 141-157. National Council of Teachers of Mathematics.

Moore, K. C., Silverman, J., Paoletti, T., Liss, D., & Musgrave, S. (in press). Conventions, habits, and U.S. teachers' meanings for graphs. *Journal of Mathematical Behavior*, doi.org/10.1016/j.jmathb.2018.08.002

Moore, K. C., & Thompson, P.W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N.E. Infante, K. Keene & M. Zandieh (Eds.), *Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education* (pp.782-789). Pittsburgh, PA: RUME

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T.A. Romberg, E. Fennema, T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69-100). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Musgrave, S., & Thompson, P. W. (2014). Function notation as an idiom. In P. Liljedahl & C.C. Nicol (Eds.), *Proceedings of the 38th Meeting of the International Group for the Psychology of Mathematics Education* (Vol 4, pp. 281-288) .Vancouver, BC: PME. Retrieved from http://bit.ly/1p08TCG.

Nagle,C. R., & Styers, J. L. (2015). Putting mathematical tasks into context. *Mathematics Teacher, 106*(3), 206-213.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Governors Association center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Retrieved from http://www.corestandards.org

O'Connor, J.J., & Robertson, E.F. (2005). History topic: The function concept. In *MacTutor History of Mathematics*. Retrieved Sept. 20, 2016, from http://www-history.mcs.st-andrews.ac.uk/history/index.html

Oehrtman, M. C., Carlson, M. O., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), Making the connection: Research and practice in undergraduate mathematics (pp. 27-42). Washington, DC: Mathematical Association of America.

Philipp, R. (1992). The many uses of algebraic variables. *The Mathematics Teacher, 85*(7), 557-561.

Piaget, J. (1970). *The child's conception of movement and speed.* New York: Basic Books, Inc.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development* (T. Brown & K. J. Thampy, Trans.). Chicago: The University of Chicago.

President's Council of Advisors on Science and Technology. (2012). Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics. Retrieved from https://obamawhitehouse.archives.gov/sites/default/files/microsites/ostp/pcast-engage-to-excel-final_2-25-12.pdf

Sajka, M. (2003). A secondary school student's understanding of the concept of function - a case study. *Educational Studies in Mathematics, 53*, 229-254.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective:Simultaneous continuous variation. In S. B. Berenson, K. R. Dawkins, M. Blanton, W. N. Coloumbe, J. Kolb, K. Norwood, & L.Stiff (Eds.), *Proceedings of the 20th annual meeting of the psychology of mathematics education North American chapter*(pp.298-303). Raleigh, NC: North Carolina State University.

Schoenfeld & Arcavi (1988). On the meaning of variable. *Mathematics Teacher, 81*, 420-427.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-the case of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 59-84). Washigton, Mathematical Association of America.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel, & E. Dubinsky(Eds.), *The concept of functions: Aspects of epistemology and pedagogy* (pp. 25-28). United States: The Mathematical Association of America.

Sierpinska, A. (2004). Research in Mathematics Education through a Keyhole: Task Problematization. *For the Learning of Mathematics, 24*(2), pp.7-15. FLM Publishing Association.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197-209.

Smith, J. P., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Stacey, K., & MacGregor, M. (1997). Ideas about symbolism that students bring to algebra. *Mathematics Teacher*, *90*, 110-113.

Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (pp. 443-466). Thousand Oaks, CA: Sage Publications, Inc.

Stalvey, H. E., & Vidakovic, D. (2015). Students' reasoning about relationships between variables in a real-world problem. *The Journal of Mathematical Behavior*, *40*, 192-210.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarios, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., & Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Mathematics, Science, and Technology Education, 1*(1), 81-104. doi:10.1080/14926150109556452

Tasova, H. I., Stevens, I. E., & Moore, K.C. (2018). A framework for analyzing written curriculum from a shape thinking and covariational reasoning perspective. In (Eds.) A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown, *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1224-1233), San Diego, CA: RUME.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, *25*, 165-208.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26, 229-274.*

Thompson, P. W. (1994c). Students, functions, and the undergraduate curriculum. *Research in Collegiate Mathematics Education. I. Issues in Mathematics Education, 4,* 21-44.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlain & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for wisdom^e* (Vol. 1, pp. 33-56). Laramie, WY: University of Wyoming College of Education.

Thompson, P. W. (2013). "Why use $f(x)$ when all we really mean is y?". *OnCore, The Online Journal of the AATM* (Fall).

Thompson, P. W. & Ashbrook, J. (2016). Calculus: Newton meets technology. Retrieved from http://patthompson.net/ThompsonCalc/

Thompson, P.W., & Carlson, M.P. (2017). Variation, covariation, and functions: Foundational ways of mathematical thinking. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P.W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior, 48,* 95-111.

Thompson, P. W., & Milner, F. (2017). Teachers' meanings for function and function notation in South Korea and the United States. In H.G. Weigand, W. McCallum, M. Menghini, M. Neubrand & G. Schubring (Eds.), *The Legacy of Felix Klein- looking back and looking ahead.* Berlin: Springer.

U.S. Department of Commerce Report - STEM: Good Jobs Now and for the Future. (2011, July 22). *Entertainment Close-up*. Retrieved from http://link.galegroup.com.aurarialibrary.idm.oclc.org/apps/doc/A262232746/ITOF?u=auraria_main&sid=ITOF&xid=55525553b

Usiskin, Z. (1988). Conceptions of school algebra and uses of variable. In A. F. Coxford & A. P. Shulte (Eds.), *The Ideas of Algebra, K-12, 1988 Yearbook* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Van de Walle, J. A., & Lovin, H. (2006). Teaching Student-Centered Mathematics, Grades 5-8. Boston: Pearson Education.

Wagner, S. (1981). An analytical framework for mathematical variables. *Proceedings of the Fifth PME Conference* (pp.165-170). France, Grenoble.

Wagner, S. (1999). What are these things called variables? In B. Moses (Ed.), *Algebraic thinking, Grades K-12: Readings from the NCTM's school-based journals and other publications* (pp.316-320). Reston, VA: National Council of Teachers of Mathematics.

Wolcott, Harry F. (1994). Transforming qualitative data: Description, analysis, and interpretation. Thousand Oaks, CA: Sage Publications.

Yin, R. K. (2006). Case study methods. In J. L. Green, G. Camilli & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 111-122). Mahwah: Lawrence Erlbaum Associates, Inc.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, IV* (Vol. 8, pp. 103-127).

Zazkis, R., & Hazzan, O. (1999). Interviewing in Mathematics Education Research: Choosing the Questions. *Journal of Mathematical Behavior, 17*(4), 429-439.

APPENDIX A

Pre Interview Schedule

1. What comes to your mind when you think of a function? P1: I don't know. Follow up: Would it help if I gave you something more specific? Sometimes in math, we have graphs, tables, and equations. Would any of these help? P1: yes. Prompt: 2. Please read each statement out loud and explain what each statement means. I will also provide tables and graphs and ask them if the tables/graphs helped to clarify the statements.	This question is to know what students think about functions and what do they mean by a function and general function notation. Students could define the function as a correspondence (For every x, there is an output y). They may define a function using a graphical representation or a symbolic representation. Students often think a function must be defined by a single algebraic formula (Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992). The students may graph a function and write a notation. They may pick points or just graph a function without picking numbers.
• Given $y = f(x)$, for every input x , there is exactly one output y .	A student may say this is the definition of function because we have $y = f(x)$. Students learn this in schools.
• Given $x = f(y)$, for every input y, x is the output.	Here the notation is different, and participants may think that this is not a function due to the order of the variables in the notation $x = f(y)$. Students may pick different points to interpret the notation.
• Given $g = r(y)$, as y increases, g decreases. • Given $y = g(r)$, as r increases, y increases and then decreases.	These are definitions of function from a covariation perspective. Students are less familiar with this definition and may not consider these as function definitions. Students may express each statement graphically and try to match the variable on the parentheses to match with the variable on the horizontal axis and the variable on the other side of the equal sign to be on the vertical axis. Students may look for an expression because students often think a function must be defined by a single algebraic formula (Carlson, 1998; Clement,

2001; Even, 1990; Even, 1993; Sierpinska, 1992).

3. Do these represent functions?

The following table defines y as a function of x, denoted $y = f(x)$.

x	-2	-1	0	1	3	4
y	8	2	-3	4	2	7

Students are familiar with tables and a student may say this is the definition of function because it is given as $y = f(x)$.

It is also possible that students may say this is not a function because it is not one-to-one. $f(-1) = 2$ and $f(3) = 2$ as well.

The following table defines x as a function of y, denoted $x = f(y)$.

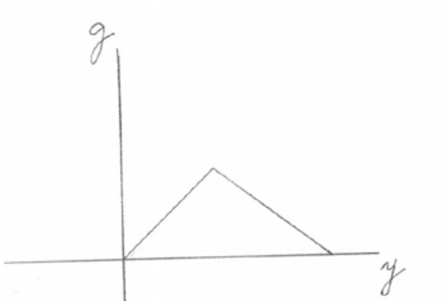
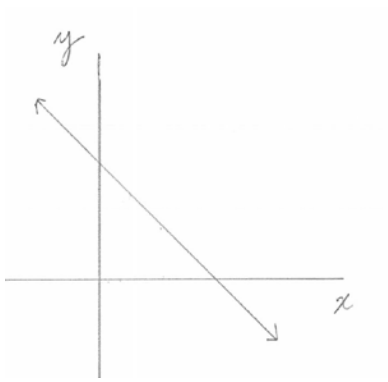
x	-3	-3	2	0
y	1	2	3	5

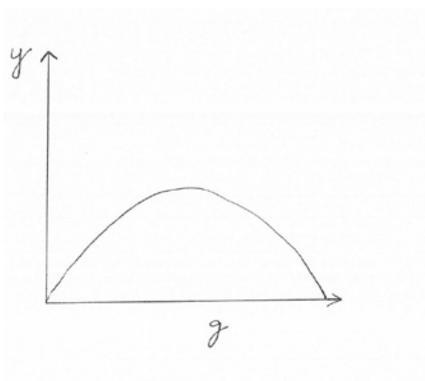
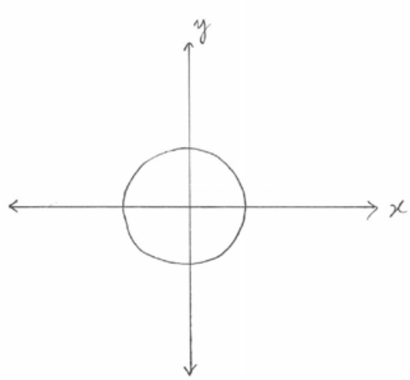
P1: same input of -3 gives outputs of 1 and 2.

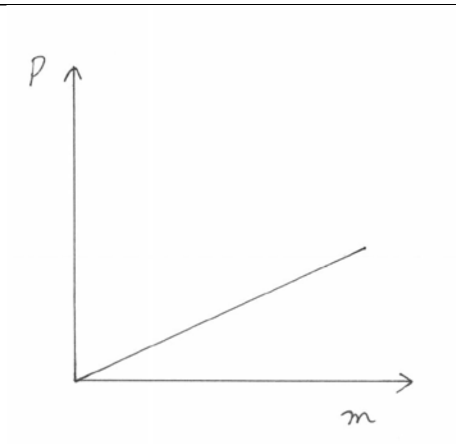
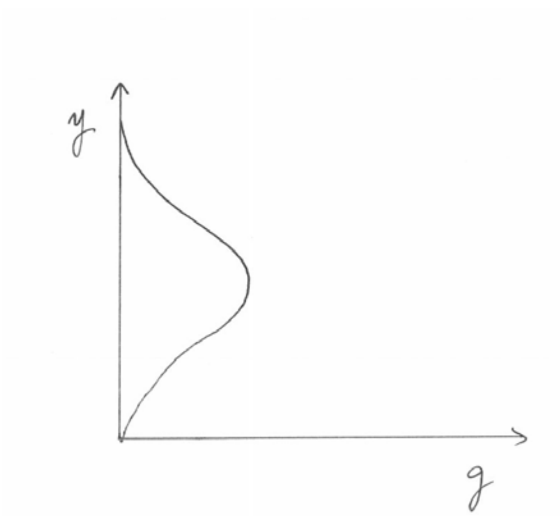
A student may still consider y as a function of x and say that the same input of -3 gives two outputs 1 and 2. I will prompt them to read the statement first.

