MANIFESTATIONS OF ELEMENTARY MATHEMATICS TEACHERS' SHIFT TOWARDS SECOND-ORDER MODELS

by

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Manifestations of Elementary Mathematics Teachers' Shift Towards Second-Order Models

Thesis directed by Professor Ron Tzur.

ABSTRACT

This study examined a shift in teachers' explanations of their students' mathematical reasoning, from being based primarily on the teachers' own mathematical knowing (first-order model, abbreviated FOM) to also attributing mathematical knowing that differs from that of the teacher (second-order model, abbreviated SOM, Steffe, 2000). In particular, the following two research questions were addressed:

(1) What changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift from mostly relying on their first-order model to teach mathematics?

(2) What may be manifested in elementary mathematics teachers' work and explanations, as they shift from using only first-order models towards differentiating between their first-order model and students' mathematical reasoning?

Results indicated four manifestations of teachers' shift towards SOM: (a) Juxtaposition of Thinking, which refers to the teacher's experience of contrast between what she or he believes the mathematics to be and what the teacher interprets as the students' reasoning; (b) Cogitation, which refers to a teacher's deepening ability to think about students' mathematical reasoning; (c) Distinction, which refers to a teacher's enhanced ability to depict student mathematical reasoning, and (d) Mindfulness, which refers to a teacher's growing intention to use interpretations of students' reasoning to facilitate instruction. I discuss important implications of the four manifestations for fostering a shift in teachers' perspective on knowing and learning

(Simon, Tzur, Heinz, & Kinzel, 2004) and Student-Adaptive Pedagogy (Steffe, 1990; Tzur, 2013), additional facets of what has been termed SOM, and similarities and differences in researchers' SOM versus teachers' shift towards SOM.

The form and content of this abstract are approved. I recommend its publication.

Approved: Ron Tzur

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CHAPTER I

CONCEPTUAL FRAMEWORK

> Human knowledge is essentially active. To know is to assimilate reality into systems of transformations. I find myself opposed to the view of knowledge as a copy, a passive copy of reality (Piaget, 1970, p. 15).

My research purpose was to examine the process of change in elementary school mathematics teachers as they make a shift to an instructional approach that includes using a second-order model (Steffe, 1995, 2000)[1] of their students' mathematical thinking. A second-order model (SOM), also referred to as a model of someone else's mathematical reality (Steffe, 2000), consists of inferences an observer—a teacher and/or researcher—makes of a student's mathematical understanding based on observed behaviors (Steffe & Thompson, 2000; Thompson, 2000; Ulrich, Tillema, Hackenberg, & Norton, 2014). The word observer refers to the person interpreting someone else's mathematical activity, be it a researcher or a teacher (Maturana & Varela, 1980).

An SOM differs markedly from the observer's own mathematics, to which Steffe referred as one's first-order model (FOM). Often, teachers conceive of their students' mathematics through the lens of their own FOM (Tzur, 2010); that is, the teacher understands a mathematical concept in a particular way and interprets students' mathematics by using her FOM. Such a teacher likely assumes that, through instruction, students will come to understand mathematical concepts the same way the teacher does (Simon, Tzur, Heinz, & Kinzel, 2000). Thus, the teacher

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[1] Because expressing the research problem of this study requires articulating key constructs of the conceptual framework that underlie formulating this problem, it is presented in Chapter II.

uses instructional techniques driven by her FOM understanding of mathematics. At issue is that FOM, while making complete sense for the teacher, may be insufficient for fostering student learning of the intended mathematics (Steffe, 2000).

To convey the importance of teachers' shift towards an SOM, in this chapter, I define and illustrate SOM and FOM and outline key components necessary for one to operate with an SOM. Beginning with how learning occurs, I use a constructivist framework to describe how a learner may advance from not knowing to knowing a new mathematical concept. I then use this conceptual framework to highlight advantages for using both FOM and SOM in teaching as opposed to just FOM. My goal is to highlight the importance of studying teacher shift towards SOM. I explicate the theoretical underpinnings in the literature about SOM and provide examples of SOMs. I then discuss in detail two elements of the conceptual framework that underlie the study of teacher shift towards SOM: (a) a teacher's perspective on knowing and learning and (b) a Student-Adaptive Pedagogy approach that corresponds to one of those perspectives. Finally, I present the research questions for this study.

To further explain the importance of SOMs for teaching mathematics in ways that foster learners' conceptual learning, I turn to articulating key constructs of the constructivist framework that guide this dissertation study. I begin by depicting learning as a cognitive process by which learners come to know what previously has been unknown to them.

The Learning Process

Cognitive change. I conceive of learning as an active cognitive change process rooted in a learner's experience. This stance is rooted in a constructivist perspective (Piaget, 1985; Simon, 1995; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 1995, 2010; Tzur, 2014; von Glasersfeld, 1995). According to this perspective, the mathematical understanding an individual has cannot be assumed to "mirror" reality, nor be equated with her observable behaviors (actions and/or language, including explanations provided by observed subjects). Rather, one's understandings can only be inferred from observable behaviors. My study builds on Piaget's (1971, 1985, 2001) explanation of the process of knowledge construction and how the human mind moves from not knowing to knowing a concept. Von Galsersfeld (1995) later built on Piaget's theory and provided substantial literature postulating how cognitive changes occur.

Von Glasersfeld (1995) asserted that Piaget's goal was to create a coherent model of human cognition and its development. In that model, learning is conceived of as an internal activity of the human mind. As the mind organizes one's experiences of the world, it is simultaneously organizing itself through shaping and coordinating three-part schemes of action and operation (Piaget, 1985; von Glasersfeld, 1995). Next, I elaborate on the notion of scheme, which constitutes a central construct about knowing, and thus of what can be used to articulate first- and second-order models.

Scheme. What constitutes available/new knowledge, and what is changing when learning takes place, are mental structures called schemes (Piaget, 1967; von Glasersfeld, 1995). According to von Glasersfeld (1995), schemes consist of three parts: recognition of a situation that leads to a goal set by the cognizing subject, activity to accomplish that goal, and a result (or expected outcome). The first part of the scheme involves recognition of a certain situation, to which Steffe (2002) referred as a "recognition template". Existing schemes provide the starting point of learning, which occur through mentally reorganizing the existing schemes at a higher level. Learners' mental systems are capable of assimilating[2] ongoing experiences into the recognition template of available schemes, which in turn may bring forth the scheme's activity.

[2] I further discuss assimilation in section: Assimilation.

For example, if a learner has available a counting-on scheme for adding two given numerosities, she could recognize a situation that involves adding 8+7 as one that calls for adding, to the first addend taken as is, singletons (1s) that constitute the second addend.

The second part of a scheme is a mental activity brought forth to accomplish the goal set out by the scheme's recognition template. In the example above, the goal would be to find the total of 1s in the combined collection, and the activity could be counting 1s while starting at one addend, not at 1 (e.g., 8; 9-10-11-12-13-14-15). This scheme also includes anticipating the need to actively keep track of accrual of 1s so she can stop the count after all 1s of the second addend are added (e.g., raising all five fingers on one hand and two fingers on the other) (Risley, 2016; Tzur & Lambert, 2011). Throughout this dissertation study, I use the term *activity* in reference to the internal, mental activity required to accomplish one's goal as set by the recognition template of her scheme(s) (Simon et al., 2004; Simon & Tzur, 2004; von Glaseresfeld, 1995). As mental activities may be accompanied by an observable action (behavior), such behaviors allow an observer to infer the internal activity and thus create an SOM of the other person's mathematics.

The third part of a scheme is a result the mental system anticipates will ensue from the activity. Based on previous "like" experiences, the learner can eventually predict this result without running the activity (Simon, 2015; Simon et al., 2010). Continuing the example, the learner would arrive at 15 singletons and understand that this is the resulting combination of both 8 and 7. If asked to explain how she arrived at that answer, she would likely call upon and express the counting-on strategy.

It is important to note that a scheme can be available to a learner as is, as well as be able to undergo changes, which would constitute learning. If a situation has previously been experienced by the learner and is similar enough in the outcome, the learner is likely to use available schemes, and those available schemes are unlikely to change (Steffe & Olive, 2010). On the contrary, if a situation has not been experienced by the learner, she may call on available schemes and, due to the gap between previous and current situations, potentially alter the existing schemes to create an outcome of new learning. I further discuss this change process in the following section on the Reorganization Hypothesis.

In summary, a scheme is a threefold mental structure that can indicate available knowing and can undergo transformation (reorganization), that is, learning (von Glasersfeld, 1995). In the remainder of this chapter, the term *scheme* will always be referred to as a foundational conceptual structure capable of changing and therefore part of a learning process. I will use the terms *available scheme, assimilatory scheme*[3]*,* and *mathematical concept* interchangeably to refer to available mathematical knowing of a learner. The next section elaborates on how learning may occur through reorganization (change) of available schemes.

Reorganization hypothesis. The change from not knowing to knowing a mathematical idea, that is, learning, occurs by reorganizing previously available schemes (Dewey, 1938; Piaget, 2001; Steffe, Liss, & Lee, 2014; Steffe & Olive, 2010, Tzur, 2014). Such reorganization reflects Dewey's (1938) principle of continuity: "From this point of view, the principle of continuity of experience means that every experience both takes up something from those which have gone before and modifies in some way the quality of those which come after" (p. 35). This principle stresses that every experience is twofold in nature: it is understood based on previous experiences and shapes subsequent experiences.

[3] Assimilatory scheme (Piaget, 1967, 1985) is further discussed in the Assimilation section of this chapter.

Piaget (2001) drew on a similar principle, which he termed “Reflective Reorganization.” Reorganization in this sense begins with available cognitive structures and proceeds through altered/changed cognitive structures. These reorganized cognitive structures constitute learning and occur through learners’ reflective processes. The reflective process expands and transforms available schemes by way of reconstructing them.

I contend that both the principle of continuity and reflective reorganization principle underlie Steffe and Olive’s (2010) reorganization hypothesis: “a new scheme is constructed by using another scheme in a novel way, the new scheme can be regarded as a reorganization of the prior scheme” (p. 1). They go on to say, “New schemes can be regarded as reorganizations of the preceding schemes because operations of the preceding schemes emerge in a new organization and serve a different purpose” (p. 2). In this sense, one learns a new mathematical concept through building on an available scheme, and learning something new is thought of as a reorganization of what is already known. Piaget (1971, 1985) articulated two interrelated mental processes, assimilation and accommodation, which constitute reorganization, to which I turn next.

Assimilation. *Assimilation* (Piaget, 1971, 1985) refers to a mental process by which available schemes organize a person’s experience. The organization is based on existing understanding and experiences a learner has. One’s ability to organize and make sense of information involves the first part of the scheme: the recognition of the situation. The assimilation serves as a lens by which the learner then uses what she knows to proceed through the goal setting, running of the activity, and identifying the result. She can only “see” or understand the situation through the lens of what she already knows; that is, her assimilatory schemes. Assimilation is a learner’s active mental process through which she organizes the experience she is undergoing by coordinating it with previously recorded experiences. The information used to organize an experience may be input from the senses or from within the mental system (Piaget 1967; Tzur, 2011).

When encountering a mathematical task, a learner will assimilate the task into what is already known; that is, into available schemes to which Piaget (1967, 1985) referred as "assimilatory schemes." In thinking about the previously exampled learner, when presented with a task of 8+7, the available scheme into which this task was assimilated, called counting-on, involved recognizing two collections of 1s, setting a goal to figure out the total of 1s, starting to count at the first addend (8), moving through 7 additional, figural singletons in the observable form of seven fingers, and arriving at a result of a known number word standing for the total of 1s (here, 15). From this perspective, for an experience to be one that promotes learning, a learner must first structure a situation by mentally bringing forth previous like-experiences in the interpretation of the current experience (Dewey, 1938). This process could be automatic and mostly unconscious to the learner (Piaget, 2001, Simon et al., 2004). In this sense, assimilation allows a person to construct a new scheme by "bringing forth" and building on previous knowledge (Clark, 2005; Jin & Tzur, 2011a, 2011b).

Accommodation. *Accommodation* refers to the cognitive change processes through which available mathematical concepts may be reorganized into new ones (Piaget, 2001; Simon et al., 2004). Accommodation is related to von Glasersfeld's (1995) second and third parts of a scheme: the activity and the expected outcome. A learner entering into a situation requiring activity will have a prediction of an outcome based on previous "like" experiences (her own assimilation of the task at hand) (Simon et al., 2004). If the actual outcome differs from the predicted outcome, a perturbation may occur (Piaget, 1985). Tzur (in press) when referring to Skemp's (1979) articulation of a perturbation described this as, “a person’s experience of a disparity between a current state and a goal state” (p. 6). According to this stance, a perturbation is a crucial component in commencing the cognitive change process.

Perturbations can come in two forms, a constraint or a variation in activity, both of which require a need for change in a learner’s predicted outcome. A constraint perturbation (Piaget, 1985) may occur if the learner is unable to achieve her predicted outcome. For example, a learner who tries to add 8+7 and is used to only counting-on with fingers (5) of one hand would sense a perturbation in that she cannot complete her counting activity to add the last two 1s.

Alternatively, a variation perturbation may occur if actual outcomes differ from expected/predicted outcomes of available schemes. In thinking about the exampled learner, once she reached the fifth finger of one hand she would pause, and then hold up the two fingers of the other hand. In this case, the perturbation is sensed in holding up the additional two fingers, something the learner may have not done until presented with the problem of 8+7.

Recent research has further explained perturbation as part of the learning process. Simon et al. (2004) suggested that a perturbation could then lead to dis-equilibration of the cognitive system in its current state. In other words, a perturbation is a catalyst for learning (Simon et al., 2004): “Perturbation is commonly understood as cognitive conflict, that is, learners’ experiences of an event not fitting with their current conceptions or lack of fit among the conceptions they hold” (p. 307). This imbalance or disequilibrium occurs because the prediction of outcome made by the individual (what the learner predicted and expected as the result) differs from the actual outcome of the activity that occurred, thus causing a questioning within the individual (Piaget, 1985, 2001; Simon et al., 2004; Tzur, in press).

In a learner's effort to reestablish equilibrium after experiencing a perturbation, two scenarios seem possible: (a) the individual avoid the perturbation without resolving it, which would result in no reorganization of new knowledge, or (b) the individual could resolve the perturbation by transforming available schemes (used for previous assimilations) to adapt to her new experience (Piaget, 1985). When such a change occurs, it is viewed as an accommodation or transformation of knowledge; that is, an internal reorganization of what was known into new knowledge.

Consequently, a learner's need to continually restore equilibration, due to the unexpected aspects of an experience of perturbation, can bring forth new learning from results the activity produces (von Glasersfeld, 1995). In the exampled learner, she would conclude that finding the sum of 8+7 could be accomplished by using additional fingers with which to count, which also reorganizes her figural representation of numbers larger than 5 (e.g., raising five fingers on one hand and the remaining singletons on the other hand). This alteration would be brought forth by her ability to take the new, changed activity and connect it into previous experiences by altering the predicted outcome, and transfer it into a new experience. A perturbing situation may also become unbearable for the learner; in which case, the learner is likely to give up and not take in any new knowledge/schemes (Hackenberg, 2010; Piaget, 1985; Steffe & Tzur, 1994; Tzur, 1996). In this case, the available scheme would be part of what the learner knows and understands. The next section discusses how the accommodation of a scheme takes place through the mental mechanism of Reflection Abstraction (Piaget, 2001), on which Simon et al. (2004) elaborated with their construct of Reflection on Activity-Effect Relationship.