P2: Two different inputs of 1 and 2 map to the same output of -3, which does not represent a function.

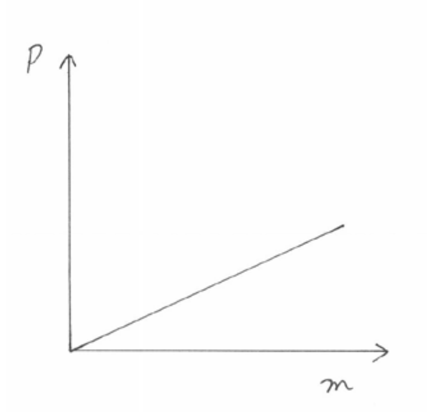
Students may conceive of one-to-one functions as functions and may not consider this onto function as a function.

 P1: The graph passes a vertical line test so it is a function. P2: Two inputs map to the same output, so it is not a function.  P1: The graph passes a vertical line test, so it represents a function. P2: One input has an output, so it represents a function.	Students may use the vertical line test to decide if the graph represents a function. Student may think that the on-to graph does not represent a function. Students may use the vertical line test to decide if the graph represents a function. Student may think that a one-to-one graph represents a function.
4. What does $u = r(s)$ mean? How do you make sense of it? P1: This is like y equals f of x but with different	I will ask the student to explain what each variable means to them. I may ask them to

letters.	graph the rule to know how they connect the rule to the graph.
5. Given a set of 4 graphs, which are functions? Which are not? why? P1: A student may say that the parabola is a function because it passes the vertical line test and the linear graph is a function because all linear graphs are functions. Follow up: Could you explain why? P1: A circle is not a function and the last graph is not a function because these graphs do not pass the vertical line test. Follow up: Could you explain a different way (without the vertical line test) why this is not a function?   	Here I expect them to explain that one input is giving two different outputs, and therefore it is not a function. This question will help me to know how students conceive of graphs that represent functions and graphs that do not represent functions.

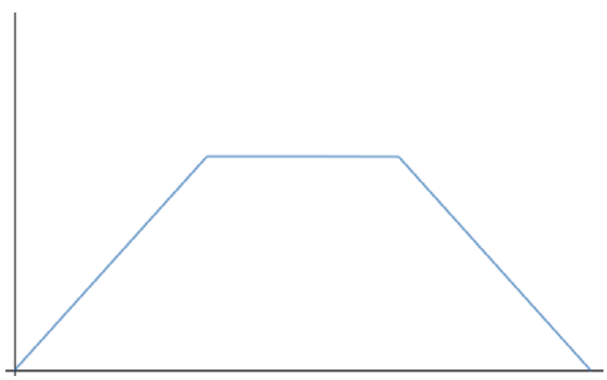
  	If we think about quantitatively, then the parabola opening sideways represents a function, where g is a function of y. The goal of this task is to see how students conceive of a function and if they engage in quantitative reasoning specifically for this graph.
6. Can you use any of these formulas to describe the graphs presented in 2? You can use formulas more than once or not at all. $g = r(y)$ $y = r(g)$ $y = f(x)$ $x = f(y)$ $m = t(p)$ $p = t(m)$ P1: A student may say that a circle can be defined as $y = f(x)$.	A student may say that a circle can be defined as $y = f(x)$ because the horizontal axis is labeled x and the vertical axis as y. (Even though the circle is not a function).

P2: A student may say $y\ = r(g)$ for the first graph because the the horizontal axis is labeled g and the vertical axis as y. P3: A student may say $p\ = t(m)$ for the linear graph because the horizontal axis is labeled m and the vertical axis as p. P4: A student may say that for the last graph $y\ = r(g)$ because the horizontal axis is labeled g and the vertical axis as y.	Student matched the variable in the parentheses to the variable in the notation I have chosen a linear graph because it is one-to-one. The students are most familiar with linear graphs, and most real-life situations can be translated to linear graphs. Moreover, there is less complexity in terms of the notation. For linear functions, we can express the situations as either $y = f(x)$ or $x = f(y)$. We can know the value of x if we know y and we can know the value of y if we know x. (Every x has an output y). So, for a linear graph, both $p = t(m)$ and $m = t(p)$ are okay. From a quantitative reasoning perspective, input can be on any axis as long as it satisfies the definition of function. This tells me that the labels of axes matter to students when they think of a notation. However, $g = r(y)$ for the last graph because g is a function of y. Here y is the input variable and g is the output variable. From a quantitative reasoning perspective, input can be on any axis as long as it satisfies the definition of function.
7.Sam said that both $m = t(p)$ and $p = t(m)$ can be used to describe the following graph. Why that made sense to that person? What do you think? P1: $p = t(m)$ is true because the horizontal axis is given by the variable m and it is the independent variable.	Student matched the variable on the horizontal axis to the variable in the parentheses and the variable on the vertical axis to the variable on the other side of equal sign to conceive of the notation. This is how students learn in school.

<table>
<tr>
<td>

P2: As m increases, p increases and as p increases m increases, so both notations can be used.</td>
<td>Thinking covariationally, as one quantity increases the other also increases, so both notations can be used. A student may combine covariational reasoning to the correspondence approach and say that one input has one output and as one quantity increases, the other also increases and therefore we can write notation both ways.</td>
</tr>
<tr>
<td>8. Given the situation below, interpret the graph.

a) Suppose that an airplane takes off from Denver International Airport. As the plane covers the distance along the ground, its altitude changes. Here is a graph representing the distance along the ground and the altitude of the airplane. Please interpret the graph.



P1: I don’t know. It looks like a hill but flat in the middle.

Prompt: Could you please tell me how the distance is changing?</td>
<td>This tells me that the student is paying attention to the overall shape of the graph. I will prompt the student to know how he /she thinks about the quantities, distance and altitude (separately and changing together).</td>
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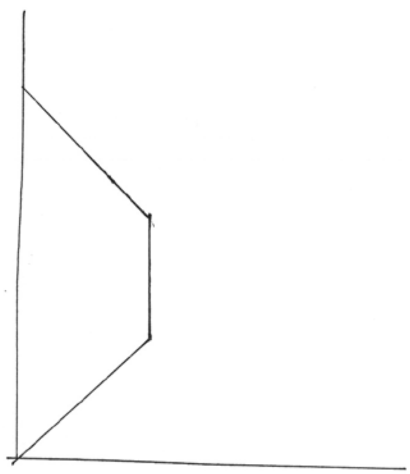
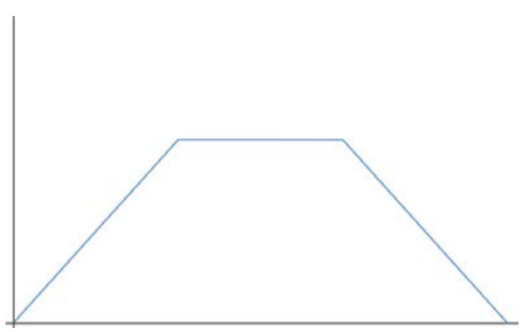
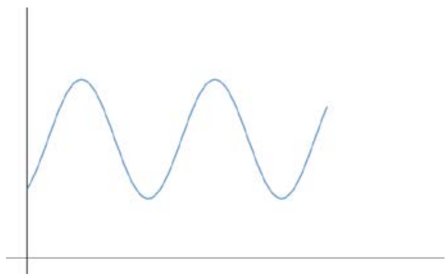
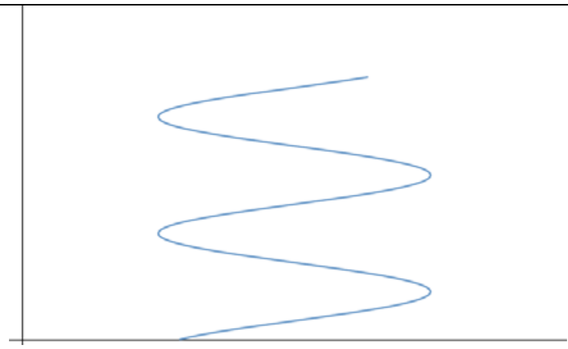
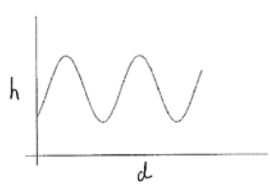
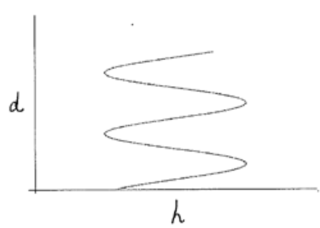
</table>

How is the altitude changing? How are both distance and altitude changing together? b) We have the same situation, but the attributes are on different axes. Please interpret the graph.  P1: I don’t know. It looks like a hill, flat in the middle and is sideways. Prompt: Could you please tell me how the distance is changing? How is the altitude changing? How are both distance and altitude changing together?	This tells me that the student is looking at the physical object. These prompts will help me to know what students think about the quantities separately and also changing together (covariation). It is possible that the student may be thinking about two quantities and notice that the relationship between two quantities stays the same. This task (representing same attributes on different axes) would allow me to explore how students are conceiving of two quantities.
Is it possible to write the situations above as $a = f(d), d = f(a)$?	

P1: A student may say that 8a) can be written as $a = f(d)$ and 8b) as $d = f(a)$. Please explain why you think that?	This tells me that the student wants to match the notation with how the axes are labeled. If we think about the notation, we cannot say that both $x = f(y)$ and $y = f(x)$. For example, if we have the distance along the x-axis and altitude along the y-axis, then we can say that $a = f(d)$, but we cannot say that $d = f(a)$ (with d along the x-axis and a along the y) because one altitude corresponds to two different distances and is therefore not a function. However, if we switch the axes, we can still say that $a = f(d)$ (with a on x-axis and distance on the y-axis). Students may say $d = f(a)$ because labeling the axes differently changes the notation (just like changing alphabets gives a different number for an equation (Wagner, 1981).
P2: A student may say $a = f(d)$, $d = f(a)$ does not make sense because there is no formula. R: Can you tell me what you think should be there? P3: I don't know. Maybe $f(x) = something$.	The student may want to see something like $f(x) = 3x + 4$. This question will help me to know if students relate notation to axes or if they relate quantities to notation, or if thinking about notation is separate than thinking about the quantities.
9. Nat said that for the first situation, the graph can be written as both $a = f(d)$ and $d = f(a)$. What do you think? 	

P1: With d on the horizontal axis, one d corresponds to a height, so h equals f of d. One h corresponds to two different d's and so d equals f of h is not true.	The student used a correspondence approach to justify the notation.
10. Given the situation below, interpret the graph. b) Suppose that a child has been swinging on a swing for some time. Here is a graph representing the total distance traveled and the height of the swing. Please interpret the graph.  P1: I don't know. It looks like a hill. Prompt: Could you please tell me how the distance is changing? How is the height changing? How are both distance and height changing together?	This tells me that the student is paying attention to the overall shape of the graph. I will prompt the student to know how he /she thinks about the quantities, distance and height (separately and changing together).
b) We have the same situation, but the attributes are on different axes. Please interpret the graph.	

 P1: I don’t know. It looks like a hill that is sideways. Prompt: Could you please tell me how the distance is changing? How is the height changing? How are both distance and height changing together?	This tells me that the student is looking at the physical object. These prompts will help me to know what students think about the quantities separately and also changing together (covariation). It is possible that the student may be thinking about two quantities and notice that the relationship between two quantities stays the same. This task (representing same attributes on different axes) would allow me to explore how students are conceiving of two quantities.
Is it possible to write the situations (in 10 above) as $h = f(d)$, $d = f(h)$? P1: A student may say that 10a) can be written as $h = f(d)$ and 10b) as $d = f(h)$. Please explain why you think that?	This tells me that the student wants to match the notation with how the axes are labeled. If we think about the notation, we cannot say that both $x = f(y)$ and $y = f(x)$. For example, if we have the distance along the x-axis and height along the y-axis, then we can say that $h = f(d)$, but we cannot say that $d = f(h)$ (with d along the x-axis and h along the y) because one height corresponds to two different distances and is therefore not a function. However, if we switch the axes, we can still say that $h = f(d)$ (with h on x-axis and distance on the y-axis). Students may say $d = f(h)$ because labeling the axes differently changes the notation (just like changing alphabets gives a different number for an equation (Wagner, 1981).

P2: A student may say $h = f(d)$, $d = f(h)$ does not make sense because there is no formula. R: Can you tell me what you think should be there? P3: I don't know. Maybe $f(x) = something$.	The student may want to see something like $f(x) = 3x + 4$. This question will help me to know if students relate notation to axes or if they relate quantities to notation, or if thinking about notation is separate than thinking about the quantities.
11. For the swing situation, Pat said that both graphs can be written as $h = f(d)$. What do you think? P1: The graph in the first situation can be written as h equals f of d and the graph with distance along the vertical axis can be written as d equals f of h.  	student paid attention to the axes labels. Distance depends on height regardless of what axis it is on. Student engaged in

P2: Both graphs represent distance increasing, one along the horizontal axis and the other along the vertical axis, so h depends on d. h equals a function of d. P3: h depends on d and each distance value corresponds to a height, so h equals f of d is the correct notation.	quantitative reasoning. Thinking quantitatively, in both cases h depends on d and also using a correspondence approach (Smith, 2003) h equals f of d is the correct notation. In other words, the student may combine the covariational perspective to the correspondence approach.

APPENDIX B

Ferris Wheel Interview 1 Schedule

<table>
<tr>
<td>Students will see a Ferris wheel (without animation) and I will ask them how the distance is changing.

P1: The student may look at the moving word.
Follow up: Please show this by pointing to the screen.

P2: The distance increases.

Follow up: The student will be asked to show this on the Ferris wheel.

P3: The distance keeps going around the circle.

Follow up: Please show this on the Ferris wheel.

If a student gets stuck, they’ll be prompted as follows:
Let us say you begin from the start, and then travel around the circle. What is happening to the distance?</td>
<td>I will ask the students about a single changing quantity which is the distance. Students may respond two different ways as identified by Thompson & Carlson, in press)
Pre-variation: If a student is looking at the physical objects, then they are at the pre-variation stage.

Variation: Students are at a “variation” stage if they look at a single changing quantity.</td>
</tr>
<tr>
<td>Students will see a Ferris wheel (without animation) and I will ask them how the height is changing.
P1: The height also goes up.
Follow up: So, does the height go up all the way?
Can you point to the Ferris wheel and show this?
P2: The height goes up and when it reaches its highest point, then it starts to decrease.
Follow up:
Can you point to the Ferris wheel and show this?
P3: The height goes up and then down.
Follow up: Can you explain and show where it is going up and where it is going down?</td>
<td>I will ask the students about a single changing quantity which is the height. Students may respond two different ways as identified by Thompson & Carlson, in press)
Pre-variation: If student is looking at the physical objects, then they are at the pre-variation stage.

Variation: Students are at a “variation” stage if they look at a single changing quantity.</td>
</tr>
</table>

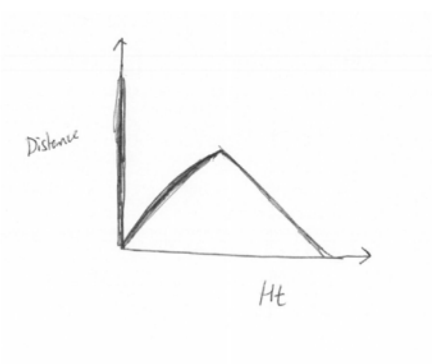
<table>
<tr>
<td>I will ask the students to run the animation to see what is happening with the height and distance. I will ask them again to explain to me how distance is changing and how height is changing.
P1: do not know.
Prompt: If you start from the ground, what is the distance and height at the start?
Follow up: If you are going far from the ground, how do the distance and height change?

P2: The student may look at the moving words distance and height.
Follow up: Please show what you mean by pointing to the screen.

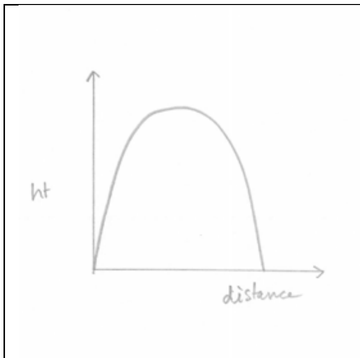
P3: The distance keeps going around the circle, and the height goes up and down.

Follow up: Where is the distance increasing and where is the height going up and down?</td>
<td>Here I will ask students about both distance and height. Here, a student may be at a pre-variation level or variation level as described above. It is also possible that the student may be paying attention to both quantities together called covariation.

The ultimate goal is to see how their thinking impacts the way they graph and conceive of the function notation.</td>
</tr>
<tr>
<td>I will ask the students to predict and then graph the relationship between distance and height. Please label the axes. Students explain why their graph makes sense.
P1: The student may graph two separate graphs, one for distance and one for height.



P2: The student may graph an iconic graph that looks like the Ferris wheel.</td>
<td>Here, there are several things to pay attention to. Students may label their axes differently, and their reasoning may be different based on how they define their axes. Graphical representation will allow me to see how students interpret the two changing quantities and how they demonstrate their thinking in terms of the graph.</td>
</tr>
</table>

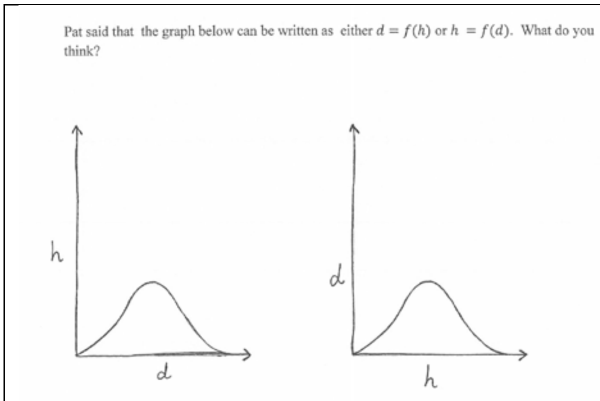
 P3: The student may sketch a graph that looks like a hill.  P4: The students may label their axes differently. **Follow up:** Why you chose the labels the way that you did?	
Could you express the graph as h equals f of d? Why? P1: No. Follow up: Why not? P1: I do not know the formula. R: Can you tell me what you think should be there? P1: I do not know. I'm not sure. **Follow up:** Could you express it like $h = f(d)$ P1: No. Follow up: Why not? P1: $h = f(d)$ does not make sense. It needs to be $f(x)$ = something. Follow up: Could you tell me what you are expecting there? P1: Maybe $y =$ something.	The students may want to see something like $f(x) = 3x + 4$. The student may be expecting different variables (Wagner, 1981) where different variables mean different numbers. If the student says $y =$ something, maybe he/she is thinking of an equation and not being able to distinguish between the equation and function (Chazan & Yerushalmy, 2003).

Follow up: is there a difference between y or $f(x)$? P1: No. $y = f(x)$. For the notation $h = f(d)$, I will ask the student about each variable. So, what does h mean? What does f mean? What do we mean by d? A student may say I have no numbers and I do not know what to do. Follow up: Is it necessary to have numbers in $h = f(d)$? Does it mean anything without numbers? P2: $h = f(d)$ or $d = f(h)$ **Follow up:** So, does it matter how we write the notation? P3: $h = f(d)$ because distance keeps going and the height depends on it. When we know the distance, we can predict the height. **Follow up:** Is it possible to write this as $d = f(h)$? Why or why not? Explain. Does it matter if we write $h = f(d)$ or $f(d) = h$?	If the student says, $y = f(x)$ then $y = f(x)$ is just a label or they think that y equals $f(x)$. They may be using an equal sign as an equivalence (Knuth et al., 2006, 2011) I am using Thompson (2013) definition of function notation to see what students think about the notation $h = f(d)$. I may ask them to pick a few numbers and tell me what that means. Then I will ask if they can think without numbers. The notation $h = f(d)$ is only meaningful if the unknown view is left behind and the variable view is made significant (Chazan; Usiskin). This will help me to know if their prior knowledge about the equations interferes with their function knowledge. This question is to see what students think about the notation if we switch $f(d)$ and h. If students are still thinking like they used to in algebra, somewhat similar to generalized numbers (x, y where $x + y = y + x$….. (Usiskin, 1988) I have mismatched variables and I want to know if the students think about notation idiomatically (Musgrave & Thompson, 2014)
Students will see the animated distance segment along the x-axis, and I will ask them to explain to me what the length on the x-axis	Here the student will see the distance segment alone. This will help me to know how students think about the distance represented by a

means in terms of the Ferris wheel. P1: The segment keeps going like the distance. **Follow up:** How is the distance changing? P2: I do not know. Prompt: Which quantity is represented by this segment? **Follow up:** How is this quantity changing? P3: Distance, but I am not sure why this is on the x-axis? **Follow up:** Do you think the segment should behave differently? Are you thinking about the graph?	segment. This will also tell me if students are bothered by the segment because a graph represents two quantities and the students are familiar with graphs of two quantities and they may wonder why this segment along the x-axis?
Then students see the height along the y-axis, and they will be asked to explain what the length on the y-axis means in terms of the Ferris wheel. P1: The segment goes up and then comes back like the height. **Follow up:** How is the height changing? What is height doing? P2: I do not know. Prompt: Which quantity is represented by this segment? **Follow up**: How is this quantity changing? P3: Height, but I am not sure why this is on the y-axis? **Follow up:** Do you think the segment should behave differently? Are you thinking about the graph?	Here the student will see the height segment alone. This will help me to know how students think about the distance represented by a segment. This will also tell me if students are bothered by the segment because a graph represents two quantities and the students are familiar with graphs of two quantities and they may wonder why this segment along the y-axis?
Show them animation of both distance and height segments together. Ask them how are both distance and height changing together? P1: Distance keeps going. **Prompt:** What about the height? P2: Height goes up and down. **Prompt:** What about the distance? P3: Distance keeps going and the height goes up or down. **Follow up**: Please show this by pointing to the screen.	Here I will ask the students about both distance and height segments together. By looking at the animated segments that change together may help them to think about two changing quantities.
Press Show Ferris wheel, then Press Show	Here students are going to explain and

trace. Tell students to watch the point and trace changing. Ask students what the trace means in terms of this situation. Ask students to compare their graph predictions to what they are seeing now. P1: They may not change their graph at all. Prompt: How does this trace compare to what you sketched before? P2: They may be thinking about one quantity before but may shift to two quantities and may want to change their graph. **Follow up:** Why do you think the graph should look like the way it does? P3: They may be thinking about two quantities to begin with. **Follow up:** Please explain what you think.	compare trace with what graphs they sketched. Their explanation will tell me if the moving segments changed how they conceived of quantities.
Could you express the graph as h equals f of d or d equals f of h? P1: No. Follow up: Why not? P1: I don't know the formula. R: Can you tell me what you think should be there? P1: I do not know. I am not sure, but $h = f(d)$ and $d = f(h)$ does not make sense. It needs to be $f(x) =$ something. Follow up: Could you tell me what you are expecting there? P1: Maybe $y =$ something. Follow up: is there a difference between y or $f(x)$? P1: No. $y = f(x)$.	The student may want to see something like $f(x) = 3x + 4$. The student may be expecting different variables (Wagner, 1981) where different variables mean different numbers. If the student says $y =$ something, maybe he/she is thinking of an equation and not being able to distinguish between the equation and function (Chazan & Yerushalmy, 2003). If the student says, $y = f(x)$ then $y = f(x)$ is just a label or they think that y equals $f(x)$. They may be using an equal sign as an equivalence (Knuth et al., 2006, 2011) Using Thompson (2013) definition of function notation to see what students think about the

I will ask about each variable in the notation h=f(d). So, what does h mean? What does f mean? What do we mean by d? A student may say I have no numbers and I don't know what to do. Follow up: Is it necessary to have numbers in $h = f(d)$? Does it mean anything without numbers? P2: $h = f(d)$ or $d = f(h)$ **Follow up:** So, does it matter how we write the notation? P3: $h = f(d)$ because distance keeps going and the height depends on it. When we know the distance, we can predict the height. **Follow up:** Is it possible to write this as $d = f(h)$? Why or why not? Explain. Does it matter if we write $h = f(d)$ or $f(d) = h$?	notation $h = f(d)$. **At the same time looking at how this task helped them to see the relationship between the quantities. (as I am asking about notation again after the tasks are complete).** I may ask them to pick a few numbers and tell me what that means. Then I will ask if they can think without numbers. The notation $h = f(d)$ is only meaningful if the unknown view is left behind and the variable view is made significant (Chazan; Usiskin). This would help me to know if their prior knowledge about the equations interferes with their function knowledge. This question is to see what students think about the notation if we switch $f(d)$ and h. If students are still thinking like they used to in algebra, somewhat similar to generalized numbers (x, y where $x + y = y + x$….. (Usiskin, 1988)
Pat said that the graph below can be written as either d=f(h) or h=f(d). What do you think?	