Reflective abstraction. The reorganization process involves activity, anticipation, and reflection. Piaget (2001) asserted that such a reorganization occurs through a process he termed "reflective abstraction." Reflective abstraction was explained by Piaget (1980) as a two-phase process for a learner to construct a more advanced concept by transforming one that already exists. First, the projection phase allows a learner to take the goal-directed actions produced by a lower-level scheme as an object of reflection for the next, higher level. Second, in the reflection phase, the learner reorganizes the objects reflected on into a higher-level concept. Through reflective abstraction, a learner comes to also know the logical necessity of mathematical relationships (Simon, 2006).

Piaget (2001) contrasted reflective abstraction with that of empirical abstraction. In empirical abstraction, learners associate properties that are perceived. In other words, learners pick up on input and output as perceptual properties or patterns of physical objects. For Piaget, this was considered a limited source of knowledge. Simon (2006, 2015) broadened the idea of empirical abstraction through defining an empirical learning process to also include processes not only based on objects or actions. When discussing the empirical learning process, Simon (2006) articulated, "Students learn *that* the pattern exists. The phenomenon that generates the pattern may remain a black box to students" (p. 365). (In this quote I have italicized "that" to emphasize empirical learning as something that does not focus on knowing why, but just *that* something occurs.)

Simon et al. (2004) elaborated on Piaget's notion of reflective abstraction as a person's ability to recognize a difference between what actually happened and what was anticipated to happen through an activity, reflect on those differences, begin to recognize patterns in the differences, and mentally abstract them as new learning/knowledge (Simon et al., 2004, Tzur & Simon, 2004). Simon et al. (2004) termed this mental mechanism, "Reflection on Activity-Effect Relationship" (hereafter abbreviated as Ref*AER).

Ref*AER articulates how a learner moves from not having to having a scheme by linking this change to von Glasersfeld's (1995) three-part notion of scheme. Within the first part, the learner sets her goal(s) based on her assimilation of the task at hand. That is, the learner's goals are based on her available schemes.

Accordingly, the learner calls on the activities of the available scheme(s) to solve the task. As learners go through the activity, they attend to the effects and begin to distinguish between positive and negative effects in the sense of actual effects fitting or not with the anticipated result, respectively (Simon et al., 2004). In the latter case, learners may make adjustments to their activity. Simon et al. (2004) referred to these adjustments as the "effects of the activities" (p. 319). It is the learner's reflection on the effects that allows for available schemes to be reorganized into new ones, a process comprised of two types of reflection (Tzur, 2011).

A learner's comparison between the anticipated effect of the activity and the actual effect is termed "Reflection Type I" (Tzur 2011). Reflection Type I supports the cognitive change when a learner experiences a perturbation between the anticipated effect of the activity and the actual effect. This, in turn, leads to a new (to the learner) activity-effect dyad.

A second type of reflection that constitutes the Ref*AER mechanism allows the learner to compare among newly linked, yet provisional, records of experience (activity-effect dyads). This Reflection Type II (Tzur 2011) involves the learner comparing across similar instances of using the scheme and abstracting what in those instances seem to have remained unchanged (invariant across various situations). The learner begins to cement a new link between the activity-effect relationship (dyad) and situations in which they are used by comparing across mental records of available schemes, comparing the negative and positive results, and accommodating those into a new scheme. That is, a Reflection Type II process may result in linking the new "dyad" to a situation that brought it forth and could thus be brought forth in future experiences. Abstracting the new activity-effect dyad, and linking it anew to a recognition template, constitutes the reorganization of available schemes into a new one; that is, an accommodation of previous into a new mathematical concept.

The mechanism of Ref*AER underlies learners advance from not knowing to knowing (Tzur, 2011; Tzur & Lambert, 2001). Through Ref*AER, the learner establishes her anticipation of the invariant relationship between the mental/predicted activity and its effect. Reflection Type I allows learners to notice a difference between what was anticipated and the actual effect. Reflection Type II allows a learner to compare anticipation of activity-effect dyads across previous, similar situations. As learners create new schemes by reorganizing available schemes through the two-type reflective mechanism, two stages in a learner's construction of a new scheme may be observed: participatory and anticipatory (Tzur & Simon, 2004). These two stages, to which I turn next, seem to be critical components for teachers to include in their SOM.

In the process of reorganizing a new scheme from available ones, the participatory stage may be observed. The participatory stage (Simon, Placa, & Avitzur, 2016; Tzur & Simon, 2004) is inferred when a learner is not yet able to solve a task spontaneously and independently; that is, the learner can engage in a particular activity, can develop knowledge of a mathematical concept, and can anticipate an expected effect. However, when presented with other tasks requiring the same mathematical concept, the learner can anticipate the effect only after somehow being prompted (e.g., through interacting with a teacher) for the activity that leads to that result. Alternatively, the participatory stage may be inferred when a learner, in the process of solving the mathematical problem, has recognized her own mistake (of anticipation) and corrected it through what she, and an observer, may identify as the “oops” experience. Whether the prompts are external or internal, in the participatory stage, they are necessary for the learner to solve the mathematical task at hand. At the participatory stage, both the recognition of a situation and the goal-directed activity exist, but anticipation of the effect cannot occur independently.

Tzur & Lambert (2001) have linked this mental stage in the construction of a new scheme to Vygotsky’s (1986) notion of Zone of Proximal Development (ZPD). Their reason was that at the participatory stage, the scheme is not yet fully constructed and not available upon assimilation (Vygotsky’s Zone of Actual Development). The reason for this lack of access to the new scheme is that the activity-effect dyad has yet to be integrated with the situation of the scheme.

Unlike at the participatory stage, at the anticipatory stage (Simon, Placa, & Avitzur, 2016; Tzur & Simon, 2004) the learner can spontaneously and independently anticipate and justify the link between an activity and its effects. Having explained the learning of new mathematics as a cognitive process of reorganization, rooted in two types of reflection and consisting of two stages, in the next section, I turn to defining an SOM and connecting it with a teacher’s ability to operate with an SOM.

Second-Order Models: Knowing and Learning Must Be Inferred

Defining learning as the internal cognitive change process the human mental system may undergo entails that knowing and coming to know can at best be inferred by an observer (Cobb & Steffe, 1983; Maturana & Varela, 1980; Steffe, 1992, 1995). The complexities of this internal reorganization process cannot be understood as parallel to or directly “mirroring” observed behaviors (Cobb & Steffe, 1983; Steffe, 1995). Rather, those mental processes must be modeled through interpreting the learner’s actions and explanations.

Inferences about other people's available schemes and changing schemes constitute SOMs. An SOM is a set of inferences an observer may make about a learner's mathematical thinking and activity (Steffe, 1995). Specifically, SOMs are "hypothetical models observers may construct of the subject's knowledge in order to explain their observations (i.e. their experience) of the subject's states and activities" (Steffe, 1995, p. 495). SOMs differ from FOMs in that, with FOMs, the observer's interpretations are essentially attributing mathematical knowing to the learner being observed through only the lens of the mathematics used by the observer herself. The observed learner may be considered to have or not to have a concept, but such an attribution, with the frequent reference to "not having" as "misconception," is likely rooted in the observer's contrasting of the other person's mathematics through assimilating it into one's own FOM.

It should be noted that all observers operate with an FOM (Steffe, 1995). Recognizing that learning is an internal process and cannot be directly accessed entails contrasting FOMs with SOM of another person's ways of knowing (schemes). By definition, the ability to create an SOM of another person (e.g., a student) is always subjective and based on the observer's own subject knowledge (Steffe, 1995): "Second-order models are understood as springing from the conceptual operations that are available to the observer, along with their modifications in the context of interacting [with the observed person]" (p. 496).

To shift to SOM, observers must begin to distinguish between their mathematics and inferences into (analysis of) the learner's mathematics (Steffe, 1992). Having an SOM allows an observer to better understand a learner as a self-organizing system: someone who has unique ways of operating that are different from one's own (Steffe, 1992). With this gained ability, instructional focus may shift from an effort of transmitting, or "showing" to learners, mathematical aspects of one's FOM (existing mathematical knowledge) – to fostering learners' construction of their own mathematical, mental operations. That is, constructing an SOM allows an observer to take on a perspective of others' mathematical ways of operating (Steffe, 1995; Ulrich et al., 2014).

In shifting to an SOM, a person needs to recognize that (a) her FOM is not the same as the learners' mathematics; (b) behaviors in and of themselves do not constitute the learner's mathematics; and (c) when interpreting with an SOM, it is precisely that: an inference into another person's mathematical thinking (not a piece of objective reality). Without such inferences, intentional support for the reorganization process to take place is not likely to happen (or, at best, if it did, there's no way for the observer to explain why, let alone change the teaching if it did not). Before finalizing my discussion of SOM, I turn to discussing the importance of SOMs in teaching mathematics.

Second-order models in teaching mathematics. To promote others' learning of mathematics (i.e., reorganization in their schemes), it seems crucial for a teacher to develop appreciation for and facility with creating SOM (Tzur, 2014; Ulrich et al., 2014). This stance is rooted in the understanding of learning as a process that begins by assimilation into available schemes, which can only be inferred. Considering the learning process outlined thus far, to promote learning of a new mathematical concept, I provide my summary of what an observer should become accustomed to doing based on the existing literature discussed:

a) Infer what a learner's existing understanding is by distinguishing the learner's assimilatory schemes from the teacher's own understanding.
b) Predict what effects a learner might anticipate from an activity based on the learner's existing understanding.
c) Capitalize on the expected learner's assimilation of what is known (available schemes).

d) Implement tasks wherein learners notice intended effects that are slightly different than what they anticipated in the first place in order to foster Reflection Type I that can begin an accommodation/transformation process.

e) Recognize the differences in what the learner expected to happen and what actually happened (new effects for the learner).

f) Use the new effects (for the learner) to orient the learner in Reflection Type II to shift the learner's anticipation into a new, transformed, understanding.

g) Determine the extent to which a learner can anticipate an outcome at a participatory or anticipatory stage and use that, in turn, to determine to what extent a new assimilatory scheme has been learned.

Shifting towards an instructional approach that includes an SOM underscores the first part of the learning process as explained above: inferring the available schemes a learner already has constructed, participatory and/or anticipatory, while clearly distinguishing those from the teacher's own understanding and assimilatory schemes (her FOM). Once a learner's existing available schemes are inferred, hypotheses can be articulated as to how learning can progress through accommodation (Simon, 1995; Simon & Tzur 2004). Thus, for an SOM to be used, a teacher needs to develop the ability to infer a student's existing understanding of the mathematics and distinguish that understanding from the teacher's own mathematical understanding. As Tzur (2010) succinctly pointed out, "The assimilation principle requires teachers to understand students' mathematics as qualitatively different from the teachers' understanding and, thus, as the conceptual force that constrains and affords the mathematics students can "see" in the world" (p. 50). Shifting towards an SOM can allow a teacher to understand that when learners learn mathematics, they are creating their own experiential reality, which may or may not be compatible with that of the teacher's (Ulrich et al., 2014).

Specifically, while using the Ref*AER model for cognitive change through two types of reflection, SOMs seem needed for teachers to foster perturbations that may lead to learning. A teacher who uses SOM can intentionally and successfully also orient reflections to foster students' transition to the participatory stage, and then to the anticipatory stage. A teacher who shifts towards an SOM can better orient a child's reflection leading to the intended cognitive change, because she can pinpoint, and link, available schemes and those that need to be developed. To this end, SOMs would include articulation of possible gaps students may experience between the anticipated effect and actual effects (Reflection Type I). Similarly, SOMs are important for fostering the comparison across instances of using invariant activity-effect relationships (Reflection Type II). Subsequently, having an SOM would also allow a teacher to choose tasks that may both bring forth known mathematics through assimilation *and* lead to transforming that mathematics through the two types of reflection.

A teacher who promotes learning through using an SOM can also infer (and assess) to what extent students have learned a particular way of thinking mathematically. In particular, a teacher can use her SOM of a student to determine if a student is at a participatory or anticipatory stage of constructing a new scheme as a way to determine what mathematical goals for learning the teacher should have for the student next. This is important, because if a student has only a participatory stage of a new scheme, a teacher's SOM could imply the student has yet to establish a new scheme. This would allow a teacher to further work with the learner to move along the reorganization process so that the new scheme would become anticipatory. In addition, a teacher with SOM could also determine if a student had constructed a particular scheme at an anticipatory stage, then be able to use this new learning as a new available (anticipatory) scheme. In short, SOMs provide the teacher with a tool for creating new learning situations that can advance the learner's mathematics to more advanced schemes (or stages).

All people, including researchers and teachers, when observing a learner, use their FOM to assimilate the other person's mathematics (Steffe, 1995). An ability to shift towards an SOM means that an observer (e.g., teacher) is adding an awareness of the need to infer what could a learner's experience with mathematics be and distinguish what she infers from her own understanding of the mathematics. Shifting towards an SOM increases the likelihood a teacher can foster students' bringing forth, through assimilation, relevant available schemes they already have and transforming those into the intended mathematics.

In contrast, a teacher who is operating only from her own FOM would, at best, set developmentally appropriate goals for students' learning. Additionally, a teacher operating from an FOM may not make an inference into differences in student conceptions from her own, as it could be assumed that the understanding of the learner has, essentially, either already reached a stage of being the same as the teacher's, or "not yet there" (often referred to as the learner's "misconception"). An SOM allows inference into the current understanding a learner has, which then allows for understanding of assimilatory schemes that could be triggered within the student, thus, situating a learning opportunity based on what the learner already knows.

The next section presents examples that can illustrate and clarify development and important differences in teaching for conceptual change between those who use only FOM and those who use both FOM and SOM. To this end, I first articulate a person's initial operation on FOM and then move towards understanding/interpreting an SOM of another person, by providing a real life example of one partaking in a book club. This example is meant to illustrate how one might initially recognize that her and others' mathematics differ. This recognition can constitute the beginning of developing an SOM (Cobb & Steffe, 1983; Steffe, 1992, 1995). Later in this chapter, a more detailed example of how SOM can be used for conceptual change is provided as it pertains to learning mathematics.

Illustrating a shift towards a second-order model. In a book club, people read a book or certain parts of a book on their own and then get together with other members to discuss what they read. When a reader reads independently, she creates and interprets her own reality of the book and its themes, main ideas, characters and their traits. This is the reader's FOM of what is going on within the context of the story as she makes sense of the book and as the story unfolds. At this point, the reader only has her own experience with the literature, and therefore, this experience serves as a guide for how the reader assumes the experience and interpretation of the story may be for others reading the same book. That is, until members meet to discuss the book and the reader has an opportunity to hear others' interpretations, she can only interpret what she read using her FOM. Once such an opportunity occurs, particularly if another reader interpreted it differently, the reader may experience a perturbation and react in one of three ways. First, the first reader may try to persuade the other person of her own understanding. For this, she may use specific examples from the story to share her interpretations and try to impart her interpretations onto the other person as a way for them to understand the theme similarly. In doing so, the reader is continuing to operate from her FOM, while trying to have the other person interpret the passage at issue in the same way as the reader's FOM. Second, she may resolve the perturbation by being persuaded with the other reader's interpretation and change her own FOM accordingly. It should be noted that, in both cases, her interpretations (original or transformed) are rooted in her FOM.

A third resolution to her perturbation between her original FOM understanding of the book and the other's interpretation could be to consider two possible frames of mind. This could thus become an initial shift in the reader's operation from her FOM to also include an SOM by attempting to understand another's FOM of the literature. This may provoke questions for the other reader, such as, "Why do you think that?" or "How did that develop for you throughout the story?" It is then possible the first reader will start trying to infer into the other person's understanding to better interpret what underlies the different understanding. When this occurs, the first reader may be attempting to make sense of another person's model, which could lead to an SOM of the other person's experience. It is important to note that it was not until the other person shared her understanding of the story, and the first reader listened to it and noticed the difference, that she could realize her FOM was different. That is, through interactions with other people, the reader may recognize her interpretation as idiosyncratic, while others could likely have different understandings.

Considering this scenario through the lens of assimilation, when the first reader was reading the book on her own, there was one reality for the reader's interpretation and understanding of the theme. This would often be the reader's default position until there was a contrast of another person's experience with the book. Both the reader and the other person experienced the book as a function of what they knew and their previous experiences, which led to different interpretations; that is, assimilation of the book's theme was unique to each reader. Thus, a reader's reality of the theme persisted at least until her interpretation of the book and of the other person's interpretation conflicted in her mind. Once confronted with a different interpretation, it might have been difficult, initially, to understand how the other person came up with her interpretation. Hence, the first reader's attempt to try to make the other interpret the theme the way she did made sense to her, as well as to someone (e.g., book club leader) who considers a need for the first reader to accept other readers' interpretations. The two ways in which two different readers assimilated the "same story" were at odds until a perturbation was created for the first reader through (a) interacting and sharing interpretations and (b) recognizing differences in these interpretations.

It is important to note that the book club example helps illustrate the issue of FOM and SOM in a domain and human experience in which most people would expect different interpretations from others. The issue becomes more complex in mathematics, because most people seem to accept it as a single body of knowledge available identically to all (Cobb & Steffe, 1983; Ernest, 1989; Steffe, 1995). Next, I turn to an example that illustrates the issue of FOM and SOM in mathematics.

Mathematical example of a second-order model. To convey the importance of SOM versus FOM within mathematics education, I now provide an example of two learners embarking on multiplicative reasoning. I portray these two learners in their journey of learning as depicted by one teacher operating from an FOM and another operating from both an FOM and an SOM. My purpose for providing this example is threefold. First, it helps emphasize the difference between such teachers in terms of what a child can assimilate and use as a springboard for further learning. Second, it provides a contrast between understanding of a student's mathematical reality a teacher can attain from operating from an FOM (e.g., the first two ways of the book reader's resolution to her perturbation) versus a teacher operating from an FOM and an SOM (the third way of resolving the book reader's perturbation). Third, it emphasizes the mathematical understanding that a teacher can promote in her students when operating only with an FOM versus with both an FOM and an SOM. Thinking about these two examples of different learners, I will then describe how such teachers may differ in their interpretations of each student's thinking. In doing so, my goal is to underscore the advantage educators may gain when shifting to an SOM along with their FOM.

For the example, consider two students who both solved the following multiplication problem correctly, arriving at 20 pieces of gum: "You have 5 packs of gum; each pack has 4 pieces of gum. How many pieces of gum do you have in all?" Learner A solved the problem by holding up five fingers and counting by four for each finger arriving at the total of 20 pieces of gum (e.g., 4-8-12-16-20). Learner A then explained, "I used my fingers to represent each pack of gum and counted each of the pieces of gum. I knew to stop at my fifth finger because that was the last pack of gum." On the other hand, Learner B explained, "I drew five boxes and then four dots in each. I then counted all the dots, like this: 1-2-3-4; 5-6-7-8; 9-10-11-12; 13-14-15-16; 17-18-19-20. So, 20 pieces of gum."

One example of a teacher operating only from an FOM is that she would determine both learners understand multiplication because they both arrived at the correct answer of 20 pieces of gum. In addition, a teacher operating from an FOM may also think of Learner A as being more advanced than Learner B due to the solution of skip counting versus counting dots. In both cases, as this teacher is relying heavily on her FOM, both learners can solve correctly multiplicative situations. Yet, this teacher's FOM seems to underlie her interpretations of both learners' solutions as having the concept like she does. Therefore, this teacher is unlikely to recognize, and distinguish, the assimilatory schemes that each learner used to arrive at the solution and use this as a basis for what the learner can and should learn next.

On the other hand, a teacher operating from an SOM and an FOM is likely to interpret the work of the two learners as conceptually different from one another – and from the teacher's own frame of reference. A teacher using an SOM may infer that Learner A can assimilate a situation into a multiplicative scheme with figural items (use of fingers), and thus distribute units of four pieces of gum into each of the 5 packs. In this sense, the teacher may attribute to Learner A the multiplicative double counting (mDC) scheme (Tzur et al., 2013). This scheme includes purposely keeping track of and coordinating accrual of both the composite units (5 packs) and the unit rate of pieces of gum (4 pieces in each pack) to arrive at the total amount of single pieces.

Similarly, using an SOM, the teacher would also interpret Learner B to have a different scheme. From Learner B's actions, the teacher may infer that she assimilated the task into a scheme in which units of 1 are being distributed—not composite units. Accordingly, a teacher operating with an SOM would select different learning goals for each learner, as well as select tasks to foster their mathematical progress (e.g., foster Learner A's abstract operations in multiplicative situations and Learner B's strengthening of her conception of number as composite unit).

This teacher's ability to create an SOM, including differentiating between her scheme of multiplication with abstract objects (numbers) and Learner A's use of figural objects (fingers) underlies her different inferences into each learner's assimilatory schemes. Thus, the teacher could differentiate her instruction to promote progress in both learners.

Types of second-order models. I now present an overview of types of SOMs that researchers may create. Ulrich et al. (2014) postulated three types of SOMs to be beneficial for teaching: Emerging, Developed, and Elaborated. In the *Emerging SOM,* researchers could understand and have insight into students' mathematical thinking. In such a stage, instructional adaptations do not necessarily occur because the model is either still being constructed or not quite accurate.

The *Developed SOM* builds on the Emerging Type in that it uses the student's mathematical understanding to anticipate and plan interactions with students. At this stage appropriate tasks can be used that fit with the student's way of thinking in order to promote further learning.

> Model is based on interactions with one student to show how this can be used to anticipate an effective sequencing of tasks to engender accommodations in students' schemes and can be done efficiently…Using previous SOMs can predict student difficulties and reactions to student difficulties and reactions to perturbations with much greater accuracy (Ulrich et al., 2014, p. 338).

In the third type, an *Elaborated SOM*, the teacher/researcher is able to determine a viable understanding of what the student's existing mathematical understanding is prior to the interaction. Consequently, the teacher can plan a whole set of tasks that would allow the student to advance in their mathematics. This paves the way for more effective planning that makes sense of student interactions and reacts meaningfully by creating what Ulrich et al. (2014) connected to the epistemic subject: "Generalization of SOMs to epistemic subjects which enable the teacher/researcher to situate student responses in a much broader framework of potential student responses and ways of operating" (p. 335). Next, I link the distinction between FOM and SOM to perspectives on mathematical knowing, learning, and teaching (Simon et al., 2004).

Shifting towards Second-Order Model: Perspectives on Knowing, Learning, and Teaching

In this section, I describe teacher perspectives that have been articulated in existing literature (Ernest, 1989; Jin & Tzur, 2011a, 2011b; Simon et al., 2000). Each of these perspectives affords and/or constrains a teacher's ability to create an SOM of their students. In describing each of the perspectives, I outline those affordances or constraints.

Simon et al. (2000) contended that underlying teachers' work are certain perspectives on knowing and learning into their classrooms of which they may or may not be aware. These perspectives are manifested in instruction and impact student learning of mathematics. Like Simon et al. (2000), I define one's pedagogical perspective as not necessarily what teachers do within a classroom, but what teachers think about their practice, the motivations behind methods they use, and the intentions that drive their instructional moves. Prior research has defined four main types of teaching perspectives on learning and knowing: (a) Traditional Perspective (Ernest, 1989), (b) Perception-Based Perspective (Simon et al., 2000), (c) Progressive Incorporation Perspective (Jin & Tzur, 2011a, 2011b), and (d) Conception-Based Perspective (Simon et al., 2000). I further describe each of those perspectives below, to highlight how the fourth (CBP) is founded on SOM.

A Traditional Perspective (TP) (Ernest, 1989) views mathematical knowledge as existing outside of the learner's experience. The view of learning is that it is a result of passive reception, so that the learner comes to accept math as it is for every person – a "mirror" of reality. Teaching from this perspective is seen as a transmission from one individual to another. Hence, a teacher with this perspective teaches mostly through presentation of mathematical information, and she expects to "transmit" the universally available mathematical knowledge to the learners.

A teacher with a TP on learning is likely to operate from an FOM. This teacher uses her understanding of the mathematics to convey the same meaning to the learners. This can often be observed in the "I do, we do, you do" model of instruction. In this model, the teacher first demonstrates how to do the mathematics (I do from an FOM), then provides time for assisted practice of doing what was modeled (we do seen through FOM). Eventually, the teacher lets learners do the math on their own (you do, to master what's in the teacher's FOM) from the "I do" and "we do" and continued "you do" practice. The learners, in turn, either get the mathematics as the teacher does, or they do not, and the teacher tries to reteach the mathematics using similar methods. It should be noted that the view of mathematics entailed here is one of essentially doing, that is, no questioning of the mental processes that may underlie such one's own or others' doing.

Second is a Perception-Based Perspective (PBP) (Simon et al. 2000), in which a teacher still views mathematics as existing outside and independent of human mental activity. However, unlike TP, with PBP learning mathematics is considered an active process that requires some sort of hands-on experience so learners can "see," or "discover," the mathematics to be learned. Accordingly, PBP entails that the mathematics, once discovered by each learner, is the same for both the teacher and each learner. Therefore, a teacher with this perspective teaches mathematics by promoting students' active discovery of concepts and eventually explaining through active perception/understanding of the explanations (Simon et al., 2000).

Like a teacher with TP, a teacher with PBP is likely to operate from an FOM. Although the means to accomplish students' understanding differ from TP teachers, a PBP teacher is inferred to still focus on getting students to understand the mathematics in the universal way everybody understands it (teacher included). This can sometimes be inferred during a "we do, you do" lesson, where the teacher introduces an activity for learners to engage in that is based on how the teacher understands the mathematics, and possibly came to understand it himself or herself. The learners and teacher then go through the activity (we do) until, eventually, the teacher removes herself and allows learners to do it on their own (you do). If learners still struggle at the end of the “you do” portion, teachers with PBP are likely to revert back to transmitting the mathematics to the learners, perhaps by a learner who the teacher believes got the math (Simon et al., 2004). In the teacher’s mind, activity/exploration is necessary to bring about the “seeing” (perception) of the mathematics the way the teacher sees it through her FOM. Thus, a teacher with PBP may notice a student struggling with the mathematics, but she may have no alternative instruction to “fall back on” when students do not see the mathematics.

Third is a Progressive Incorporation Perspective (PIP) (Jin & Tzur, 2011a, 2011b), in which a teacher guides and creates opportunities for students to be active in the learning via problem solving to extend known knowledge into new knowledge. Deriving from studies of Chinese teaching practices (Jin & Tzur, 2011a, 2011b), PIP takes a dialectical view on knowing and learning as being both outside and inside the learner. Importantly, teachers with PIP largely operate from an FOM. They use a lesson structure that connects mathematical concepts developmentally and aim to promote learning through incorporating the new, intended knowledge into what students already know. Specifically, they initiate the new learning by bridging tasks, which are geared toward bringing forth students’ available knowledge. They then move to variation tasks (of problems, of solutions), which are geared toward gradually incorporating the new ideas into the old. The final part of each lesson involves the teacher summarizing the learning for the students.

Unlike PBP, in PIP what students know is considered key to what they can learn. Thus, the lesson structure is set to first bring forth what the student understands (bridging) and then promote it into new knowledge (variation). At the end of each lesson, when the teacher reverts to the role of an explainer of the mathematics, the entire teaching-learning interactions being rooted in FOM becomes apparent.

Fourth is the Conception-Based Perspective (CBP) (Simon et al., 2000), in which the teacher views mathematics is as being created by the learner through the use of existing understanding (anticipation), mental activity, and reflection. CBP centers on the notion of assimilation: one can only “see” (know, or learn) what one has conceptualized. Accordingly, a CBP includes the following three elements outlined by Simon et al. (2000):

> 1. Mathematics is created through human activity. Humans have no access to mathematics that is independent of their ways of knowing.
>
> 2. What individuals see, understand, and learn is constrained and afforded by what they currently know (current conceptions).
>
> 3. Mathematical learning is a process of transformation of one’s knowing and ways of acting. By using the term transformation, we mean to indicate that learning involves modification of existing ideas, not just the accumulation of additional ideas (p. 583).

A teacher who adheres to the Conception-Based Perspective acts as a facilitator to their students’ learning. The teacher continuously provides opportunities for engaging students in problem solving adapted to their existing understanding and orienting reflection on those activities (Tzur, 2013).

Teachers with CBP mostly operate with both an FOM and an SOM, while recognizing that their interpretation of the student’s SOM involves the lens of their own FOM. A CBP teacher not only understands learning as an active process (like PBP and PIP) but also as dependent on what each learner already knows and thus may assimilate, and transform, in any learning opportunity. For a teacher with CBP, learning can only occur by learners first assimilating and then transforming *their* available schemes.

It is important to note that it is possible for a teacher to have an SOM and not have CBP. A teacher with just an SOM would be able to determine the mathematical operations to which a learner has. This teacher would be able to determine what a learner's conceptions are. However, a teacher with the ability to just determine an SOM may not know how to promote learning as a cognitive change once she created her SOM. This is demonstrated in Chapter IV of this dissertation, for both case study teachers. In contrast, a teacher with a CBP would not only be able to determine the SOM of the student but also create a learning path for the student based on a Hypothetical Learning Trajectory[4] (Simon, 1995; Simon & Tzur, 2004) of how a student may accommodate the existing assimilations and reflect on those to create new (to the learner) knowledge. That is, an SOM seems necessary but insufficient for CBP; a teacher who can construct an SOM may or may not adhere to CBP, whereas a teacher who adheres to CBP dynamically strives to create SOMs as a tool for designing instruction. In this sense, studying teachers' shift towards an SOM is important, because it affords teachers a necessary tool for moving toward CBP and the corresponding Student-Adaptive Pedagogy approach (Steffe, 1990; Tzur, 2013). Next, I turn to describing the Student-Adaptive Pedagogy approach as it further accentuates the importance of this study's contribution to possible shifts towards an SOM[5].

Student-Adaptive Pedagogy

Drawing on Steffe's (1990) notion of adaptive teaching, Tzur (2013) introduced the notion of *Student-Adaptive Pedagogy* to depict an approach in which the teacher's instructional moves are designed to advance a learner from her existing (assimilatory) to new mathematical schemes. This approach is likely to be observed with teachers who adhere to CBP, although the

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[4] Hypothetical Learning Trajectory is described in the next section.
[5] The importance of Student-Adaptive Pedagogy to this study is further explicated in Chapters II and III.

two do not always go hand in hand. A Student-Adaptive Pedagogy approach requires teachers have an SOM of their students in order to advance mathematical conceptions through reorganization of their assimilatory schemes[6]. Following, I outline key components of the Student-Adaptive Pedagogy approach. I then provide articulation as to how these components translate to teaching by further defining Student-Adaptive Pedagogy. Finally, I recap the importance of SOMs for the Student-Adaptive Pedagogy approach.

The Student-Adaptive Pedagogy approach is a way of teaching in which the teacher tailors instructional moves to learners' existing mathematical conceptual understanding (Steffe, 1990; Tzur, 2013). A teacher who adheres to this approach understands the goals set for the students' learning are distinct from the goals that drive the student's activity. Tzur (2017) has defined this approach as follows:

> Student-Adaptive Pedagogy is an approach rooted in a constructivist stance on mathematical knowing and learning. It draws on the core constructs of assimilation and reorganization (construction). Assimilation entails anything a person (learner) experiences is afforded and constrained by her or his available conceptions. Thus, one cannot be given new knowledge, but rather has to construct it as reorganization of available conceptions. Such reorganization is explained as a reflective process of abstraction, which consists of two types of mental comparison and two stages. For teaching, this stance on conceptual learning entails a cyclic, reflective process consisting of three principal activities: Analyzing (diagnosing) students' available math conceptions that could be reorganized, articulating the math intended for students' learning as a result

[6] Student-Adaptive Pedagogy requires that one have an ability to operate using an SOM of their learners. However, if one has an ability to operate using an SOM it does not mean that they are using the pedagogical approach of Student-Adaptive Pedagogy.