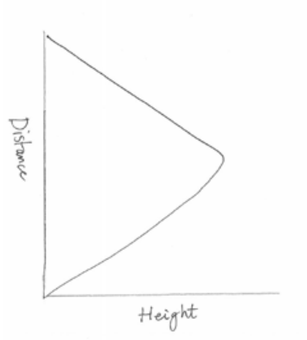
	
P1: The graph on the left can be written as h equals f of d and the graph on the right can be written as d equals f of h.	Student is paying attention to how the axes are labeled.
P2: Both graphs represent distance along the horizontal axis and we cannot interchange variables.	Student engaged in quantitative/covariational reasoning.
P3: Distance keeps going and the height varies. Also, each distance value corresponds to a height value, so h equals f of d is the correct notation.	Student has combined the covariational perspective to the correspondence approach to conceive of the notation.

APPENDIX C

Ferris Wheel Interview 2 Schedule

<table>
<tr>
<td>I will ask the students again to run the animation to see what is happening to the height and distance. . I will ask them again to explain to me how distance is changing and how height is changing.</td>
<td>This will serve as a quick overview of what they did in their previous interview (FW task 1)</td>
</tr>
<tr>
<td>I will ask the students to predict and then graph the relationship between distance and height represented on different axes (distance on y-axis and height on the x-axis). Students explain why their graph makes sense.
P1: The student may graph two separate graphs, one for distance and one for height.

Follow up: Please explain what you just sketched.
P2: The student may graph an iconic graph that looks like the Ferris wheel with axes labeled differently.
</td>
<td>Graphical representation will allow me to see how students interpret the two changing quantities and how they demonstrate their thinking in terms of the graph. I also want to know if representing attributes on different axes changed how they were thinking about the quantities. It is also quite possible that their graph does not represent how they actually think about quantities. This task will help me explore student's thinking.</td>
</tr>
</table>

Follow up: Please explain what you just sketched. P3: The student may sketch a graph that looks like a hill with axes labeled differently.  **Follow up:** Please explain what you just sketched.	
Could you express the graph as h equals f of d or d equals f of h? P1: My variable height was on the horizontal axis and distance was on the vertical axis, so $d = f(h)$. **Follow up:** Could you explain a little more.	Here a student is trying to match the axes with the notation and expects to see some consistency of the notation with the labels of axes. Student may **conceive of the notation and graph** where whatever is in the parentheses is always on the horizontal axis and whatever is on the other side of an equal side is always on the vertical axis.
P2: No. Follow up: Why not? P1: I do not know the formula. R: Can you tell me what you think should be there? P1: I do not know. I am not sure, but $h = f(d)$ and $d = f(h)$ does not make sense. It needs to be $f(x) =$ something.	The students may want to see something like $f(x) = 3x + 4$. The student may be expecting different variables (Wagner, 1981) where different variables mean different numbers.
Follow up: Could you tell me what you are expecting there? P1: Maybe $y =$ something. Follow up: is there a difference between y or	If the student says $y =$ something, maybe he/she is thinking of an equation and not being able to distinguish between the equation and function (Chazan & Yerushalmy, 2003).

$f(x)$? P1: No. $y = f(x)$.	If the student says, $y = f(x)$ then $y = f(x)$ is just a label or they think that y equals $f(x)$. They may be using an equal sign as an equivalence (Knuth et al., 2006, 2011)
I will ask about each variable in the notation h=f(d). So, what does h mean? What does f mean? What do we mean by d?	I am using Thompson (2013) definition of function notation to see what students think about the notation $h = f(d)$.
A student may say I have no numbers and I do not know what to do.	I may ask them to pick a few numbers and tell me what that means. Then I'll ask if they can think without numbers.
Follow up: Is it necessary to have numbers in $h = f(d)$? Does it mean anything without numbers?	The notation $h = f(d)$ is only meaningful if the unknown view is left behind and the variable view is made significant (Chazan; Usiskin). This would help me to know if their prior knowledge about the equations interferes with their function knowledge.
P2: $h = f(d)$ or $d = f(h)$ **Follow up:** So, does it matter how we write the notation? P3: $h = f(d)$ because distance keeps going and the height depends on it. When we know the distance, we can predict the height. **Follow up:** Is it possible to write this as $d = f(h)$? Why or why not? Explain.	
Does it matter if we write $h = f(d)$ or $f(d) = h$?	This question is to see what students think about the notation if we switch $f(d)$ and h. If students are still thinking like they used to in algebra, somewhat similar to generalized numbers (x, y where $x + y = y + x$….. (Usiskin, 1988)
Students will see the animated distance	Here the student will see the distance segment

segment along the y-axis, and they'll be asked to explain what the length on the y-axis means in terms of the Ferris wheel. P1: The segment keeps going like the distance. **Follow up:** How is the distance changing? P2: I do not know. **Prompt:** Which quantity is represented by this segment? **Follow up:** How is this quantity changing? P3: Distance, but it was height before. **Follow up:** Do you think the segment should behave differently? Are you thinking about the graph?	alone. This will help me to know how students think about the distance represented by a segment. This will also tell me if students are bothered by the segment because a graph represents two quantities and the students are familiar with graphs of two quantities and they may wonder why only this segment along the vertical axis? The student may be bothered that we now have distance along the y-axis instead of height. The student may say that the distance will go up and down just because they had height along the vertical axis before.
Then students see the height along the x-axis, and they'll be asked to explain what the length on the x-axis means in terms of the Ferris wheel. P1: The segment goes up and then comes back like the height. **Follow up:** How is the height changing? What is height doing? P2: I don't know. Prompt: Which quantity is represented by this segment? **Follow up**: How is this quantity changing? P3: Height, but I'm not sure why this is on the x-axis? **Follow up:** Do you think the segment should behave differently? Are you thinking about the graph?	Here the student will see the height segment alone. This will help me to know how students think about the height represented by a segment. This will also tell me if students are bothered by the segment because a graph represents two quantities and the students are familiar with graphs of two quantities and they may wonder why only this segment along the horizontal axis? The student may be bothered that we now have height along the horizontal axis instead of distance. The student may say that the height keeps going just because they had distance along the horizontal axis before and they may still be thinking about the distance.
Show them animation of both distance and height segments together. Ask them how are both distance & height changing together?	Here I will ask them about both distance and height segments together. By seeing the animated segments that change together may help them to think about two changing quantities.

<table>
<tr><td>Press Show Ferris wheel, then Press Show trace. Tell students to watch the point and trace changing. Ask students what the trace means in terms of this situation.
Ask students to compare their graph predictions to what they are seeing now.

P1: They may not change their graph at all.

Prompt: How does this trace compare to what you sketched before?

Student may want to change what they graphed.

P2: They may be thinking about one quantity before but may shift to two quantities and may want to change their graph.
Follow up: Why do you think the graph should look like the way it does?

P3: They may be thinking about two quantities to begin with.
Follow up: Please explain what you think.</td><td>Here students are going to explain and compare trace with what graphs they sketched. Their explanation will tell me if the moving segments changed how they conceived of quantities.</td></tr>
<tr><td>Could you express the graph as h equals f of d or d equals f of h?
P1: No.
Follow up: Why not?
P1: I don’t know the formula.
R: Can you tell me what you think should be there?
P1: I do not know. I am not sure, but $h = f(d)$ and $d = f(h)$ does not make sense. It needs to be $f(x) =$ something.
Follow up: Could you tell me what you are expecting there?
P1: Maybe $y =$ something.

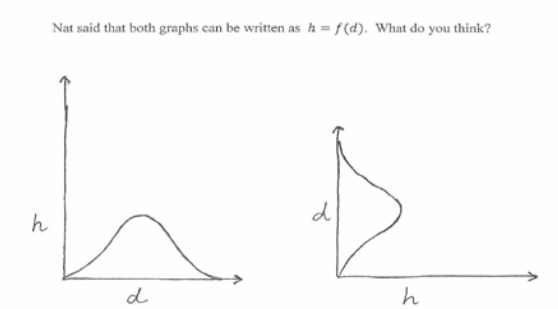
Follow up: is there a difference between y or $f(x)$?</td><td>The student may want to see something like $f(x) = 3x + 4$.

The student may be expecting different variables (Wagner, 1981) where different variables mean different numbers.

If the student says $y =$ something, maybe he/she is thinking of an equation and not being able to distinguish between the equation and function (Chazan & Yerushalmy, 2003).

If the student says, $y = f(x)$ then $y = f(x)$ is just a label or they think that y equals $f(x)$.</td></tr>
</table>

P1: No. $y\ = f(x)$. I will ask about each variable in the notation. what does he mean? What does f mean? What do we mean by d?	They may be using an equal sign as an equivalence (Knuth et al., 2006, 2011) I am using Thompson (2013) definition of function notation to see what students think about the notation $h\ = f(d)$. **At the same time looking at how this task helped them to see the relationship between the quantities. (as I am asking about notation again after the tasks are complete).**
A student may say I have no numbers and I don't know what to do.	I may ask them to pick a few numbers and tell me what that means. Then I will ask if they can think without numbers.
Follow up: Is it necessary to have numbers in $h = f(d)$? Does it mean anything without numbers? P2: $h = f(d)$ or $d = f(h)$ **Follow up:** So, does it matter how we write the notation? P3: $h = f(d)$ because distance keeps going and the height depends on it. When we know the distance, we can predict the height. **Follow up:** Is it possible to write this as $d = f(h)$? Why or why not? Explain.	The notation $h = f(d)$ is only meaningful if the unknown view is left behind and the variable view is made significant (Chazan; Usiskin). This would help me to know if their prior knowledge about the equations interferes with their function knowledge.
Does it matter if we write $h = f(d)$ or $f(d)\ = h$?	This question is to see what students think about the notation if we switch $f(d)$ and h. If students are still thinking like they used to in algebra, somewhat similar to generalized numbers (x, y where $x + y\ =\ y + x$….. (Usiskin, 1988)
P2: The student may say $d = f\ (h)$ because we labeled the axes differently. Height is now along the horizontal axis and distance is along the vertical axis. **Follow up:** Could you please explain this by pointing to your graph?	Students may say $d = f\ (h)$ because labeling the axes differently changes the notation (just like changing alphabets gives a different number for an equation (Wagner, 1981).

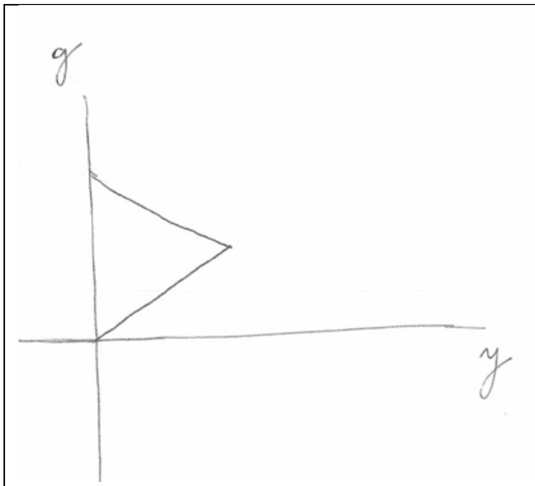
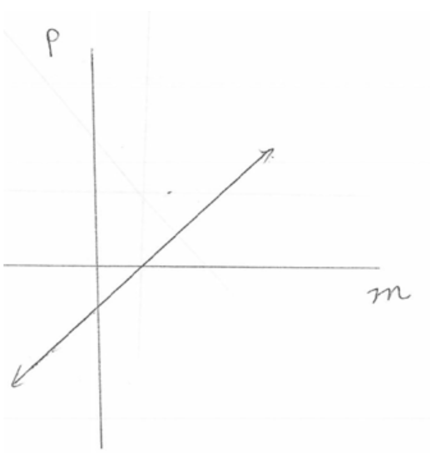
P3: A student may say $h = f\ (d)$. **Follow up:** Why do you think that? Explain.	It is possible that the student may be thinking about two quantities and notice that the relationship between two quantities stays the same so ($h = f(d)$). This task (representing same attributes on different axes) would allow me to explore how students are connecting quantities to the notation.
Nat said that both graphs can be written as h =f(d). what do you think?  P1: The graph on the left can be written as h equals f of d and the graph on the right can be written as d equals f of h. P2: The graph on the right cannot be written as d equals f of h, because one height input corresponds to two different distance outputs, so the graph does not represent a function. This graph does not pass the vertical line test. P3: Both graphs represent distance increasing, one along the horizontal axis, and the other along the vertical axis. The height increases and decreases while the distance keeps increasing. Each distance value corresponds to a height, so h equals f of d is the correct notation.	Student is paying attention to how the axes are labeled. Student engaged in quantitative/covariational reasoning and also used a correspondence approach (Smith, 2003).