> of the reorganization, and devising tasks and activities that have the reasoned potential to promote the desired reorganization of available into intended conceptions. Those three principal activities comprise seven teaching practices (or "steps" in the cycle) teachers can develop to enact and understand the Student-Adaptive Pedagogy approach [sic].[7]

A crucial facet of the above depiction is that a teacher who adheres to the Student-Adaptive Pedagogy approach (hereafter, I shall refer to it as an *AdPed teacher*) is aware of students' mathematical behavior being rooted in existing schemes a student *does have* (see, for example, Hunt & Tzur, 2017). This recognition occurs through the teacher's ability to infer into the student's understanding, that is, to create an SOM of the student's mathematical understanding. As a result, an AdPed teacher can identify which mathematical concept the student could and should construct next, and how that new concept can evolve through reorganization of the existing understanding the student has.

The Student-Adaptive Pedagogy approach underpinnings have been developed based on three main components: (a) the Hypothetical Learning Trajectory (HLT) (Simon, 1995; Simon & Tzur, 2004), (b) the seven instructional steps of a teaching cycle (Tzur, 2008), and (c) the Teaching Triad (Tzur, 2010). I present, in detail, these three components to further explain the underpinnings of the Student-Adaptive Pedagogy approach. These main components are all reliant on a teacher's ability to create an SOM of her students in order to promote further mathematical understanding. By further describing each component, I intend to point out the importance of studying a shift towards an SOM in teachers.

Hypothetical learning trajectory (HLT). The HLT (Simon, 1995; Simon & Tzur 2004) is a construct that underlies designing instruction based on a constructivist perspective. HLTs

[7] The mentioned seven teaching practices are outlined in Tzur, 2008.

consist of three elements: (a) the goal for student learning that denotes task direction, (b) mathematical tasks that promote such learning, and (c) a hypothesis/prediction about how student thinking and understanding may evolve throughout the learning activity (Simon & Tzur, 2004). The HLT is based on an understanding of the student's acquired mathematical knowledge and, thus, is dependent on a teacher's ability to form an SOM of her students. The HLT is modified regularly by the teacher and is used as a vehicle for fostering learners' understanding of the intended mathematical concept(s). In designing an HLT, a teacher would take into consideration the mechanism of cognitive change (e.g., Ref*AER) by which learners can transform their available concepts.

Seven instructional steps for teaching. A component of the Student-Adaptive Pedagogy approach is a cycle of seven key steps that further explicate the HLT construct. Tzur (2008) stated,

> For teaching mathematics students must be engaged in tasks that serve three principle functions: (a) fostering assimilation of tasks into their available, relevant conceptions, (b) fostering orientation of their focus of attention so that they notice effects of their work on the task intended by the teacher, and (c) fostering students' reflection on a distinction/formation of the new, intended conception (p. 140).

To carry out these three functions as a teacher, Tzur proposed seven steps, which became part of the Student-Adaptive Pedagogy approach. I will now briefly describe the first 3 steps, as I see them as a support for the rationale of the need to study teachers' shift towards SOM. (Note: Steps 4-7 rely on the first three while focusing on other aspects of the teaching-learning process, and thus are not discussed here.)

The first step within this seven-step cycle is "Specifying Students' Current Conceptions"

(Tzur, 2008, p. 141). It requires one to determine what the learner can assimilate based on what is already known to the learner, as what is known affords and constrains what can be reorganized. In this first step, a teacher needs to infer the available schemes based on a learner's actions and explanations. Indeed, it requires the teacher to separate those inferences from her own understanding and existing schemes (that is, from her FOM).

The second step is "Specifying the Intended Conception" (Tzur, 2008, p. 141). This involves the teacher decomposing the conceptions learners are expected to learn. As part of this process, a teacher connects between what the student has available and how it builds/can be reorganized into a new mathematical understanding for the learner. As Tzur (2008) stated, "To promote the intended learning effectively the teacher also needs to specify the differences between and transformations (shifts in awareness) needed from a current state to an intended state" (p. 142). This requires the teacher to have an SOM of her learners. Without an SOM, a teacher is unlikely to attend to the effects that a learner would notice from a task designed to promote learning through reflection on those effects.

The third step, "Identifying an Activity Sequence" (Tzur, 2008, p. 142), involves a teacher envisioning the mental activity sequence that a learner may go through when she assimilates a task. This, essentially, seems compatible with Simon's (1995) notion of hypothesizing how change in the learner's available mathematics would lead to the intended mathematics; that is, articulating an HLT. If unable to create an SOM of a learner's mathematics, one cannot successfully envision a mental activity sequence through which the cognitive change might unfold. This claim draws on the premise that, "students construct their understandings, they do not absorb the understandings of their teachers" (Simon, 1995, p. 122). Therefore, the (mental) activity sequence is unique to the learner based on the existing conceptions that the teacher infers the learner has.

The teaching triad. The *Teaching Triad* (Tzur, 2010) is a model for instructional design that was created as a truncated, practical guide for teachers based on the seven-step instructional cycle. In place of the seven steps, it highlights three principle functions of the cyclical, teaching-learning process. The Teaching Triad takes into consideration what students should learn next based on what they already know and the ability for a teacher to thoughtfully select tasks.

The first "vertex" of the triad consists of the teacher's ability to infer students' existing, available mathematical conceptions based on their actions and language. For example, a teacher would be able to decipher two types of epistemic subjects (Piaget, 1966; Ulrich et al., 2014) in two learners who are presented with the following multiplicative situation: "5 packs of gum, each with 4 pieces of gum, how many pieces of gum in all?" The teacher notices that Learner A solved the problem by holding up five fingers and counting by four for each finger arriving at the total number of cans as 20. Learner A then explained, "I used my fingers to represent each pack and counted each of the pieces of gum. I knew to stop at my fifth finger because that was the last pack." Learner B, on the contrary, answered with the number 9 and explained: "The question said 'in all', so I added 5 and 4 and got 9 total cans. I held up five fingers on one hand and four on the other and counted 1-2-3-4; 5-6-7-8-9." In the first part of the Teaching Triad, a teacher would infer from Learner A's actions and explanation that she operated on the composite unit of 4 and seemed to determine an ability of tracking units to know when to stop counting (Risley, 2016; Risley, Hodkowski, Tzur, 2015; Risley, Hodkowski; Tzur, 2016). Similarly, the teacher would infer that Learner B possibly lacked an understanding of a concept of number and the ability to operate on composite units multiplicatively. The teacher would infer that, at this point, Learner B is only able to operate on units of one and, perhaps, begin operating additively on composite units.

Key to the first part of the Teaching Triad is the teacher's ability to diagnose the students' existing conceptions (assimilatory schemes) through creating an SOM (Steffe & Thompson, 2000; Thompson, 2000). That is, a teacher infers the students' understandings of the mathematics (SOM) and distinguish (bracket) those understandings from the teacher's own mathematical understanding (FOM). This loops back to the assimilation brought forward from the constructivist theory on knowing and learning: "The assimilation principle requires teachers to understand students' mathematics as qualitatively different from the teachers' understanding and, thus, as the conceptual force that constrains and affords the mathematics students can "see" in the world" (Tzur, 2010, p. 50).

In the second "vertex" of the Teaching Triad, the teacher draws on "research-based accounts of expert-intended (first-order) mathematical understandings" (Tzur, 2010, p. 58) and decides on mathematical goals for teaching *based on the learner's existing assimilatory schemes*. This vertex uses what the student can assimilate (the first part of triad) as the foundation to then determine what type of cognitive reorganization may take place to create new learning. For example, when considering Learner A and Learner B discussed earlier, a teacher would recognize that Learner A seemed to operate on a composite unit of 4 and may attribute this to a potential early stage of multiplicative double counting (mDC) (Tzur et al., 2013). As a result, the teacher could determine to move Learner A to a similar problem with different numbers to develop further the mDC scheme (Risley et al., 2015; Risley et al., 2016). On the other hand, the teacher would recognize that Learner B, who counted by ones, needed further assessment by the teacher and instruction to strengthen her concept of number in order to begin operating on a composite unit before moving into any multiplicative situation. Without an SOM of the learner's mathematics, a teacher cannot tailor goals for student learning to their available schemes.

The third, and final, "vertex" of the Teaching Triad entails that a teacher should choose appropriate tasks for a student's learning based on *both* the teacher's created SOM (Steffe, 1995, Steffe & Thompson, 2000; Thompson, 2000; Tzur, 2010) and the mathematical concepts the teacher intends to teach. Here, a teacher explicitly reasons how the tasks may draw on the student's ability to assimilate and use reflection on activity-effect relationship as a way to reorganize existing understanding to move toward more advanced concepts. In the example of Learner A, the teacher may choose a task (e.g., a variation of the Please Go Bring for Me game) that would be appropriate for this child to advance her mDC scheme. On the other hand, for Learner B the teacher would recognize that to push that child further into mDC would be a daunting, perhaps impossible goal at that time. The teacher could, instead, work with Learner B to begin operating on composite units by way of operation on ones (e.g., the How Far From the Start game). The Teaching Triad is conceptualized using the HLT (Simon, 1995; Simon & Tzur, 2004).

In summary, a teacher who adheres to a Conception-Based Perspective can mindfully use the Student-Adaptive Pedagogy approach to teaching. This will involve using the 7-step teaching cycle (or the Teaching Triad as its abbreviated version), including the creation of HLTs as a necessary practice. Importantly, CBP and a Student-Adaptive Pedagogy necessitate an SOM. SOM allows the teacher to determine where the students' existing conceptions are and how to foster next mathematics based on what the learner understands.

Research Questions

This dissertation study addressed the lacuna of research about how elementary teachers may shift to SOMs of their students. That is, the study focused on a shift in teachers' thinking about her students' work when beginning to recognize her own mathematics as being different from the students' mathematics and starting to articulate what the latter may be. By examining this lacuna, we can better understand, and link, how teachers construct their own understanding of mathematical realities (Cobb & Steffe, 1983) with how they may shift from using their own mathematical realities to distinguishing and using another individual's mathematical realities as a basis for teaching. Specifically, this dissertation study focused on aspects of teachers' shift towards SOM as teachers: (a) began to infer what a learner's existing understanding was while distinguishing it from the teacher's, (b) began to predict what a learner might anticipate from an activity based on the learner's existing understanding, and (c) began to capitalize on the expected learner's assimilation of what she already knows to guide subsequent instruction.

To this end, this study addressed the following questions:

1. What changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift away from mostly relying on their first-order models (FOMs) to teach mathematics?
2. What may be manifested in elementary mathematics teachers' work and explanations, as they shift from using only first order models towards differentiating between their first order model and students' mathematical reasoning?

CHAPTER II

INTRODUCTION

> To regard the mathematics and physics of education as second-order models rather than first-order models is more demanding because we have to know so much more to give currency to the voices of children, and thus to a broader spectrum of knowing not just constructing (Steffe, 1995, p. 505).

In this study, I intended to explain a process of change in elementary mathematics teachers' thinking as they make a shift towards an instructional approach that includes SOM. In this chapter, I discuss the importance of the study in terms of three topics. First is the apparent need to promote a teacher's use of SOM to help overcome prevalent underachievement in mathematics within the United States. Second is the potential to understand how an SOM supports teachers in pinpointing and thus possibly building on students' assimilation to increase learning compared to existing teaching practices. Third is the contribution to explaining how teacher development of SOM may serve in providing effective mathematical teaching as implied by the Student-Adaptive Pedagogy approach. To explicate these three topics, I first discuss current practices in mathematics education. Then, I examine how these existing practices warrant a change to SOMs. Finally, I present an alternative pedagogy to existing practices, which requires an ability to have an SOM.

Significance of the Proposed Study – Why Change Current Teaching Practices?

My main contention for the significance of this study is that examining teachers' shift towards SOMs can provide an alternative foundation for addressing the prevalent underachievement in mathematics in the United States. Students in the United States, across grade bands and social-cultural subgroups, have shown low mathematics achievement scores (NAEP, 2015). In an attempt to raise students' mathematical achievements, introduction and implementation of standards have been repeatedly enacted (CCSS, 2010; NCTM 1989, 1991, 2000). However, in and of themselves, those standards and reform oriented practices do not seem sufficient. In fact, this underachievement may be due in part to three current issues within education: the way mathematical standards and curricula are implemented, teacher mathematical understandings, and some common teaching practices promoted for pre-service and existing teachers. In the following subsections, I provide further descriptions of each of these issues and discuss how promotion of SOMs may help addressing them. In order to do so, I first present an image of an elementary mathematics classroom, which is used throughout the descriptions of each of the issues to illustrate how SOMs may serve as a solution.

Image: A mathematics classroom. To link each of the issues to the need for SOMs, I provide an image of an elementary mathematics teacher and her classroom. In this classroom there are roughly three groups of students to whom she is teaching mathematics. First, there are students at grade-level. These students seem to be those for whom the recommended mathematics "clicks" and they may succeed. Second, there are students who are above grade-level. These students seem to breeze through what is being taught and, at times, may be bored. Third, there are students who are below grade-level. These students typically struggle to understand the content they are being taught and may, at times, appear frustrated with their learning or lack thereof. I now move to describing each of the three issues while connecting them to the image of these three groups of students and explicating how promoting a shift towards an SOM can help.

Issue 1: Mathematical standards and curricula. One issue of the underachievement in mathematics may be related to the Common Core State Standards (CCSS, 2010). The CCSS defined what students should understand and be able to demonstrate in different mathematical areas. As CCSS (2010) stated, these new standards were developed to provide "research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time" (para 4). Unfortunately, the CCSS seem deficient in allowing teachers to determine where a student's existing understanding is and thus link with suitable goals for those students (Tzur, 2011).

Instead, by and large, efforts to advance students' mathematical achievements (CCSS, 2010; NCTM, 1989, 1991, 2000) seem to have focused on increasing expectations and fitting them to age groups and/or grade-levels, not to individual students. In this sense, the standards reflect their authors' FOMs. Accordingly, curricula developers have designed lessons for progressions that involve student interaction with the required grade-level material (McGraw-Hill Education, 2012; New York State Education Department, 2014; Pearson Education, 2012; The Math Learning Center, 2014; University of Chicago School Mathematics Project, 2007). This can be seen in reform-oriented teaching characterized by a Perception-Based Perspective, as explained in Chapter I (Simon, Tzur, Heinz, & Kinzel, 2000). Such interaction through teaching promotes students' mathematical understanding through immersing them in hands-on tasks and activities that, the developers and teachers believe, will bring about the "seeing" of the intended mathematics.

According to such an approach (PBP), students would "get" the mathematics by participating in the activity. However, such an approach seems to mask differences in how students assimilate the tasks/activities and thus can access the prerequisite mathematical ideas and engage in reasoning about intended ones. Dewey (1902) asserted that such an approach, which is not focused on student available (assimilatory) conceptions, lacks a direct relationship to the child's present or past experience and understanding. He considered this as a manifestation of three evils for student *learning*: (a) lack of organic connection to content, (b) lack of motivation, and (c) loss of quality in the learning process. In other words, with standard- and reform-oriented teaching, little to no attention is given to students' existing conceptions. Accordingly, there seems to be no attention to how teachers can determine what students can assimilate and therefore use as a basis for future, more advanced, mathematical concepts for the student.

In the aforementioned image of three groups within a classroom, if a standards-driven curriculum is implemented, only students who have the conceptual prerequisite for the intended learning are likely to gain from the enacted curriculum. This claim is rooted in the stance on learning as the advancement from not knowing to knowing a mathematical concept by reorganizing one's existing schemes (detailed in Chapter I). Thus, the current approach to teaching, promoted through typically used, standard-based curricula, can at best cater to the reorganization of assimilatory schemes of one of the three groups of students (e.g., the ones considered at grade-level). Unfortunately, the other two groups of students are not likely to learn the intended mathematics (e.g., it is "above their heads," or they already have the intended mathematics). Specifically, for the students below grade-level, the content is so far removed that they are mostly unable to assimilate the tasks/activities and, thus, are unlikely to make progress. Tzur et al. (2017) have recently provided a demonstration of this issue, by showing the predictive power of the strength of a child's conception of number as composite unit on her ability to engage in reasoning with the multiplicative double counting (mDC) scheme (Tzur et al., 2013).

Efforts to raise student achievements in mathematics seem to fit students to the required grade-level standard and curriculum. They thus, at best, seem to foster learning by students with "grade-level assimilatory schemes." What seems lacking in these efforts is an approach that fits curricular goals and activities to every student's available conceptions as a way to further promote each of their mathematical conceptions (Tzur, 2008). This latter approach necessitates understanding how teacher may shift towards an SOM so they can base their instruction on a student's existing mathematical realities so it promotes assimilation and conceptual reorganization suitable to every individual student.

Issue 2: A teacher's mathematical understanding. A second issue pertains to teachers' mathematical knowledge for teaching and how it effects students' learning and achievements (Hill, Rowan, & Ball, 2005). Mathematical knowledge for teaching involves much more than teachers' ability to solve mathematical equations and arrive at the correct answer. Teachers also need to understand the underpinnings of mathematical concepts, to both analyze and foster students' solutions and explanations (Hill et al., 2005).

Furthering a teacher's mathematics may not be sufficient to increase their pedagogical content knowledge for teaching. However, as I articulated in Chapter I, it is a necessary component for teaching mathematics, shifting towards an SOM, and creating a foundation upon which to build pedagogical content knowledge. Ball (1991) stated, "a teacher's understanding of mathematics is a critical part of the resources available which comprise the realm of pedagogical possibility in teaching mathematics" (p. 52). Without such understanding, the realm of pedagogical possibility is insufficient (Ball, 1991, 2000).

In order to create an SOM of a student, a teacher needs to have an in depth understanding of the mathematics to which she tries to promote (Cobb & Steffe, 1983). In particular, the teacher's mathematics allows for two things: (a) inferring what the student can assimilate and (b) creating a model of the student to predict possible ways to help the student reorganize the existing mathematics into more advanced concepts (Steffe, 1995). If teachers do not understand a mathematical concept for themselves, it is highly unlikely they could recognize it in others.

In order to foster a student's reorganization of her existing understanding into more advanced mathematics, the teacher has to first recognize the existing mathematical mental actions of the student. This requires the teacher, or the more knowledgeable other, understand the mathematics more deeply than the student. Consider, for example, fostering a student's shift from counting-on to mDC (Tzur et al., 2013). To do this, a teacher needs to first recognize that the student views both addends in an additive problem as units composed of ones. Hence, the student anticipates that when combining two quantities, she can begin with one of those quantities as a thing and count-on from it while intentionally keeping track of the second quantity to know when to stop counting (Tzur & Lambert, 2001). In the 3-group classroom image, this would mean, for example, a teacher's ability to identify children who are yet to construct a conception of number (e.g., they count-all), or have constructed a weak conception (e.g., counting-on), or have constructed a strong conception of number (e.g., break-apart, make a ten, or BAMT strategy; see Tzur et al., 2017).

With this recognition, the teacher can then adapt instruction to each student in those groups. For those with BAMT, the teacher can work to foster their construction of the mDC scheme; for those with counting-on the teacher can foster construction of decomposition of composite units (e.g., BAMT); and for those with counting-all the teacher can foster construction of number as composite unit. However, if a teacher has yet to construct a conception of mDC herself (as opposed to reasoning additively about multiplication, as in "repeated addition"), she is unlikely to recognize and foster this way of reasoning in her students. To operate using both an SOM and an FOM requires that one can recognize, and therefore make, an inference to the mental activity of a student's mathematical understanding based on her observed behaviors and explanations. Noticing and interpreting these behaviors and explanations greatly depends on a teacher's FOM (Cobb & Steffe, 1983). Simply put, a teacher's FOM seems to serve as an "upper cap" for her SOM.

In general, if a teacher does not have a conceptual understanding of the mathematics being taught, there is, at best, the possibility for one group of students to learn the mathematics. The "at best" notion here refers to the likely possibility of none of the groups learning the mathematics properly (e.g., all would come to know multiplication as repeated addition, like the teacher). Moreover, if one of these students enters into a place where they can no longer assimilate and reorganize the mathematics, a teacher who does not understand the mathematics is not likely to recognize this. Similarly, the teacher is unlikely to pinpoint why the below grade-level students are struggling with the math and, therefore, be unable to provide intervention (I demonstrate these possibilities with the cases of Charlie and Sam, analyzed in Chapter IV). While the teacher may recognize the above grade-level students need more advanced ways of learning, the teacher's lack of understanding of the mathematics would limit the nature and effectiveness of extensions to their learning of the mathematical concept.

Issue 3: Teachers' awareness of students' mathematics. A third issue involves efforts that have been made to design teacher development toward the ability to notice what students are attending to (Jacobs, Lamb, & Philipp, 2010; Mason, 1998, 2008). In their view, the focus seems placed on awareness and noticing of specific students' behaviors and strategies, but not on inferences as to what mental activities underlie the noticed behaviors (SOMs) and, therefore, why particular behaviors occur and make sense for the students.

Accordingly, the work of Mason (1998, 2008) and Jacobs et al. (2010) suggested that teachers notice what students attend to, but the notion of assimilation seems missing in their view. In contrast, if a teacher is operating with an SOM, she can better understand why students are attending to what they do and behave the way they do. For example, consider the example (from Chapter I) of a teacher who works with a child while solving the problem, “You have 5 packs of gum; each pack has 4 pieces of gum. How many pieces of gum do you have in all?” The teacher may notice the child is adding ones as opposed to counting and tracking numbers larger than ones. A teacher who uses an SOM would not only attend to it (as Mason and Jacobs et al. emphasized) but also understand that units of one, rather than composite units, may be what is mentally available to the student. Consequently, that teacher would be able to consider that a student can assimilate counting by ones and thus bring forth this concept, and transform it, when teaching a more advanced mathematical conception.

While the efforts of Mason (1998, 2008) and Jacobs et al. (2010) provided a good first step, they seemed to pay little attention to the need for teachers to infer into the mental activity that may underlie what students attend to and why they behave the way they do. That is, those researchers seemed to overlook assimilation as a construct that explains what facilitates the students’ attending. In this sense, noticing is necessary but insufficient for fostering student learning and reasoning, because it falls short of specifying students’ assimilatory schemes. An SOM takes noticing further by using it as a basis for inferring the conceptual roots for why this student attends to it (assimilation). As I explained in Chapter I, such inferences provide the basis for fostering further learning as a reorganization of students’ available schemes (Piaget, 1971, 1985, 2001; Simon, 1995; Simon, Tzur, Heinz, & Kinzel, 2004; Steffe & Olive, 2010; von Glasersfeld, 1995).

Considering the three-group classroom again, the teacher would notice that the at grade-level students seem to be working through the problems and making some progress toward the curricular goals (standards). The teacher would also notice the particular struggles of students who are below grade-level (e.g., with calculations or with correct strategies). However, the teacher would not be able to pinpoint assimilation affordances and constraints in students' work, nor would she be able to use those as a way to build a model of the student's thinking (which is different from the teacher's) that explains why the students may be struggling (I demonstrate this with the cases of Charlie and Sam in Chapter IV). Similarly, the teacher would notice the above grade-level students excelling at the task without being able to pinpoint the assimilation affordances and constraints these students have. Thus, the teacher is likely to engage those students in tasks that would not foster further mathematical growth for them.

I have discussed three potential issues within education of the mathematics underachievement: standards and curricula, teacher mathematical understanding, and teacher development of noticing what students are attending to. I will now explain how SOMs can contribute to addressing these three issues and why my study of a shift towards SOM can make a contribution to change in the mathematics underachievement by fostering mathematics teacher education.

When a shift towards SOM occurs, teachers can begin to separate students' mathematical understanding from their own and therefore recognize that a student's experience with the learning can be very different from the teacher's. (This is further discussed with the case of Charlie in Chapter IV.) Such a separation may create a way of reasoning through the standards and curricula, which can then lead the teacher to decipher through tasks and lessons. Specifically, the deciphering could allow teachers to make choices for altering the tasks and lessons based on their inferences into how students may assimilate a task, not how they (the teachers) would. Therefore, creating SOMs of students can allow teachers to think differently about the reform-oriented tasks and make necessary adjustments that could cater to every students' available conceptions and allow students to advance via reorganization (see Chapter I) of their existing understanding.

Teachers' ability to create an SOM of their students is capped by their own understanding of the mathematics (Cobb & Steffe, 1983; Steffe, 1995). To shift towards SOM entails teachers' better understanding of the mathematics. Therefore, a shift towards SOM can increase teachers' mathematics. This in turn may create an enhanced ability for the teacher to infer into students' thinking and allow the teacher to use that inference to predict possible ways in which the student may reorganize her mathematics into more advanced concepts.

To shift towards SOM moves beyond simply noticing what students attend to. It takes into account student assimilation and uses it as a basis for instruction. In thinking about how one learns (Chapter I), a shift towards SOMs can develop a teacher's ability to inference into the students' mental activity and use it to then create tasks which are tailored to students' conceptual understandings (not above or below their heads).

I now turn to an example of my own development as a teacher. This example manifests all three issues, as I gradually shifted from operating solely on my FOM to operating on both an SOM and an FOM.

My Journey towards a Second-Order Model

By examining teachers' shift towards SOMs of their students' mathematical thinking, this study can create a foundation for teacher educators' work as they attempt to foster teachers' ability to learn to identify student existing understanding. This can, in turn, foster a transformation of existing teaching practices into that which may be better adapted to students (Tzur, 2001). I, for example, experienced this non-trivial shift in my own efforts as a 4th grade elementary teacher. Although I illustrate the shift with an example from my teaching of multiplication, it permeated my entire practice.

Initially, I taught my students the application of multiplication via algorithms and facts, which essentially constituted my own mathematics (FOM). I had no idea what multiplicative reasoning was. Thus, I could only use facts and algorithms to solve problems as given in the curriculum. I thought that if students played a game that involved practicing multiplication facts, and thus interacting with the materials, it would help them learn better. I noticed what the students were attending to, in terms of the correct execution of algorithmic steps and whether or not they got the correct answer, and equated those noticed behaviors with students' understandings (or lack thereof).

What I seemed to have overlooked in my prior teaching practice, and what I know now from my growth towards having an SOM, is that I neglected teaching my students based on the ways *they* understood the mathematics. My teaching, noticing included, was driven by the curriculum and standards, not by what students already knew. I believed that if students interacted with the set materials, they would get the math, and I would be able to attend to students when they struggled. Of course, this was not enough for the students' mathematical success, though I did not know why at the time. Looking back at my teaching now, I see how what my students could assimilate was the basis for what knowledge they could then build, but I did not distinguish that from my FOM. Accordingly, I also neither considered what mathematics would be developmentally appropriate to teach next based on the math they already knew, nor which tasks would sensibly promote such learning.

Importantly, I now also recognize that, at the time, my FOM comprised a limited understanding of multiplication—mainly understanding it as repeated addition. I also had no idea of what constitutes numbers as conceptual units, or how such units serve in my students' learning to reason multiplicatively. Thus, my practice could be characterized as if I tried to facilitate students' learning of my own thinking (operation with FOM), instead of facilitating students' construction of their understanding, using what they already knew (operation with SOMs and FOM).

As I have been working with mathematics educators who focused on fostering the development of both FOM and SOM, I have gradually developed the awareness and ability to create SOMs. My perspective on knowing and learning grew toward a Conception-Based Perspective and Student-Adaptive Pedagogical approach. This growth allowed me to realize how my students' struggles, particularly in transferring what they learned to novel situations, were a reflection of teaching based on my FOM. My experience serves as an example that it is possible for a teacher to shift from a practice rooted in her FOM to a practice rooted in both an SOM and an FOM. While I can now recognize this shift in me due to the intensive, targeted work with mathematics educators, no research exists on how such a shift towards an SOM occurs for teachers in general and for elementary teachers in particular. In fact, while I can report about some aspects of my own change, the process of transformation on my own ways of modeling students' mathematics remains largely obscured.

A Possible Way Forward: Student-Adaptive Pedagogy – Teaching that Requires Second-Order Models

A teacher's ability to construct and better understand a student's mathematical reality (an SOM) can drive pedagogy, particularly if it is geared toward fostering learning as a transformation in what students already know (Cobb & Steffe, 1983). A specific approach that draws on SOMs is called Student-Adaptive Pedagogy (detailed in Chapter I) (Steffe, 1990; Tzur, 2013). This approach entails teachers strategically and constantly identify what mathematics to teach and which tasks can assist in achieving the identified goals for students' learning based on students' assimilatory schemes (further defined in Chapter I). This approach underlies teachers tailoring of key instructional moves to students' existing and/or evolving mathematical ways of reasoning.

To successfully enact the Student-Adaptive Pedagogy approach, however, a teacher must learn to explicitly distinguish her own mathematical understandings (FOMs) from the students' mathematics. Accordingly, a teacher needs to continuously infer and create models of plausible students' understanding (SOMs) based on behaviors she notices when they solve mathematical tasks. In thinking about the three-group mathematics classroom at the beginning of this chapter, a teacher who adheres to the Student-Adaptive Pedagogy approach, which includes an SOM of her students, can successfully promote the learning of *all three levels of students*. This is due, in large part, to her ability to create models of each student's assimilatory schemes and use those for promoting further mathematics learning. It is a teacher's understandings and interpretations of the mathematical realities of students that provide the basis for using a Student-Adaptive Pedagogical approach. In such an approach, accepted standards, such as the CCSS, are understood through the teacher's FOM and can at best serve to guide the goals (end-points) for student learning. It is the creation of SOMs of students' ways of reasoning that guide the starting points, the path, and how tasks may be designed to promote the intended progression.

In Summary

I presented three issues within our educational system and a pedagogical way forward to overcome underachievement in mathematics. Teachers' shift towards SOM can help foster mathematics teacher education, which can contribute to closing the achievement gap. Specifically, shift towards SOM can: (a) create an ability for teachers to reason through the standards and curricula in order to provide learning opportunities geared to a students' existing understanding, (b) enhance teachers' mathematical understanding so that inferences into students' mental activity can be made, and (c) allow for inferences into students' mental activity based on observed student behaviors. This then can lead to instructional practices where focus is placed on the student first and what the student can assimilate with goals for further understanding via reorganization (see Chapter I).

The proposed solution amounts to promoting the Student-Adaptive Pedagogy approach, which is rooted in a constructivist, Conception-Based stance on knowing and learning. The Student-Adaptive Pedagogy approach includes a constant cycle of three principal teaching activities, as depicted in the Teaching Triad (outlined in Chapter I) (Tzur 2010). To design tasks beneficial for students' conceptual advances, the Student-Adaptive Pedagogy approach requires regular planning and modification of the HLT (Simon & Tzur, 2004), which is the third of a seven-step cycle outlined by Tzur (2008). For teachers to enact Student-Adaptive Pedagogy suitably, they need to add SOM of their students' mathematical thinking to their FOMs of mathematics (while constantly improving the latter).