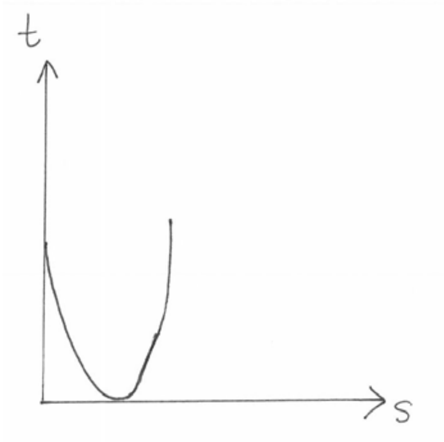
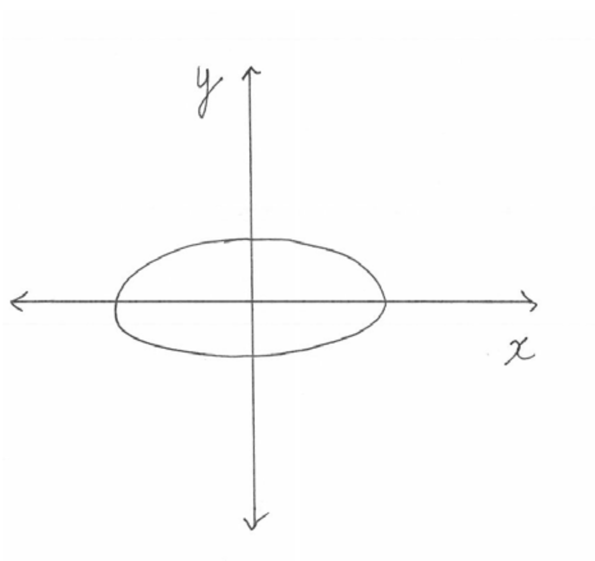
APPENDIX D

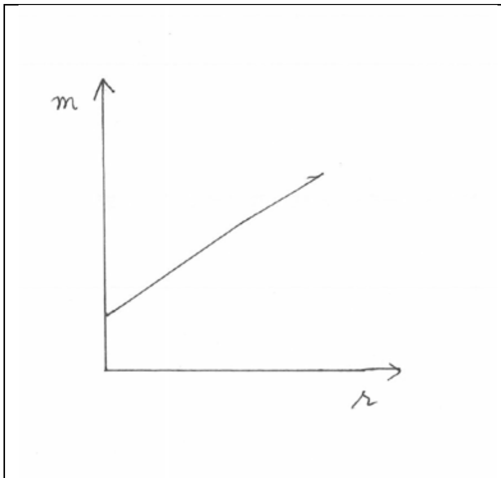
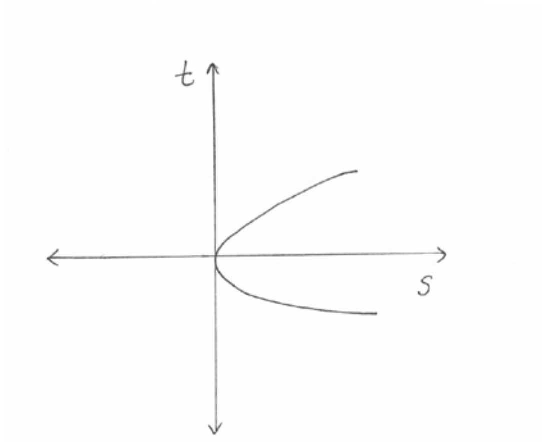
Post Interview Schedule

1. What comes to your mind when you think of a function? P1: I don't know. Follow up: Would it help if I gave you something more specific? Sometimes in math, we have graphs, tables, and equations. Would any of these help? P1: yes. Prompt: 2. Please read each statement out loud and explain what each statement means. I will also provide tables and graphs and ask them if the tables/graphs helped to clarify the statements.	This question is to know what students think about functions and what do they mean by a function and general function notation. Students could define the function as a correspondence (For every x, there is an output y). They may define a function using a graphical representation or a symbolic representation. Students often think a function must be defined by a single algebraic formula (Carlson, 1998; Clement, 2001; Even, 1990; Even, 1993; Sierpinska, 1992). The students may graph a function and write a notation. They may pick points or just graph a function without picking numbers.
Given $g = f(y)$, for every input y , there is exactly one output g .	A student may not like different variables and may want to convert to the standard y equals f of x form and then interpret the rules.
Given $x = t(y)$, for every input y, x is the output.	Here the notation is different and participants may think that this is not a function due to the notation $x = t(y)$.
Given $m = r(y)$, as y increases, m decreases. Given $y = g(p)$, as p increases, y increases and then decreases.	These are definitions of function from a covariation perspective. Students are less familiar with this definition and may not consider these as function definitions. Students may express each statement graphically and try to match the variable on the parentheses to match with the variable on the horizontal axis and the variable on the other side of the equal sign to be on the vertical axis. Students may look for an expression because students often think a function must be defined by a single algebraic formula (Carlson, 1998; Clement,

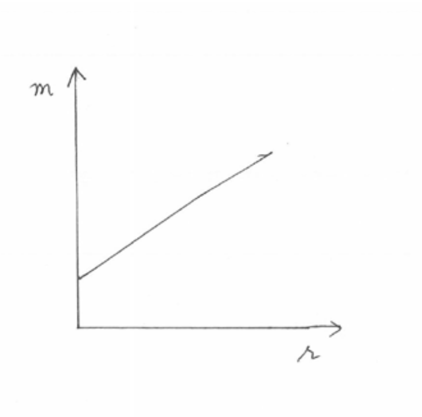
3. Do these represent functions? The following table defines y as a function of x, denoted $y = f(x)$. <table><tr><td>x</td><td>-5</td><td>-1</td><td>0</td><td>1</td><td>4</td></tr><tr><td>y</td><td>3</td><td>2</td><td>-6</td><td>4</td><td>7</td></tr></table> The following table defines x as a function of y, denoted $x = f(y)$. <table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>-2</td></tr><tr><td>y</td><td>3</td><td>2</td><td>-6</td><td>3</td><td>4</td></tr></table> P1: same input of -2 gives outputs of 3 and 4. P2: Two different inputs of 3 and 4 map to the same output of -2, which does not represent a function.	2001; Even, 1990; Even, 1993; Sierpinska, 1992). Students are familiar with tables and a student may say this is the definition of function because it is given as $y = f(x)$. It is also possible that students may say this is a function because it is one-to-one. A student may still consider y as a function of x and say that the same input of -2 gives two outputs 3 and 4. I will prompt them to read the statement first. Students may conceive of one-to-one functions as functions and may not consider this onto function as a function.

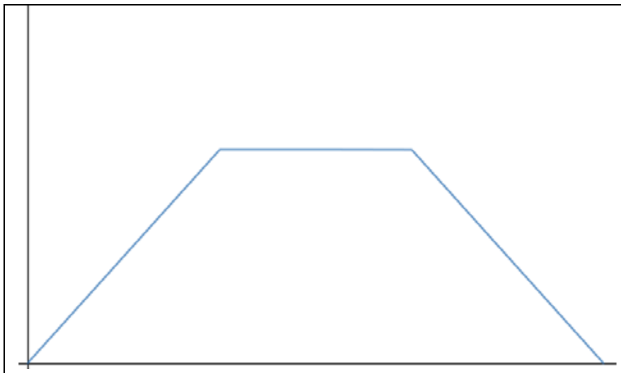
 P1: The graph does not pass a vertical line test so it does not represent a function. P2: One input maps to two different outputs, so it does not represent a function.	The student may use the vertical line test or the correspondence (Smith, 2003) approach to decide that the graph does not represent a function. If we think about quantitatively, then this graph represents a function, where y is a function of g. The goal of this task is to see how students conceive of a function and if they engage in quantitative reasoning specifically for this graph.
 P1: The graph passes a vertical line test, so it represents a function. P2: One input has an output, so it represents a function.	Student may use the vertical line test to decide if the graph represents a function.

	Student may think that a one-to-one graph represents a function.
4. What does $g = r(m)$ mean? How do you make sense of it? P1: This is like y equals f of x but with different letters.	I will ask the student to explain what each variable means to them. I may ask them to graph the rule to know how they connect the rule to the graph.
5.Given a set of 4 graphs, which are functions? Which are not? why?   	This question will help me to know how students conceive of graphs that represent functions and graphs that do not represent functions.

  P1: A student may say that the parabola is a function because it passes the vertical line test and the linear graph is a function because all linear graphs are functions. Follow up: Could you explain why? P1: An ellipse is not a function and the parabola that opens sideways is not a function because these graphs do not pass the vertical line test. Follow up: Could you explain a different way (without the vertical line test) why this does not represent a function.	If we think about quantitatively, then this graph represents a function, where s is a function of t. The goal of this task is to see how students conceive of a function and if they engage in quantitative reasoning specifically for this graph. Here I expect them to explain that one input is giving two different outputs, and therefore it does not represent a function.
6. Can you use any of these formulas to describe the graphs presented in 2? You can	

use formulas more than once or not at all. $s = h(t)$ $t = h(s)$ $y = p(x)$ $x = p(y)$ $m = s(r)$ $r = s(m)$	
P1: A student may say that an ellipse can be defined as $y = f(x)$.	A student may say that an ellipse can be defined as $y = p(x)$ because the horizontal axis is labeled x and the vertical axis as y. (Even though an ellipse is not a function).
P2: A student may say $t = h(s)$ for the first graph because the the horizontal axis is labeled s and the vertical axis as t.	Student may match the variable in the parentheses to the variable in the notation.
P3: A student may say $m = s(r)$ for the linear graph because the horizontal axis is labeled r and the vertical axis as m.	I have chosen a linear graph because it is one-to-one. The students are most familiar with linear graphs, and most real-life situations can be translated to linear graphs. Moreover, there is less complexity in terms of the notation. For linear functions, we can express the situations as either $y = f(x)$ or $x = f(y)$. We can know the value of x if we know y and we can know the value of y if we know x. (Every x has an output y). So, for a linear graph, both $m = s(r)$ and $r = s(m)$ are okay. From a quantitative reasoning perspective, input can be on any axis as long as it satisfies the definition of function.
P4: A student may say that for the last graph $t = h(s)$ because the horizontal axis is labeled s and the vertical axis as t.	This tells me that the labels of axes matter to students when they think of a notation. However, $s = h(t)$ for the last graph because s is a function of t. Here t is the input variable and s is the output variable. From a quantitative reasoning perspective, input can be

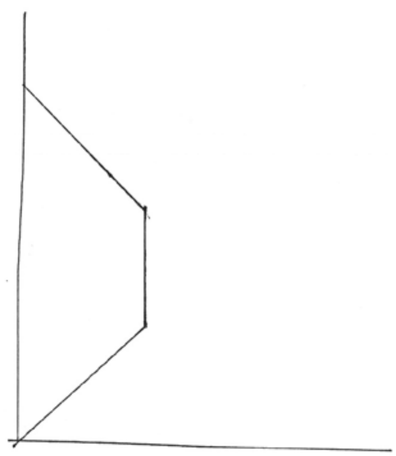
	on any axis as long as it satisfies the definition of function.
7. Max said that both $m = s(r)$ and $r = s(m)$ can be used to describe the following graph. Why that made sense to that person? What do you think? 	
P1: $m = s(r)$ is true because the horizontal axis is given by the variable r and it is the independent variable.	Student matched the variable on the horizontal axis to the variable in the parentheses and the variable on the vertical axis to the variable on the other side of equal sign to conceive of the notation. This is how students learn in school.
P2: As r increases, m increases and as m increases, r increases, so both notations can be used.	Thinking covariationally, as one quantity increases the other also increases, so both notations can be used. A student may combine covariational reasoning to the correspondence approach and say that one input has one output and as one quantity increases, the other also increases and therefore we can write notation both ways.
8. Given the situation below, interpret the graph. a) Suppose that an airplane takes off from Denver International Airport. As the plane covers the distance along the ground, its altitude changes. Here is a graph representing the distance along the ground and the altitude of the airplane. Please interpret the graph.	