CHAPTER III

METHODS

The main purpose of this study was to examine and explain a possible process of change in elementary school mathematics teachers' thinking as they make a shift towards an instructional approach that includes using a second-order model (SOM) (Steffe, 1995, 2000). While research exists on the types/levels a researcher may be able to create for an SOM (Ulrich, Tillema, Hackenberg, & Norton, 2014), this dissertation study focused on what was the possibility for an upper-elementary teacher to move towards one of the levels of SOM. To recap, this study intended to answer the following research questions:

1. What changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift away from mostly relying on their first-order models (FOMs) to teach mathematics?
2. What may be manifested in elementary mathematics teachers' work and explanations, as they shift from using only first order models towards differentiating between their first order model and students' mathematical reasoning?

Currently, there appears to be no research on the advancement of SOM within teachers. Thus, research for this study needed a design to which the researcher could both examine the phenomenon under study and generate an explanation of processes participants might go through as they make a shift towards SOM (Creswell, 2013). Accordingly, a grounded theory (qualitative) research design was used (Glaser & Strauss, 1967). In addition, to increase the likelihood of the phenomenon under study to actually take place, in this study the researcher served two roles: a coach to the teachers and a researcher, continually researching the teachers' changes through the coaching. Accordingly, this study used the Teaching Development

Experiment methodology (TDE) (Simon, 2000). In this chapter, I include the background of the larger project, within which this dissertation study has been conducted, a description of the study participants, and a description of the Teacher Development Experiment methodology.

Research Setting

This study was conducted within the context of a larger project, funded by the US National Science Foundation (NSF), titled, *Student-Adaptive Pedagogy for Elementary Teachers: Promoting Multiplicative and Fractional Reasoning to Improve Students' Preparedness for Middle School Mathematics* project (AdPed Project)[8]. This project aims to (a) support teacher growth in understanding and implementing Student-Adaptive Pedagogy through a professional development model that focuses on multiplicative and fractional reasoning, and (b) to promote/improve student learning and outcomes while measuring their growth in those mathematical domains (Tzur et al., 2015). Four main questions guide the larger study:

1. How can key dimensions of Student-Adaptive Pedagogy be identified and measured in teacher practices?
2. How can students' multiplicative reasoning be measured?
3. To what extent does the professional development model promote teachers' growth toward Student-Adaptive Pedagogy?
4. Do student outcomes increase after teacher professional development?

The larger AdPed Project involves, as subjects, both teachers and students of upper-elementary grades from elementary schools in an urban area in the western United States. Four schools, from two school districts, are part of the larger project. Each of the districts has over

[8] This study was supported by the US National Science Foundation under grant 1503206. The opinions expressed do not necessarily reflect the views of the Foundation.

75% of the student population who are English Language Learners (ELL) and receive free or reduced lunch.

As discussed in Chapters I and II, the Student-Adaptive Pedagogy approach necessitates that a teacher operate with an SOM. However, operating with an SOM does not necessarily make a teacher one who adheres to the Student-Adaptive Pedagogy approach. Studying teachers who participated in the larger project made sense, because it was my hypothesis they will have opportunities to shift towards an SOM as their development toward Student-Adaptive Pedagogy is promoted. Thus, these teachers could serve as subjects for detecting and studying the phenomenon of a shift towards SOM.

This dissertation study used a subset of data collected within the larger AdPed Project, consisting of multiple teaching episodes and learning interactions between me, the researcher-coach, other team members of the larger project, and the participating teachers. As the researcher-coach (RC), I recognize the original goal of the larger project was to promote growth in the Student-Adaptive Pedagogy approach to teaching. Thus, working with teachers who participated in the larger project provided a strategic research site for detecting a shift towards SOM. Specifically, each event of data collection in the larger project placed an explicit focus on the extent to which teachers understand students' mathematical conceptions, that is, on what each student could currently assimilate. Creating an SOM requires a hypothetical thinking (making inferences) of the student's knowledge in order to gain understanding of what the individual could assimilate (Steffe, 1995). As such, developing an SOM has become a sub-goal of my work with each of the two case study teachers[9]. Before discussing the TDE methodology, I provide background for each of the participants.

Participants

This study used a convenience and purposeful sampling scheme (Onwuegbuzie & Collins, 2007). I chose the sample of two case study teachers among the participants of the larger, AdPed Project, so they could inform the phenomenon under study (Creswell, 2013), namely, their possible shift towards SOM. Focusing on only two teachers followed a recommendation made by Steffe (personal communication), to work with a small number that provides for in-depth scrutiny of a phenomenon that has not been studied previously. To this end, the sample consisted of two upper-elementary teachers who, when joining the larger project, seemed at different entry points in regards to both SOM and Student-Adaptive Pedagogy approach (background of each participant is described below). Those entry points helped to increase variation and likelihood of detecting a shift towards an SOM. As Creswell (2013) stated, "when a researcher maximizes differences at the beginning of the study, it increases the likelihood that the findings will reflect differences or different perspectives—an ideal in qualitative research" (p. 157). By increasing sampling variation, I intended to enhance the possibility to better understand how to shift from sole use of FOM towards including an SOM in the teacher's pedagogy may occur.

One possible increased variation criterion involved prior experience with the Student-Adaptive Pedagogy approach; that is, before teachers began their participation in the larger project. In the sample, I included both a teacher who had no previous experience with the

[9] I describe each of the data collection events and their relation to development of SOM in the Data Collection section of this chapter.

Student-Adaptive Pedagogy approach and one who had attended graduate courses at the university, in which promoting the Student-Adaptive Pedagogy approach served as a major goal. Another criterion to increase variation was the experience of teaching each participant had upon commencement of the study (novice or experienced). Specifically, one teacher from each category of these two criteria was included in this study, which I now further explain.

A case study with previous experiences: Charlie. When joining the larger AdPed Project, Charlie (pseudonym) was just beginning his second year of teaching in a 4th grade classroom. At that point in time, he had received his teaching certification and Master of Arts degree in Curriculum and Instruction. As part of the Teaching Certificate program, prior to joining this study, he had taken mathematics education courses that focused on mathematical knowledge for teaching, on inferring students' conceptions, on determining next mathematics to teach based on what students already know, and on better selecting tasks to help students learn the intended math. Since Charlie's graduation, he had been taking additional courses in mathematics education at the university. Additionally, prior to joining the study, Charlie had co-taught elementary mathematics to students in his classroom with professors from the university who taught the mathematics education courses.

A case study with no previous experience: Sam. When joining the AdPed Project, Sam (pseudonym) had no prior experience with the Student-Adaptive Pedagogy approach. Sam was then starting her eighth year of teaching and was teaching in a third grade classroom. Of several possible teachers in the larger project, I chose Sam as a case study of an experienced teacher without prior exposure to the Student-Adaptive Pedagogy approach, because her growth through the larger project's activities indicated likelihood for detecting aspects of a shift towards SOM. While her growth appeared different (and possibly less pronounced) than Charlie's, Sam provided a contrasting case and thus an increased likelihood of studying different stages/phases in teachers' shift towards SOM.

Qualitative Research Methods

This study used a grounded theory (Glaser & Strauss, 1967) qualitative approach, designed to examine a phenomenon that has not yet been studied. Qualitative studies that use grounded theory are those that involve researchers seeking to generate and discover a theoretical explanation for a process or an action (Creswell, 2013). Denzin and Lincoln (1994) depicted qualitative research as,

> multimethod in focus, involving an interpretive, naturalistic approach to its subject matter. This means that qualitative researchers [are] attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them. Qualitative research involves the studied use and collection of a variety of empirical materials—case study, personal experience, introspective, life story, interview, observational, historical, interactional, and visual texts—that describe routine and problematic moments and meanings in individuals' lives. Accordingly, qualitative researchers deploy a wide range of interconnected methods, hoping always to get a better fix on the subject matter at hand. (p. 2)

I chose a qualitative method for this study because there is currently no existing research on how a shift towards operating on both an FOM and an SOM may occur in a teacher. Creating an initial depiction of this phenomenon would be supported by qualitative methods. Specifically, I chose to study teachers in their classrooms during an academic year in which they were receiving coaching through the larger AdPed Project Professional Development (PD) program, which enhanced the likelihood of this shift in each of the two case studies. Each qualitative method for data collection that I used is further described under Data Collection.

Due to the unknown phenomenon of teachers' shift to operating with an FOM and an SOM, I anticipated that, like in other qualitative studies, the interactions with the participants could lead to modifications, adjustments, and alterations of my initial hypotheses. Such modifications were expected particularly because of the reflexive relationship, between coaching the teachers and analyzing data about their shift towards SOM. This reflexive relationship is a key aspect of the TDE methodology I chose, which is describe next.

Teaching development experiment (TDE). In this section, I outline key elements of a TDE (Simon, 2000), which is the qualitative approach used in the larger project and built on in this dissertation study. I then describe how the elements of the AdPed Project and data collection of this study align with components of a TDE. This alignment provided opportunities for both promoting teachers' growth and for analysis of that growth.

Simon (2000) designed the TDE methodology for studying teachers' development by a researcher acting as both the researcher and a coach. TDE interweaves key elements of three constructivist methodologies: (a) Teaching Experiment (Cobb & Steffe, 1983; Steffe, 1991; Steffe, Thompson, & von Glasersfeld, 2000), (b) Whole Class Teaching Experiment (Cobb, 2000; Cobb, Yackel, & Wood, 1993), and (c) a case study methodology. I now briefly describe each of these in the context of TDE before defining how each of these was part of this dissertation study.

Constructivist teaching experiment/whole class teaching experiment. The TDE interweaves elements of the constructivist teaching experiment (Steffe, 1991) and of the whole class teaching experiment (Cobb, 2000). In both, the researcher serves as the teaching agent who promotes the learning she sets out to study. This allows for increasing the likelihood of reflexively fostering the phenomenon to be studied while engaging the researcher in a process of reflection, analysis, iteration, and designing instruction to reach repeatedly refined goals (Simon, 2000; Steffe, 1991).

The Constructivist Teaching Experiment (Steffe, 1991) was designed to inquire into how students construct particular mathematical ways of thinking (e.g., schemes). It focuses on formulating a model of a student's mathematical reality (i.e., SOM) and how that reality changes through teaching mathematics for conceptual growth. Researchers use teaching experiments to explore and explain goal-directed, mental, mathematical activity of students (Czarnocha & Maj, 2008; Steffe, 1991). Similar to a teaching experiment, teachers who are participating in the PD of a TDE are considered the "students." Accordingly, for the teaching experiment elements of this study I use two main methods (further explicated below): Buddy-Pairs (Tzur & Marshall, 2003; Tzur et al., 2015) and Account of Practice data sets (Simon & Tzur, 1999).

Whole class teaching experiments focus on promoting teachers' own mathematics (Simon, 2000), as well as enhancing their mathematical instruction. In this study, the targeted (enhanced) instruction is the Student-Adaptive Pedagogy approach. Promoting teachers' development of this approach focused on each vertex of the Teaching Triad (Tzur, 2010): (a) what are students' current mathematical conceptions, (b) what mathematics do I intend for my students to learn based on their current conceptions, and (c) what tasks are likely to foster the intended conceptions. A shift towards an SOM falls under the first vertex. As part of the TDE, data for this study were collected through the two Summer Institutes and through the half-day, professional development workshops (both further detailed below).

Case study. The case study portion of the TDE includes studying the real-life instruction of a teacher within her classroom (Simon, 2000). These job-embedded experiences of teacher learning allow the researcher-coach to gain insights into an individual teacher's development within the overall study. The case study portion of a TDE largely takes into account how development occurs, which is then possibly enacted in practice, within the mathematics classroom taught by the teacher (Simon, 2000). For this dissertation study, besides the variance in entry points to the study, each participant served as a case of teacher development that included learning through enacting and reflecting on teaching activities within a real-life classroom setting (Creswell, 2013).

Data Collection

The TDE consists of both whole group and individual interactions between the researcher-coach and the case study participants. For this study, both the whole group and individual components were utilized for data collection. Each of these components had multiple sessions. Each session is considered and referred to as an "episode." Episodes include a sequence of moves that were used to promote teacher change, and, in this dissertation study, focus on the nature and extent to which the shift towards an SOM took place.

The whole group episodes consisted of two Summer Institutes as well as four half-day Professional Development workshops. Individual teaching episodes consisted of both Account of Practice data sets (Simon & Tzur, 1999) and of Buddy-Pairs (Tzur & Marshall, 2003). These guided co-teaching sessions were with me or other AdPed Project team members/RCs and the case study teachers. All episodes were part of the larger AdPed Project, and key aspects of these PD components of the larger project are described below.

Each episode within a TDE methodology involves the researcher(s) as both a researcher and a coach (hereafter referred to as researcher-coach or RC). The researcher-coach enacts or observes and co-organizes/plans each of the episodes to generate and confirm (or disprove) ongoing hypotheses regarding teacher development (Simon, 2000). For each episode, I took field notes along with another team member who assisted in documenting observations, generating hypotheses, recording overall impressions, and planning future work with the teachers. All episodes were recorded on video for future analysis and reference. I now describe each of these data collection episodes in detail and describe specifically how, in each episode, there is an enhanced sub-goal for the study participants of promoting shift towards an SOM.

Summer institutes. The AdPed Project included two, five-day Summer Institutes (SI-1 and SI-2), led by PI Tzur and the AdPed Project team to promote teachers' growth toward the Student-Adaptive Pedagogy approach. SI-1 focused on five main goals:

1. Build a community of learners through building positive, caring relationships;
2. Deepen teachers' ways of reasoning multiplicatively (whole numbers, fractions);
3. Foster understanding of students' ways of reasoning that differ from the teacher's and make sense of where students are conceptually along a research-based, developmental path for:
    a. Concept of Number,
    b. Multiplicative reasoning path (whole numbers), and
    c. Fractional reasoning path;
4. Promote teachers' appreciation and understanding of student struggles in learning upper-elementary mathematics; and

5. Improving teachers' practices by promoting a shift toward the Student-Adaptive Pedagogy approach with a cyclical process of:
    a. Noticing student strategies (not just/mainly right or wrong answers);
    b. Diagnosing plausible conceptual sources of strategies (why they make sense to the students);
    c. Acting (plan-the-implement) with (a) articulated goals for students' learning along the developmental paths and (b) tasks/activities – with explicit rationale – to foster intended progress; and
    d. Reflecting on the extent/reasons that activities did (or not) bring about intended the learning.

To accomplish these goals, SI-1 engaged teachers in the daily activities as follows.

Five-Day Plan:

	Morning	Afternoon
Day 1 Tuesday, 5/31	• Introduction Activities • **MCKT** Assessment (pre-PD)	• M&M activity Part 1: (mixed school groups) • M&M (Part 2): Teach another group
Day 2 Friday, 6/3	• Noticing students' solutions (video) • Teachers solve problem your way • Begin planning to teach to this goal	• Discuss: Group M&M systems • Learn together: Attributes of system • Learn a different system
Day 3 Monday, 6/6	• Multiplicative operations required • Noticing student solutions (video) • Planning: Promote additive to multiplicative transition • **New method – PGBM**: how it works in class, rationale, key features	• First four multiplicative reasoning ways • Link 4 ways to (a) units operated on, (b) operations used, (c) how PGBM helps, (d) mathematics in curriculum it supports
Day 4 Tuesday, 6/7	• Solve fraction tasks • Notice students' unsuccessful work on a typical ("trivial") fraction task • Elaborate/discuss plans for teaching fractions (from homework)	• **New method - French Fry**: how it works in class, rationale, key features • Diagnose students' work; revise plan • First four, ways of fraction reasoning; link to intended math in curriculum
Day 5 Wednesday, 6/8	• Learn math and pedagogy for recursive partitioning ways of reasoning • Diagnose own understanding before/after learning experiences • Plan to teach this (link to curriculum)	• **MCKT** Assessment (post-1) • Reflection and evaluation – recommended pedagogical cyclic pattern (notice/diagnose –> plan –> implement/adjust –> reflect) • Next steps

Figure 3.1. Five-day plan for SI-1.

Three of the SI-1 goals were relevant for promoting, and studying, shift towards an SOM: (a) deepening teachers' ways of reasoning multiplicatively (whole numbers, fractions), (b) fostering understanding of students' ways of reasoning that differ from the teacher's and making sense of where students are conceptually along a research-based, developmental path, and (c) improving teachers' practice by promoting shift towards the Student-Adaptive Pedagogy approach with a cyclical process (noticing and diagnosing). SI-1 spent extensive time promoting teachers' development of the mathematical knowledge for teaching (MKT; see Hill & Ball, 2004). This focus on teachers' MKT began on Day 1, through activities that fostered better understanding of place value base ten systems, multiplication, division, and fractions. Activities used for promoting growth in teachers' mathematical understanding were selected so that teachers could then use them with students in their classrooms. For example, teachers learned mathematics through activities such as the *Please Go Bring for Me* multiplicative reasoning game (Tzur et al., 2013) and the *French Fry* fractional reasoning development activity (Tzur & Hunt, 2015), which were designed to promote students' learning of those ways of reasoning.

In addition, in order to create an SOM of a student, one has to recognize and identify that the student's understanding of the mathematics (i.e., what the student can assimilate) may be different from the teacher's own mathematics (Steffe, 1995). This can eventually lead to promoting further, more advanced mathematics in students, which may be different from how the teacher's own promotion of mathematics occurred. SI-1 focused on this idea in two ways. First, it involved teachers watching videos of students as the students worked through tasks, guiding teachers to infer mental units students might have used to solve a task and how they operated on those units. Second, teachers were engaged in learning about aspects of the Student-Adaptive Pedagogy approach. This approach asserts learning occurs through a reorganization of what a student already knows; that is, reorganization of her assimilatory schemes. This twofold focus (MKT, children's thinking) was emphasized throughout the five-day workshop by displaying a version of the Teaching Triad (Tzur, 2010) and discussing the importance of identifying what a student can assimilate in order for that student to reorganize more advanced mathematical understanding.

While goals for SI-2 were similar to those for SI-1, the AdPed Project team altered the plan to fit with teachers' growth over the year of PD. For example, in SI-2, promoting teachers' MKT focused on their fractional reasoning, whereas in SI-1 it focused on multiplicative reasoning with only a minimal attention to unit fractions. This included advanced ways to reason with fractions, such as reversible and recursive partitioning schemes (Steffe & Olive, 2010; Tzur, 2014). SI-2 also included more observations of students and the underlying reasoning (e.g., units and operations), in which they were solving particular tasks. Further facilitation regarding the theory of learning that underlies the Student-Adaptive Pedagogy approach was also emphasized, including a presentation of a new, improved diagram of the teaching cycle and explicit discussions of assimilation and accommodation.

Specifically, SI-2 focused on the following five main goals:

1. Build a community of curious learners with positive, caring relationships;
2. Deepen teachers' ways of reasoning multiplicatively (mostly revolving around fractions);
3. Foster teachers' understanding of students' ways of reasoning that differ from the teacher's and make sense of where students are conceptually along a research-based, developmental path for:
    a. Conception of number,

    b. Multiplicative reasoning path (whole numbers), and
    c. Fractional reasoning path;
4. Appreciate and understand student struggles in learning upper elementary mathematics; and
5. Improve teaching through a shift toward the Student-Adaptive Pedagogy approach with a cyclic process consisting of five main practices:
    a. Noticing student strategies (not just/mainly right or wrong answers),
    b. Inferring plausible conceptual sources of those strategies (why they make sense to students),
    c. Acting (plan-then-enact) in the classrooms to foster students' learning along the developmental paths, while using tasks/activities – with explicit rationale to foster intended progress, and
    d. Reflecting on the extent/reasons that activities did (or not) bring about the intended math.

To accomplish these goals, SI-2 included the following daily activities for teachers to engage in.

Weekly Overview:

	Day-1	Day-2	Day-3	Day-4	Day-5
8:30-10:15 Morning 10:30-12:00	a) Reconvening MCKT b) AdPed Gist + SI Overview: Triad & Practices Schemes (6 MR, 8 FR)	a) Practices #1-2 (Video B, Division) Language Lens b) Math for Teaching: FR-2	a) Math for Teaching: FR-3 b) Practice #3 (Video A/B, mDC & MUC & Division) + Language Lens	a) Math for Teaching: FR-4 b) Practices #4-5 (Video A/B, mDC, MUC, & Division) + Language Lens	a) Practices #4-5 (Video C, iteration-based) + Lang. Lens b) School grade-team discussion (challenges)
12:00-1:00	Lunch	Lunch	Lunch	Lunch	Lunch
1:00-2:30 Afternoon 2:45-4:30	c) Math for Teaching: FR-1 d) Practices #1-2 (Video A: mDC, SUC, UDS, MUC) + Language Lens End of day evaluation	c) School teams planning: MR, FR-iteration d) Practices #1-2 (Video C, Iteration-based) Language Lens End of day evaluation	c) School teams planning: MR, FR-iteration, early recursive d) Practice #3 (Video C, iteration-based) + Language Lens End of day evaluation	c) School teams planning: MR, FR-iteration, FR-recursive d) Practices #4-5 (Video C, iteration-based) + Language Lens End of day evaluation	c) MCKT d) SI-2 Evaluation Reflect & Look Ahead

Figure 3.2. Five-day plan for SI-2.

Half-day professional development workshops. As part of the AdPed Project's PD program, teachers in each school building were engaged in half-day workshops during the academic year. These PD workshops focused on (a) promoting teachers' own understanding of the mathematics they teach, specifically MKT of multiplicative and fractional reasoning; (b) teachers' awareness of and capacity to identify student conceptions; and (c) designing and adapting classroom situations to advance students from these conceptions to the intended ones (Tzur et al., 2015). Each workshop consisted of either one, four-hour session or two, two-hour sessions held within a week of each other. Similar to SI-1 and SI-2, the workshops focused on teachers' development of awareness to and an ability to distinguish students' reasoning from the teachers' own mathematical understanding. That is, AdPed Project PD workshops provided additional context in which teachers could shift towards an SOM, and thus another source of data for this dissertation study.

Buddy-Pairs. Dr. Tzur developed *Buddy-Pairs* as a job embedded PD method (Tzur & Marshall, 2003) as an adaptation to the Japanese Lesson Study method (Bass, Usiskin, & Burril, 2002) and the idea of ThinkingPartners (Cordova, Kumpulainen, & Hudson, 2012). Each Buddy-Pair episode consisted of two, interrelated activities: an experience of co-teaching a lesson and a reflection session about that co-taught lesson. It aimed to extend teachers' knowledge of mathematics, of their students' mathematical conceptions, and of ways to tailor future teaching (goals, activities) to students' available conceptions via reflection on their own teaching activities as well as on their buddy's activities (Tzur et al., 2015).

Throughout the school year, each participating teacher partnered (buddied) with another teacher in her school. About twice a month, each teacher either visited the buddy's classroom or hosted their buddy to observe teaching as well as students' work. Teaching mathematics during those Buddy-Pair visits was done not only by the teacher in the visited classroom, but also in collaboration with me as a researcher-coach or other AdPed Project team member (hereafter referred to as co-teaching).

Following each Buddy-Pair co-teaching session, a reflection session with both buddies was held led by me or an AdPed Project team member. This reflection session served as an opportunity for teachers to reflect on and learn more about how to better understand their students' mathematical conceptions, determine what mathematics may be best for students to learn next based on what they know, and how to get them to the new mathematical understanding. Four main questions guided each Buddy-Pair episode (Tzur et al., 2015):

(a) What mathematical idea were students supposed to learn and what does it mean to understand it?

(b) What understanding *did* students develop and how do you know?

(c) What were you doing that promoted or hindered the learning sought?

(d) What would you teach/do in the next lesson and why (explained in terms of students' available conceptions)?

A Buddy-Pair episode is specifically designed to promote teachers' Reflection on Activity-Effect relationship in their teaching (Simon, Tzur, Heinz, & Kinzel, 2004), particularly focusing on Reflection Type 1 and Reflection Type 2 (Tzur & Simon, 2004; Tzur 2011). Each buddy served as a comparison to the thinking of the other teacher. By providing sources of comparison, both types of reflection could occur, allowing for a greater likelihood of learning for the teachers.

When operating with an SOM, one must determine what a student knows conceptually and can therefore assimilate. This inference into the students' assimilation can then be used as a guide to determine what would make sense to promote in the students (i.e., accommodation/reorganization). This directly pertains to questions a, b, and c listed above that guided Buddy-Pair reflections. These questions oriented teachers' attention onto inferring students' work in each lesson, comparing the work of students within a class, and articulating the students' assimilation as a basis for learning intended mathematics. In addition, follow-up questions were asked as to "Why might a student answer like this?" or "Why, might this answer make sense for the student?" For each Buddy-Pair episode, teachers had opportunities to not only develop their own models of students' thinking, but also contrast that model by discussing it with their buddy and the RC. When recurring multiple times, teachers' reflection across instances of their teaching activities (i.e., Reflection Type 2) may promote their shift towards an SOM of their students' mathematical thinking. The next section describes a method for collecting data about each individual teacher within a TDE.

Account of practice data sets. Simon and Tzur (1999) designed the Account of Practice (AOP) methodology as a way to inquire into a teacher's practice by articulating, from the researcher's perspective, the teacher's perspective on knowing and learning mathematics. They used the term "teaching practice" to include "not only everything teachers do that contributes to their teaching (planning, assessing, interacting with students) but also everything teachers think about, know, and believe about what they do. In addition, teachers' intuitions, skills, values, and feelings about what they do are part of their practice" (p. 254). AOPs serve as a way for the researcher to articulate how a teacher makes sense of her experience of teaching mathematics—particularly the rationale she uses to select and/or adapt goals and activities for students' learning. When regular sets are conducted over time, researchers are able to construct an account of the teacher's practice as it is being developed while participating in a PD program (Simon & Tzur, 1999).

Data collection to create AOPs consists of sets that interweave interviews and classroom observations. Specifically, an AOP set may consist of an interview-observation-interview-observation-interview sequence (I-O-I-O-I), which allows linking the teacher's classroom activities with her thinking about those activities. This 5-episode sequence is considered a full AOP set. As part of the AdPed Project, some adaptations were made to include a shorter, 3-episode sequence consisting of interview-observation-interview (I-O-I). This shorter type of AOP set is considered a partial set. For this dissertation study, both types of sequences were used. It should be noted that while AOP sets provided important data for the study, the goal of the study was not to construct an account of teachers' practice—only to use the data collection and analysis method for addressing the research questions about teachers' shift towards SOM.

For this study, Account of Practice Data sets served as glimpses into a how a teacher understood the mathematics to be taught, viewed her students' mathematics, and how that may have been relevant for the students' learning of more advanced mathematics. AOPs were selected that focused on how the teachers were thinking about the mathematics the students currently have, and how intended mathematics could be promoted. In thinking about operating with an SOM, an AOP set provided evidence of the extent to which a teacher recognizes, or does not recognize, what students' can assimilate.

To summarize, in this dissertation study I utilized four different data sources: Summer Institutes, half-day PD workshops, Buddy-Pairs, and AOPs. Figure 3.3 provides an overview of the data that informed each of the case studies.

	SI-1 (Days attended)	SI-2 (Days attended)	Half-Day Workshops	Buddy-Pairs (amount, 2016/2017 school year)	AOPs (amount, timing, and type)
Charlie	5	5	4	4	Full, May 2016 Full, May 2017
Sam	5	5	4	4	Partial, May 2016 Partial, May 2017

Figure 3.3. Overview of data utilized for dissertation study.

Data Analysis

As described earlier, this study followed a qualitative grounded theory approach (Creswell, 2013; Glaser & Strauss 1967). Accordingly, ongoing and retrospective constant comparative methods of data analysis were used (Creswell, 2013; Glaser, 1965). Ongoing analysis took place during and immediately after each Summer Institute, half-day PD workshop, Buddy-Pair, or AOP set. I reviewed field notes taken for that day, and paid specific attention to interactions in which the participants seemed to focus on discussion of students' reasoning.

In order for me to infer into a teacher's shift towards an SOM, I had to create a model of the teacher's assimilatory apparatus and if that apparatus went through a change. That is, as the researcher as I analyzed the data, I created an SOM of the teachers in this dissertation study and their assimilation of their students' mathematical behaviors as well as how they considered students' mathematical knowing. In referring back to Chapter I, Conceptual Framework, my inferences into changes in the participants' assimilation of their students' mathematical behaviors is what constituted evidence of a shift towards an SOM. To note such changes, once data collection was completed, I observed and took relevant notes for all video from the Summer Institutes and half-day workshops. Although not used in the analysis chapter of my dissertation study, because evidence of teachers' assimilation of their students' mathematical behaviors was insufficient, my notes informed the development of themes and objectives/goals for learning. For example, an important goal during SI-1, SI-2, and the workshops was to foster teachers' understanding of students' ways of reasoning – operations on units. These learning activities provided background for findings of this study (see Analysis Chapter IV, Cogitation manifestation).

I then transcribed video recorded data from the Buddy-Pairs and AOP sets in chronological order. Each participant underwent separate, individual AOPs and BPs with the researcher. For Charlie, data comprised of two full AOP sets (I-O-I-O-I), a total of ten sessions, ~45-75 minutes each, and four BPs (two in Charlie's classroom) consisting of eight sessions, ~45-75 minutes each. For Sam, data comprised of two partial AOPs (I-O-I), a total of six sessions, ~45-75 minutes each, and four BPs (two hosted by Sam) consisting of eight sessions, ~45-75 minutes each (see Figure 3.3).

Once transcribed, I followed a constant comparative analysis method (Creswell, 2013; Glaser, 1965), while interweaving case-by-case with cross-case analysis. For each teacher, analysis included creation of major categories, or themes and their properties (Creswell, 2013; Glaser, 1965). To this end, I first read through each transcript of each participant's Buddy-Pair and AOP. As I was reading, I began to log what I considered to be potential evidence of operation solely on an FOM or on FOM and SOM. As I moved through the transcripts, chronologically, I began to notice contrasts in how teachers assimilated and interpreted students' behaviors, which I assimilated into a shift from using just FOM to also SOM. I noted those contrasts in a log. As my notes increased, I began to compare them across other logs, moving back-and-forth between the notes of one participant (within-case) and notes of the two participants (between-cases). When these comparisons led me to consider similarities across data segments, I noted a category for the logs I had written. I then worked to group like categories that focused on the core phenomenon of study—in this case, a teacher's shift towards an SOM. I chose this method because it allowed creation of themes in each teacher's shift (or lack thereof) towards SOM, as well as themes across participants.

Following, I analyzed (retrospectively), line-by-line, in categorical and chronological order from the transcripts all data segments that focused on the core phenomenon of SOM within each category I had created. My purpose in this last step of analysis was to chunk the categories into ideas (Glaser, 1965) and create a theme for each chunk. This also included eliminating excerpts that, while serving me well in the process of categorizing, did not provide compelling evidence in support of the themes I created. As themes were created and certain excerpts eliminated, I vetted them with my Dissertation Advisor (Dr. Tzur), to further consider what seemed to serve as compelling evidence. Consequently, although Buddy-Pair excerpts certainly served during the theme creation, I eliminated them from the data eventually presented in Chapter IV - so the most compelling evidence for each emerging theme is used. I then reviewed and categorized all themes that pertained to teachers' shift towards an SOM according to how the themes seemed to fit with one another. Finally, I developed propositions/hypotheses that convey a narrative of participants' shift towards SOM.