<table>
<tr>
<td>

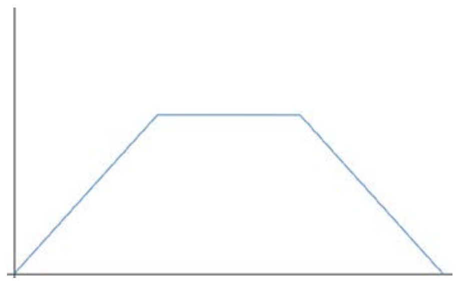
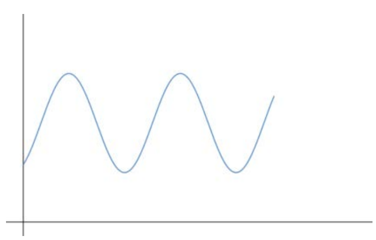
P1: I don’t know. It looks like a hill but flat in the middle.

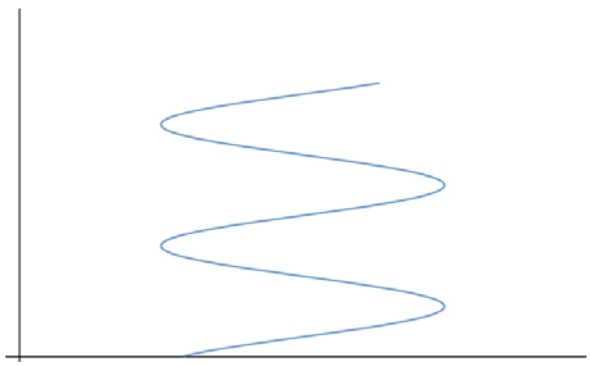
Prompt: Could you please tell me how the distance is changing?

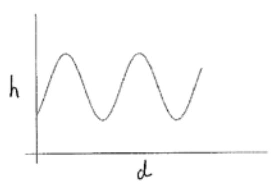
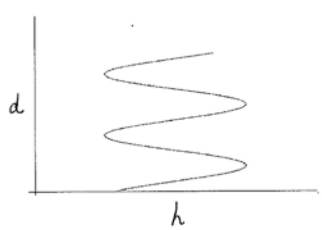
How is the altitude changing?

How are both distance and altitude changing together?</td>
<td>This tells me that the student is paying attention to the overall shape of the graph. I will prompt the student to know how he /she thinks about the quantities, distance and altitude (separately and changing together)</td>
</tr>
<tr>
<td>b) We have the same situation but the attributes are on different axes. Please interpret the graph.

P1: I don’t know. It looks like a hill, flat in the middle and is sideways.</td>
<td>This tells me that the student is looking at the physical object.</td>
</tr>
</table>

Prompt: Could you please tell me how the distance is changing? How is the altitude changing? How are both distance and altitude changing together?	These prompts will help me to know what students think about the quantities separately and also changing together (covariation). It is possible that the student may be thinking about two quantities and notice that the relationship between two quantities stays the same. This task (representing same attributes on different axes) would allow me to explore how students are conceiving of two quantities.
Is it possible to write the situations above as $a = f(d), d = f(a)$? P1: A student may say that 8a) can be written as $a = f(d)$ and 8b) as $d = f(a)$. Please explain why you think that?	This tells me that the student wants to match the notation with how the axes are labeled. If we think about the notation, we cannot say that both $x = f(y)$ and $y = f(x)$. For example, if we have the distance along the x-axis and altitude along the y-axis, then we can say that $a = f\ (d)$, but we cannot say that $d = f\ (a)$ (with d along the x-axis and a along the y) because one height corresponds to two different distances and is therefore not a function. However, if we switch the axes, we can still say that $a = f(d)$ (with a on x-axis and distance on the y-axis). Students may say $d = f\ (a)$ because labeling the axes differently changes the notation (just like changing alphabets gives a different number for an equation (Wagner, 1981).
P2: A student may say $a = f(d), d = f(a)$ does not make sense because there is no formula. R: Can you tell me what you think should be there? P: I don't know. Maybe $f(x) = something$.	The student may want to see something like $f(x) = 3x + 4$. This question will help me to know if students relate notation to axes or if they relate quantities to notation, or if thinking about notation is separate than thinking about the quantities.

9. Chris said that for the first situation, the graph can be written as both $a = f(d)$ and $d = f(a)$. What do you think?  P1: With distance on the horizontal axis, one d corresponds to a height, so h equals f of d. One h corresponds to two different d values, so d equals f of h is not true.	The student used a correspondence approach to justify the notation.
10. Given the situation below, interpret the graph. b) Suppose that a child has been swinging on a swing for some time. Here is a graph representing the total distance traveled and the height of the swing. Please interpret the graph.  P1: I don't know. It looks like a hill. Prompt: Could you please tell me how the distance is changing? How is the height changing? How are both distance and height changing together?	This tells me that the student is paying attention to the overall shape of the graph. I will prompt the student to know how he /she thinks about the quantities, distance and height (separately and changing together)

b) We have the same situation, but the attributes are on different axes. Please interpret the graph.  P1: I don't know. It looks like a hill that is sideways. Prompt: Could you please tell me how the distance is changing? How is the height changing? How are both distance and height changing together?	This tells me that the student is looking at the physical object. These prompts will help me to know what students think about the quantities separately and also changing together (covariation). It is possible that the student may be thinking about two quantities and notice that the relationship between two quantities stays the same. This task (representing same attributes on different axes) would allow me to explore how students are conceiving of two quantities.
Is it possible to write the situations (in 10 above) as $h = f(d)$, $d = f(h)$? P1: A student may say that 10a) can be written as $h = f(d)$ and 11b) as $d = f(h)$. Please explain why you think that?	This tells me that the student wants to match the notation with how the axes are labeled. If we think about the notation, we cannot say that both $x = f(y)$ and $y = f(x)$. For example, if we have the distance along the x-axis and altitude along the y-axis, then we can say that $h = f(d)$, but we cannot say that $d = f(h)$ (with d along the x-axis and h along the y) because one altitude corresponds

	to two different distances and is therefore not a function. However, if we switch the axes, we can still say that $h = f(d)$ (with h on x-axis and distance on the y-axis). Students may say $d = f\ (h)$ because labeling the axes differently changes the notation (just like changing alphabets gives a different number for an equation (Wagner, 1981).
P2: A student may say $h = f(d)$, $d = f(h)$ does not make sense because there is no formula. R: Can you tell me what you think should be there? P: I don't know. Maybe $f(x) = something$.	The student may want to see something like $f(x) = 3x + 4$. This question will help me to know if students relate notation to axes or if they relate quantities to notation, or if thinking about notation is separate than thinking about the quantities.
11. For the swing situation, Sam said that both graphs can be written as $h = f(d)$. What do you think?   P1: The graph in the first situation can be written as h equals f of d and the graph with distance along the vertical axis can be written as d equals f of h.	 Student is paying attention to the axes labels.

P2: Both graphs represent distance increasing, one along the horizontal axis and the other along the vertical axis, so h depends on d. h equals a function of d.	Distance depends on height regardless of what axis it is on. Student engaged in quantitative reasoning.
P3: h depends on d and each distance value corresponds to a height, so h equals f of d is the correct notation.	Thinking quantitatively, in both cases h depends on d and also using a correspondence approach (Smith, 2003) h equals f of d is the correct notation. In other words, the student may combine the covariational perspective to the correspondence approach.

APPENDIX E

Pre Interview Questionnaire (with additional questions)

1. What comes to your mind when you think of a function?

2. Please read each statement out loud and explain what each statement means.

- Given $y = f(x)$, for every input x , there is exactly one output y .
- Given $x = f(y)$, for every input y, there is exactly one output x .
- Given $g = r(y)$, as y increases, g decreases.
- Given $y = g(r)$, as r increases, y increases and then decreases.

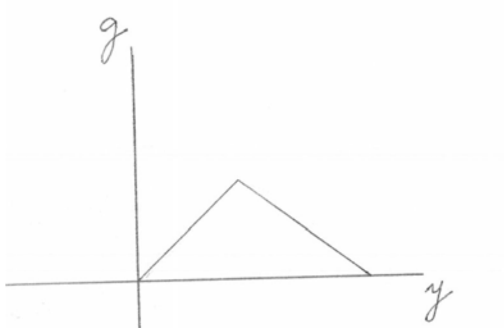
3. Do these represent functions?

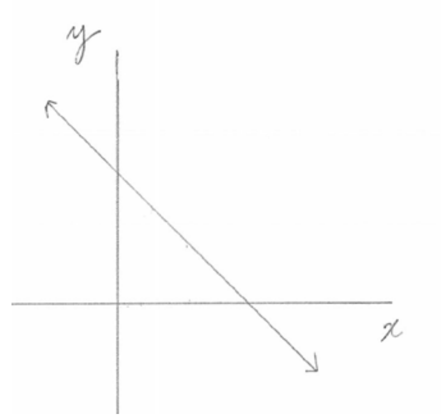
The following table defines y as a function of x, denoted $y = f(x)$.

x	-2	-1	0	1	3	4
y	8	2	-3	4	2	7

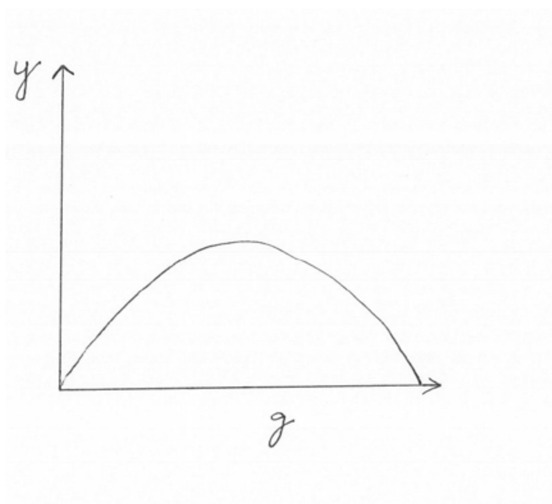
The following table defines x as a function of y, denoted $x = f(y)$.

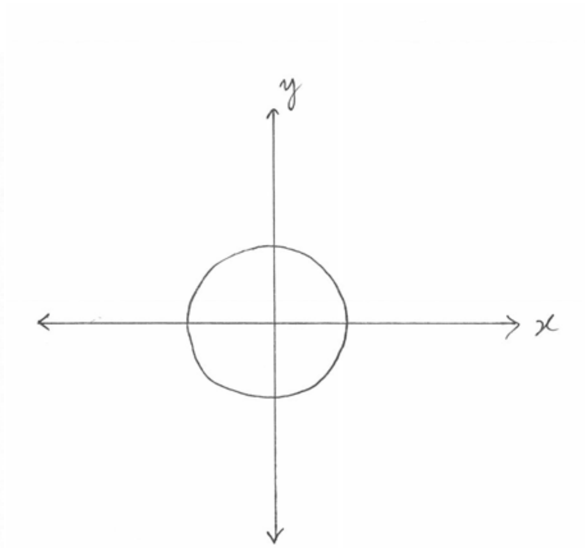
x	-3	-3	2	0
y	1	2	3	5

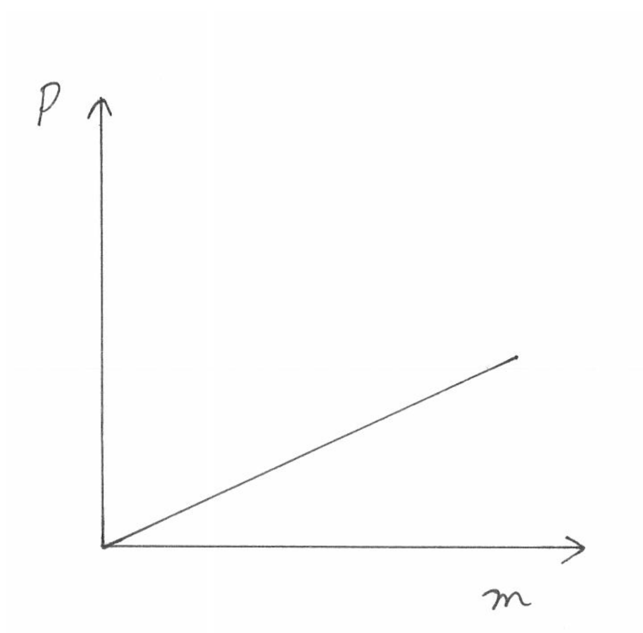


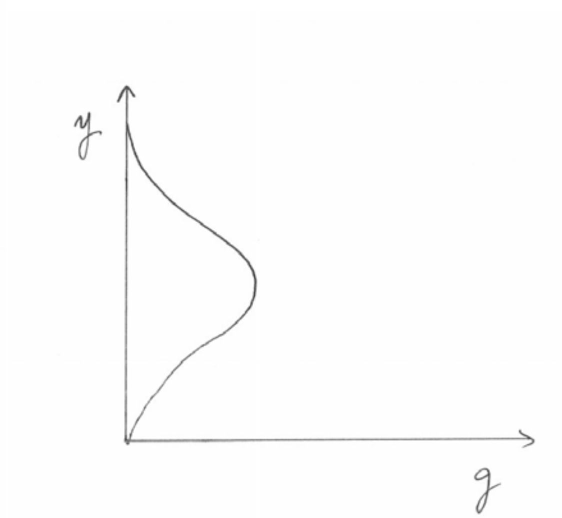


4. What does $u = r(s)$ mean? How do you make sense of it?
5. Given a set of 4 graphs, which represent functions? Which do not? why?





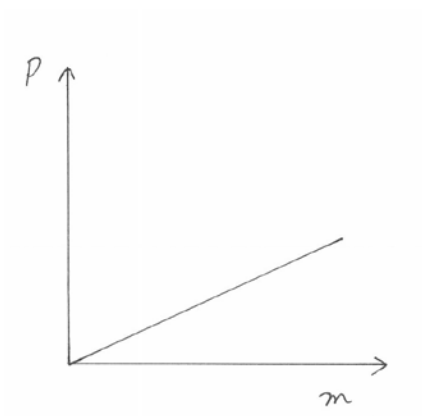




6. Can you use any of these formulas to describe the graphs presented in 2? You can use formulas more than once or not at all.

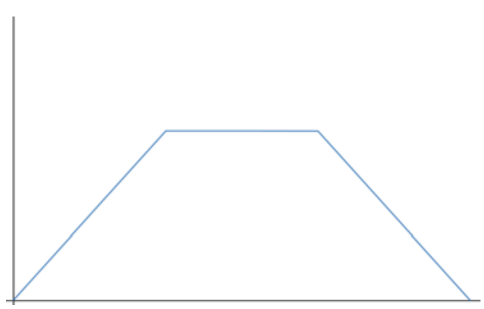
$$g = r(y)$$
$$y = r(g)$$
$$y = f(x)$$
$$x = f(y)$$
$$m = t(p)$$
$$p = t(m)$$

7. Sam said that both $m = t(p)$ and $p = t(m)$ can be used to describe the following graph. Why that made sense to that person? What do you think?

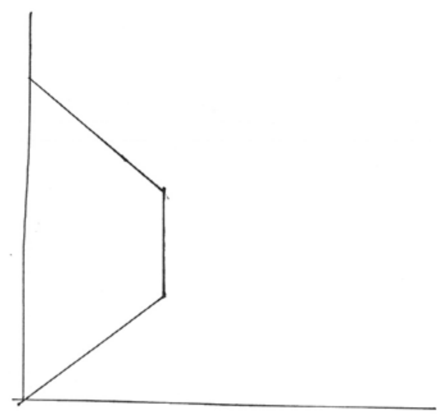


8. Given the situation below, interpret the graph.

a) Suppose that an airplane takes off from Denver International Airport. As the plane covers the distance along the ground, its altitude changes. Here is a graph representing the distance along the ground and the altitude of the airplane. Please interpret the graph.

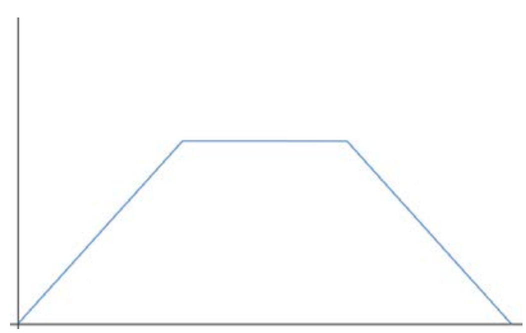


b) We have the same situation but the attributes are on different axes. Please interpret the graph.



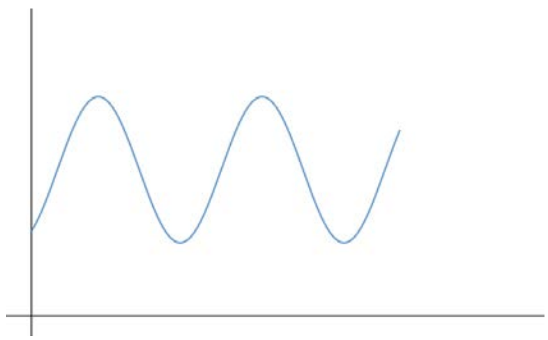
9. Is it possible to write the situations (in 8 above) as $a = f(d)$, $d = f(a)$?

10. Nat said that for the first situation, the graph can be written as both $a = f(d)$ and $d = f(a)$. What do you think?

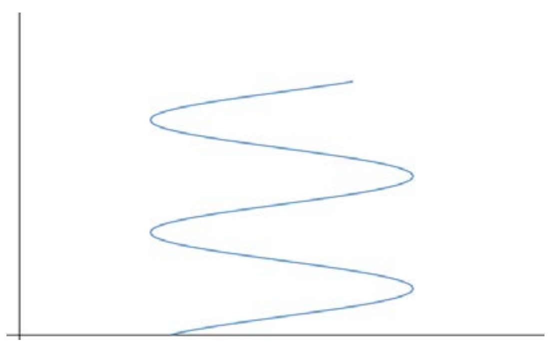


11. Given each situation below, interpret the graph.

a) Suppose that a child has been swinging on a swing for some time. Here is a graph representing the total distance traveled and the height of the swing. Please interpret the graph.

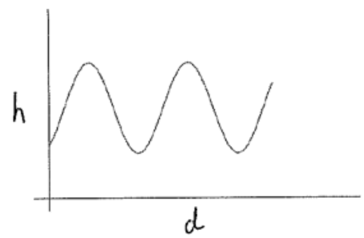


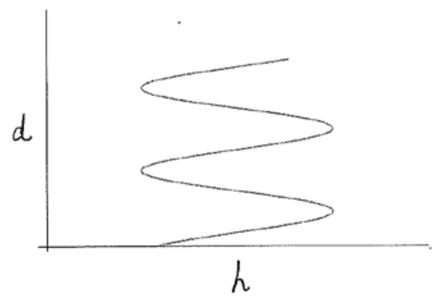
b) We have the same situation, but the attributes are on different axes. Please interpret the graph.



12. Is it possible to write the situations (in 6 above) as $h = f(d)$, $d = f(h)$?

13. For the swing situation, Pat said that both graphs can be written as $h = f(d)$. What do you think?





APPENDIX F

Ferris wheel Interview 1 Questionnaire

Introduction:
Have students talk about their experience of being on a ferris wheel.
Identifying Changing Quantities
Students will see ferris wheel without any measurements.
How is the distance changing?
How is the height changing?
The students will then be asked to run the animation to see what's happening with the height and distance. They will again be asked to explain how distance is changing and how height is changing.
Predicting Relationships between Quantities

Please predict and then graph the relationship between distance and height. Please label the axes. Students explain why their graph makes sense.

Investigating Function Notation
Ask about $h = f(d)$ or $= f(h)$?

Investigating Changing Quantities

Students see the Ferris wheel. Students click on animate point, then show them the distance segment along the x-axis. Ask students the following questions:

- Explain what the length on the x-axis means in terms of the Ferris wheel.
- How is the distance changing?
- Why does it make sense that the distance around a circle could be represented by a line segment?

Have students hide distance, then show height. Ask students the following questions:
- Explain what the length on the y-axis means in terms of the Ferris wheel.
- How is the height changing?
- What is height doing?

Forming Relationships between Changing Quantities

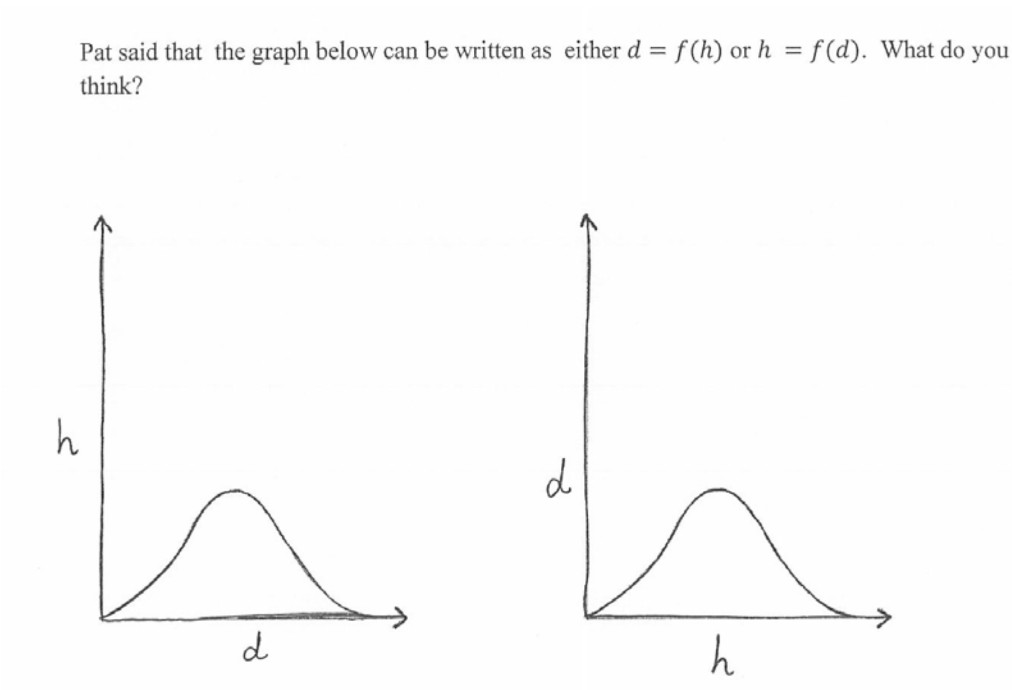
Have students show both distance and height.
Press Animate Point. How are both distance & height changing together?

Press Show ferris wheel, then Press Show trace. Tell students to watch the point and trace changing. Ask students what the trace means in terms of this situation. Ask students to compare their graph predictions to what they are seeing now. Ask them if it makes sense.

Investigating Function Notation

Ask about $h\ = f(d)$ or $= f(h)$?

Ask about a student Pat. Pat said that the graph below can be written as either d equals f of h or h equals f of d. What do you think?



APPENDIX G

Ferris Wheel Interview 2 Questionnaire (distance on the vertical axis, and height on the horizontal axis)
Show picture of the Ferris wheel and ask students how the distance and height are changing.
Ask students how they labeled the axes before. Now, tell students that they are going to sketch a graph with distance on the vertical axis, and height on the horizontal axis.

Investigating Function Notation
Ask about the notation $h = f(d)$ or $d = f(h)$? Why?
Investigating Changing Quantities

Students see the Ferris wheel. Students click on animate point, then show them the distance segment along the y-axis. Ask students the following questions:

- Explain what the length on the y-axis means in terms of the Ferris wheel.
- How is the distance changing?
- Why does it make sense that the distance around a circle could be represented by a line segment?

Have students hide distance, then show height. Ask students the following questions:
- Explain what the length on the x-axis means in terms of the Ferris wheel.
- How is the height changing?
- What is height doing?

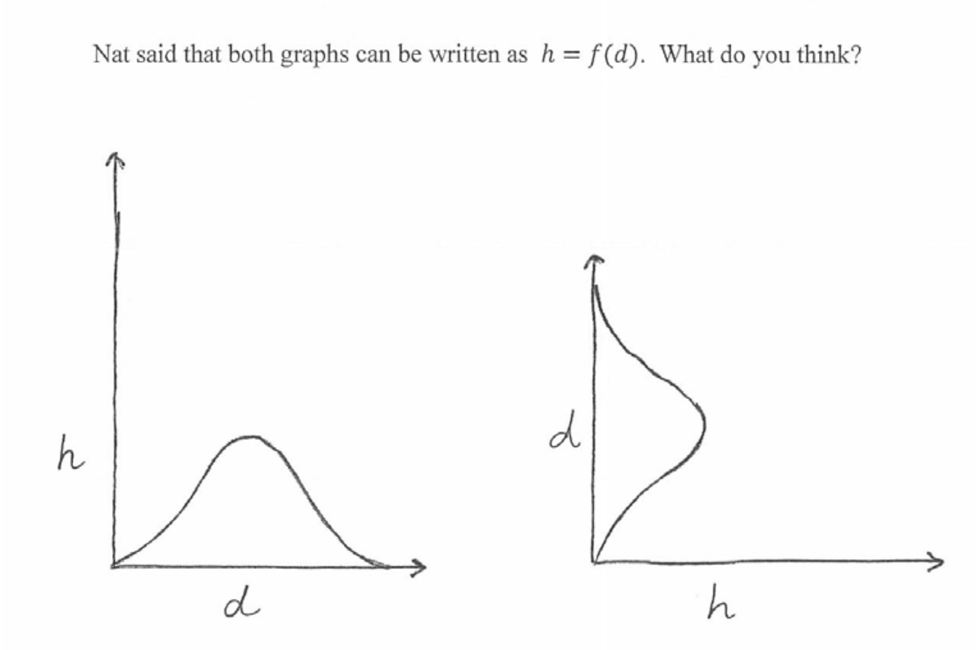
Forming Relationships between Changing Quantities

Have students show both distance and height.
Press Animate Point. How are both distance & height changing together?

Press Show Ferris wheel, then Press Show trace. Tell students to watch the point and trace changing. Ask students what the trace means in terms of this situation.
Ask students to compare their graph predictions to what they are seeing now.
Ask them if it makes sense.

Investigating Function Notation
Ask about the notation $h = f(d)$ or $d = f(h)$? Why?

Ask about a student Nat. Nat said that both graphs could be written as h equals f of d. What do you think?



APPENDIX H

Post Interview Questionnaire (with additional questions)

1. What comes to your mind when you think of a function?

2. Please read each statement out loud and explain what each statement means.

- Given $g = f(y)$, for every input y , there is exactly one output g .
- Given $x = t(y)$, for every input y, x is the output.
- Given $m = r(y)$, as y increases, m decreases.
- Given $y = g(p)$, as p increases, y increases and then decreases.

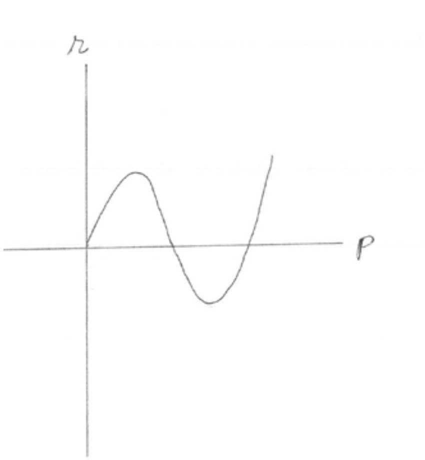
3. Do these represent functions?

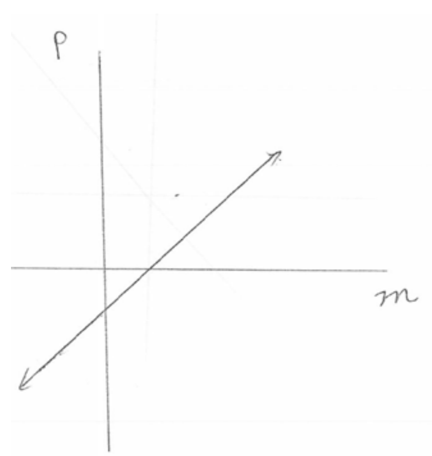
The following table defines y as a function of x, denoted $y = f(x)$.

x	-5	-1	0	1	4
y	3	2	-6	4	7

The following table defines x as a function of y, denoted $x = f(y)$.

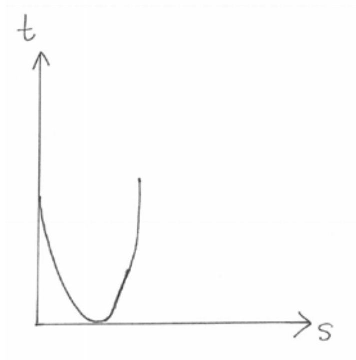
x	-2	-1	0	1	-2
y	3	2	-6	3	4

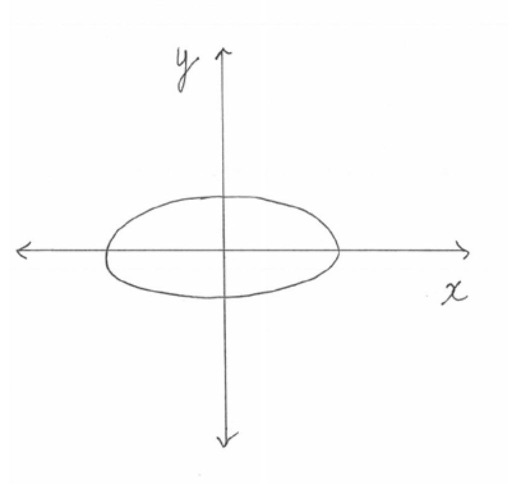


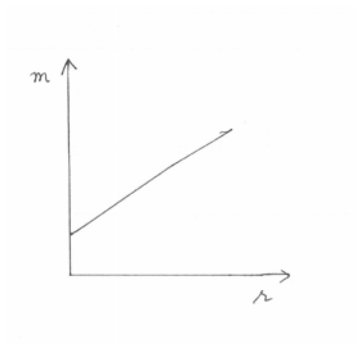


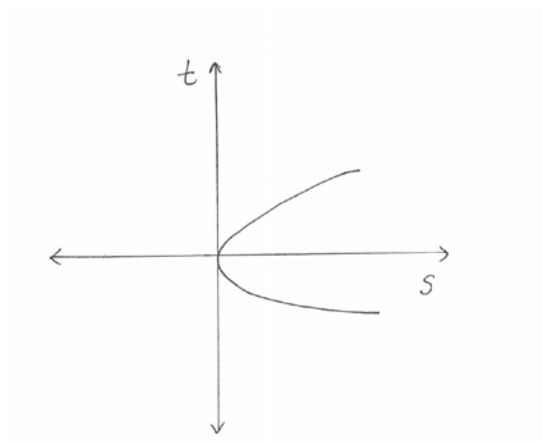
4. What does $g = r(m)$ mean? How do you make sense of it?

5. Given a set of 4 graphs, which are functions? Which are not? why?





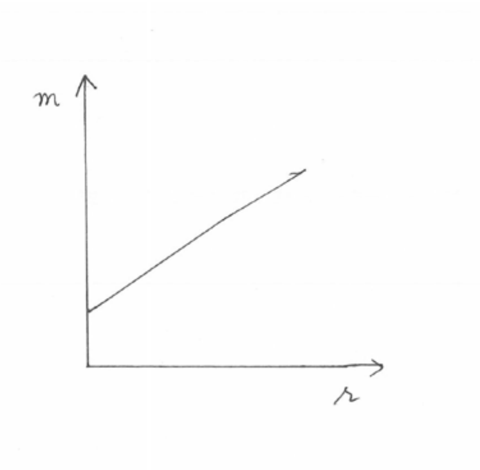




6. Can you use any of these formulas to describe the graphs presented in 2? You can use formulas more than once or not at all.

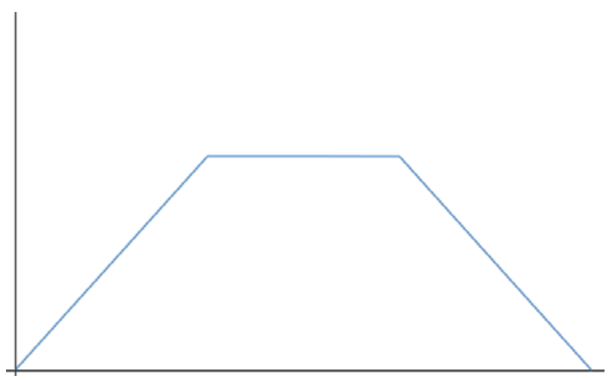
$$s = h(t)$$
$$t = h(s)$$
$$y = p(x)$$
$$x = p(y)$$
$$m = s(r)$$
$$r = s(m)$$

7. Max said that both $m = s(r)$ and $r = s(m)$ can be used to describe the following graph. Why that made sense to that person? What do you think?

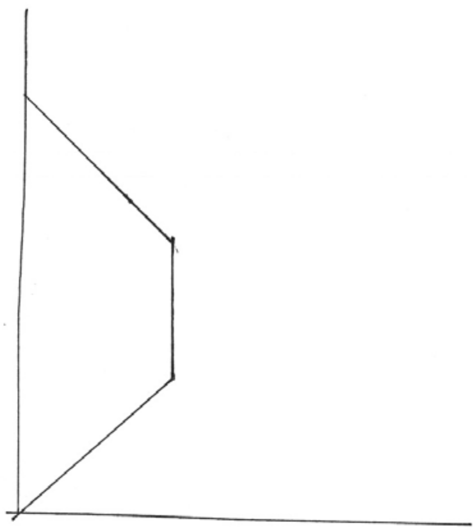


8. Given the situation below, interpret the graph.

a) Suppose that an airplane takes off from Denver International Airport. As the plane covers the distance along the ground, its altitude changes. Here is a graph representing the distance along the ground and the altitude of the airplane. Please interpret the graph.

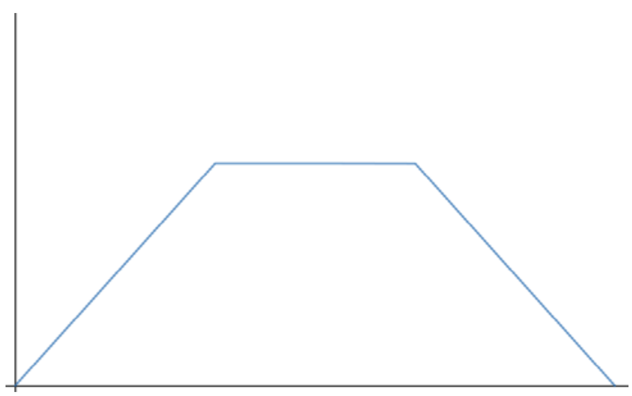


b) We have the same situation, but the attributes are on different axes. Please interpret the graph.



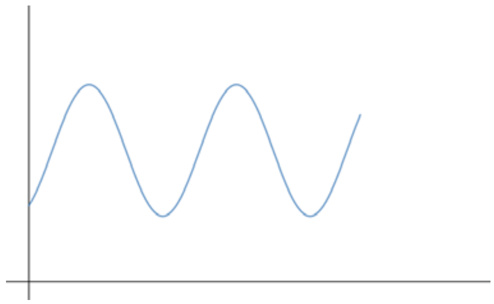
9. Is it possible to write the situations (in 8 above) as $a = f(d)$, $d = f(a)$?

10. Chris said that for the first situation, the graph can be written as both $a = f(d)$ and $d = f(a)$. What do you think?

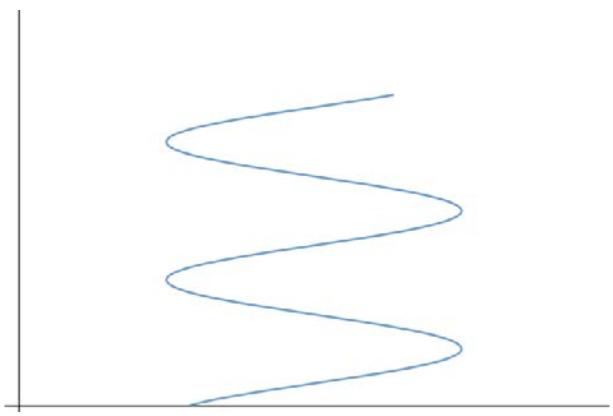


11. Given each situation below, interpret the graph.

a) Suppose that a child has been swinging on a swing for some time. Here is a graph representing the total distance traveled and the height of the swing. Please interpret the graph.



b) We have the same situation, but the attributes are on different axes. Please interpret the graph.



12. Is it possible to write the situations (in 11 above) as $h = f(d)$, $d = f(h)$?

13. For the swing situation, Sam said that both graphs can be written as $h = f(d)$. What do you think?