I again compared across the two cases to determine possible commonalities in their shift towards an SOM and themes that had developed through my line by line analysis. This organic process of moving back-and-forth from each case to cross-participant enabled me to compare nuances within evolving themes, which eventually led to casting them as manifestations. I selected and wrote about themes called manifestations in the Analysis (Chapter IV) of this study based on the extent to which they help to address one or more of the research questions.

CHAPTER IV

ANALYSIS

This study examined a shift in teachers' explanations of their students' mathematical behaviors, from being based primarily on the teachers' own mathematical knowing (i.e., FOM) to also attributing mathematical knowing that differs from that of the teacher (SOM). That is, this study aimed to describe shift in teachers' ability to construct hypothetical models of student's mathematical knowledge (underlying nature of their activities) based on and to make sense of students' observable activity (Steffe, 1995). The study addressed the following research questions:

1. What changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift away from mostly relying on their first order models (FOMs) to teach mathematics?
2. What may be manifested in elementary mathematics teachers' work and explanations, as they shift from using only first order models towards differentiating between their first order model and students' mathematical reasoning?

The data analysis presented in this chapter is gleaned from Account of Practice data sets (AOP) and Buddy-Pair (BP) sessions, which were conducted and/or observed by myself – the author of this dissertation study. The analysis on data focused on data sets from two teachers, Charlie and Sam (names of teachers, as well as of students, are all pseudonyms). Throughout the transcripts, three researcher-coaches are quoted. Myself, the author of this dissertation study, is referred to as researcher-coach 1 (RC1). Researcher-coach 2 and 3, referred to as RC2 and RC3, were team members of the larger research project.

As discussed in Chapter 1, assimilation is a central construct informing this dissertation study. Specifically, assimilation is used throughout this results chapter to infer a teacher's progression towards SOM from the researcher's perspective (Simon, 2000; Simon & Tzur, 1999; Simon et al., 2004). I focus on manifestations of changes in Charlie and Sam's assimilation of their students' mathematical behaviors and how they considered students' mathematical knowing. The manifestations reflect my inferences into Charlie and Sam's shift towards SOM; they do not connote awareness on their part.

Therefore, by manifestation, I mean the results of a shift that I inferred in the teachers' assimilation based on their observable behaviors. Specifically, to distinguish such manifestations I focused on the teachers' ability to provide a more in-depth description of their students' mental activity (SOM) that seemed separated from the teacher's first-order model (FOM). Data analysis presented in this chapter indicated two distinct phases through which the shift towards SOM may take place: (a) some realization that a student's mathematical experience differs from the teacher's own experience and, (b) an attempt to distinguish the student's mathematical experience, that is, to specify some model of the student's thinking.

It is important to note that each manifestation outlined indicates a shift towards SOM; however, these manifestations did not occur within independent activity of the case study participants. Each of the manifestations outlined are particularly contextualized, in that they were elicited as teachers' responses to questions within the interviews. Additionally, not all manifestations were observed in each case study. The purpose of the analysis is to explicate the meaning of each manifestation in a shift towards SOM while substantiating it in the data – not to claim that it is necessarily appearing in each teacher's work.

Accordingly, this chapter is organized around four manifestations, which I inferred in teachers' shift towards the twofold appearance of SOM (i.e., realizing students may think differently, specifying students' experiences). To provide a glimpse into each manifestation, I briefly discuss them here. Then, I shall further articulate them while analyzing teacher data within this chapter.

Manifestation 1: *Juxtaposition of Thinking*. I shall use this term to signify when a teacher began to contrast her own FOM experience with a student's different experience. As each participant interacted with myself or the team researchers, there were moments in which a juxtaposition of thinking occurred where teachers realized (through side-by-side comparisons between FOM and SOM) that the students' experience was not the same as the teacher's experience. For example, in the case of Charlie, by thinking about how he thought about the mathematics and how students thought about the mathematics, he was able to 'bracket' his understanding, which indicated an extension of his thinking from just FOM to FOM and SOM (further articulated below). Juxtaposing a student's experience with one's own is important for SOM, as it essentially *breaks the mirror* (discussed in Chapter 1) from a teacher's point of view. That is, a teacher is having an exchange with the researcher that indicates students' mathematical knowing differs from the teacher's FOM. The exchange(s) seemed to allow the teacher to express a novel interpretation of a student's understanding.

Manifestation 2: *Cogitation*. I shall use this term to attribute to a teacher an evolving ability to contemplate what might their students' experience of the mathematics be. Cogitation is important for SOM because it allows inferences to be made regarding the students' mathematical thinking as separated from the teacher's own mathematical understanding. For example, in the case of Sam, she was better able to think about and describe the units and operations her students were using and how this differed from her own FOM of thinking (further articulated below).

Manifestation 3: *Distinction*. I shall use this term to attribute to a teacher an evolving ability to make distinctions in their students' mathematical reasoning that go beyond the teacher's own (FOM) distinctions. My analysis suggests two main types of distinctions, one arising from comparing between students and the other from comparing two potential types of understanding within one student's reasoning. The manifestation of distinction can serve as evidence that the teacher brackets her own FOM from the inferences she makes in students' mathematical reasoning (Steffe, 1992). For example, Charlie and Sam began to distinguish between a student's ways of reasoning based on the units and operations the student could possibly be using to solve a particular problem and what could this mean on a developmental spectrum of conceptual understanding (further discussed below).

Manifestation 4: *Mindfulness*. I shall use this term to attribute to a teacher an evolving ability to shift her instructional focus, from basing practice only on an FOM to beginning to include an SOM as a force that drives fostering students' construction of intended mathematics by building on their distinct, available ways of reasoning. Said differently, Mindfulness refers to a teacher's intention to facilitate students' structuring of their own activity based on the mathematics the teacher inferred the students understood. For example, (further articulated below) teachers who began developing Mindfulness expressed a want to restructure their instruction differently so that it would have allowed for more opportunity for the students to bring the mathematics they understood and then for the teacher to build on that understanding without telling students how to "do" the mathematics. In the next section, I analyze data pertaining to the first manifestation, Juxtaposition of Thinking.

Manifestation 1: Juxtaposition of Thinking

To depict Juxtaposition of Thinking as a manifestation of a shift towards SOM, this section includes analysis of three excerpts from AOPs. For each segment, I inferred the teacher went through a state of juxtaposed thinking regarding his students' understanding of a learning experience and what the teacher thought could be the student's experience. The juxtaposed thinking illustrated in the following excerpts indicated a recognition that students' sense of the mathematics lesson is different than the teacher's, who facilitates the experience.

Charlie's juxtaposition of thinking. To further clarify the manifestation of Juxtaposition of Thinking, I begin this section with data indicating Charlie operated solely from an FOM and PBP. Then, I present data of how, through interviewing, Charlie began to juxtapose his FOM with his students' reasoning (Excerpt 3). The series of excerpts (1-3) presented here took place during Charlie's first full AOP (AOP 1), which consisted of a full data set (that is, I-O-I-O-I). To contextualize the excerpts, I provide a brief overview of Charlie's pre-interview and observation 1 before Excerpt 1.

In the first lesson, Charlie's goal for students, which he stated multiple times throughout the pre-interview and is summarized here, was to begin comparing unit and non-unit fractions on a number line. For example, he asked all students to plot $\frac{1}{3}$ and $\frac{1}{8}$ on two separate number lines and to determine which of those two unit fractions was larger. He also posed the task to students to determine an unknown point on a number line (including $\frac{2}{3}$, $\frac{2}{4}$, $\frac{4}{5}$, and $\frac{4}{6}$). This was the first time Charlie's instruction focused on the understanding of size relationships among unit fractions and among non-unit fractions with the whole class.

During the first (observed) lesson of this data set, Charlie led students through questioning and activities to do what, he seemed to anticipate, would allow them to accomplish the learning goal. He first had students provide examples of unit fractions, define what a unit fraction is, and compare a few unit fractions. For example, students responded that $\frac{1}{6}$ is a unit fraction because it has a one in the numerator. Charlie then included his additional understanding of unit fractions by leading students to state that $\frac{1}{6}$ would repeat 6 times or add 6 times to get to the whole. Similarly, Charlie then guided the discussion so that students practiced determining which unit fraction may be smaller or larger based on the number of pieces that fit into the whole.

Charlie's enactment of a plan to foster students' learning of the intended mathematics seemed guided by his FOM of unit fractions (developed, in part, in the context of the larger project's PD program). It seemed to match the logical sequence of his plan as stated in the pre-lesson interview. He started with students providing what they thought a unit fraction was, and when he thought they missed an important piece of information – he added it and asked the class to then repeat. I inferred that, for him, this rehearsal of his definition seemed adequate evidence that students understood unit fractions as he did, which could thus serve them in the logical progression to the next part of the lesson, namely, comparing two unit fractions.

Charlie then led students into the next portion of the lesson, which was to compare $\frac{1}{8}$ and $\frac{1}{3}$. The majority of students agreed that $\frac{1}{3}$ is larger than $\frac{1}{8}$, however two students disagreed. Charlie included himself in the disagree category, stating his reason that, "8 is larger than 3 so that is why $\frac{1}{8}$ is larger than $\frac{1}{3}$." He then challenged the class to prove to him which fraction was larger, which was to be done using two un-partitioned number lines he projected on the board (see Figure 4.1).

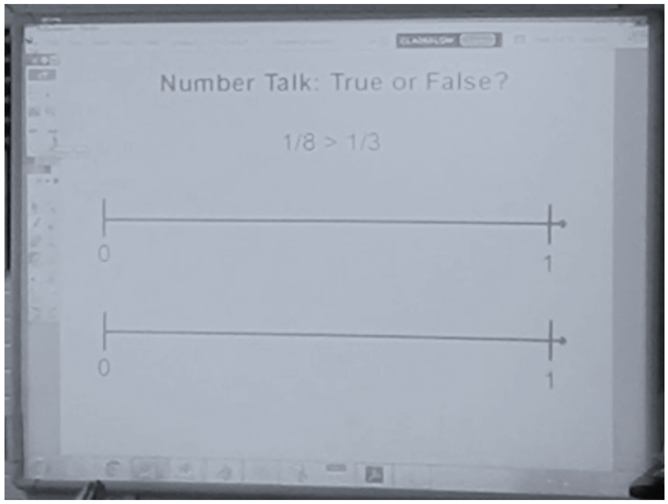

Figure 4.1. Charlie's blank number line chart presented on the white board for students to use when plotting $\frac{1}{3}$ and $\frac{1}{8}$ to determine which fraction is larger.

Then, the majority of the lesson (roughly 30 minutes) consisted of letting students take turns, with some apparent difficulties (Charlie discussed this in his post-interview), in going to the board and trying to plot both $\frac{1}{8}$ and $\frac{1}{3}$ on the two number lines provided. Charlie's earlier anticipation of the lesson progression was to move forward with plotting unit fractions on a number line based on students' understanding of unit fraction as proven to him by students repeating his definition. However, he soon realized that, when tasked with plotting, students struggled to place $\frac{1}{3}$ and $\frac{1}{8}$ on the number lines. To me, this serves as evidence that Charlie's FOM understanding of unit fractions was not available to his students. At this point, however, Charlie seemed unable to make the distinction that his FOM, which he is very clear about, is not available to students. I make this claim because his attempts to help the students seemed to be on *how* to solve the task—not on further exploring their understanding of unit fractions. That is, he seemed more focused on the overall goals for the lesson and making sure each was "accomplished" within the time of the lesson. The available data indicated no consideration of students' meanings for unit fractions based on their difficulty to plot both $\frac{1}{3}$ and $\frac{1}{8}$ on a number line. Such a consideration would be indicated had he not continued on with the lesson plan – moving into *non-unit fractions* on a number line.

Roughly thirty minutes later, Charlie then led students through an activity to help them determine a non-unit fraction point on a number line with a multiple-choice question. He introduced this portion of the lesson by defining a non-unit fraction for students: "Any fraction that has a number other than one in the numerator." Charlie again had students go through a similar, procedural process to the beginning of class, only this time with non-unit fractions: first find the unit fraction, next repeat it the number of times indicated by the numerator, and then make a mark where the non-unit fraction would fall on the number line (between 0 and 1). As each point was determined by the class, Charlie led the students through a process of elimination of the multiple choices. For example, he led the class to determine that Point A falls at $\frac{4}{5}$ through a rather directed discussion that the $\frac{4}{5}$ point is in the same place as Point A (see Figure 4.2).

For Charlie, I infer, it seemed that leading students through this activity and having them interact with the actual plotting of points meant the students now "had the understanding". I link such a conclusion on the part of the teacher to the Perception-Based Perspective (PBP), as discussed in Chapter 1. Specifically, he seemed to me to expect that the "We Do" portion of the whole class guided activity (seeing how he placed the non-unit fraction on the line) and discussion of the process leading to that placement – would yield the intended student understanding. Accordingly, to me, he also seemed to expect that they were then ready to move to a "You Do" portion of working independently on additional problems. This was indicated in Charlie's next move, providing the students with a worksheet of tasks to determine a few more fractional points on a number line. The worksheet slightly differed from the class example in that it contained multiple points on a number line, not just one as in the class example (Point A only). While students worked, Charlie moved about the class to answer questions students had about the task.

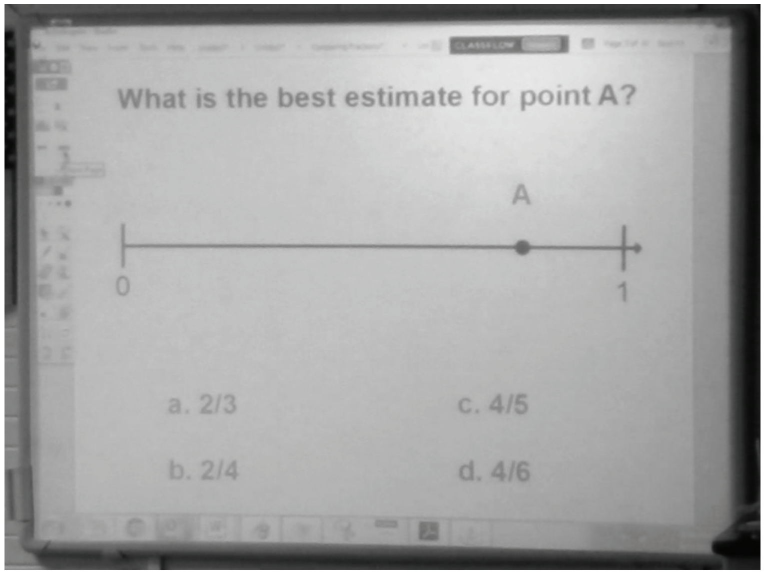

Figure 4.2. Charlie's presentation of plotting non-unit fractions on a number line with a multiple-choice question to students.

Finally, Charlie ended the lesson with a summary of what *he considered* students had *learned.* He reiterated the process of finding a non-unit fraction on a number line with a new problem of determining who (between two runners) ran further (see Figure 4.3). He reminded students of the process to find a non-unit fraction on the number line: first find the unit fraction, then repeat it a number of times determined by the numerator as a way to find where the non-unit fraction would be plotted on a number line. During this summary, Charlie then seemed to bring in an entirely new goal for students as a way to help students understand that $\frac{3}{6}$ and $\frac{1}{2}$ are equivalent. He began showing students an equation of $\frac{4}{8}$ divided by $\frac{4}{4}$ is $\frac{1}{2}$. And the lesson concluded.

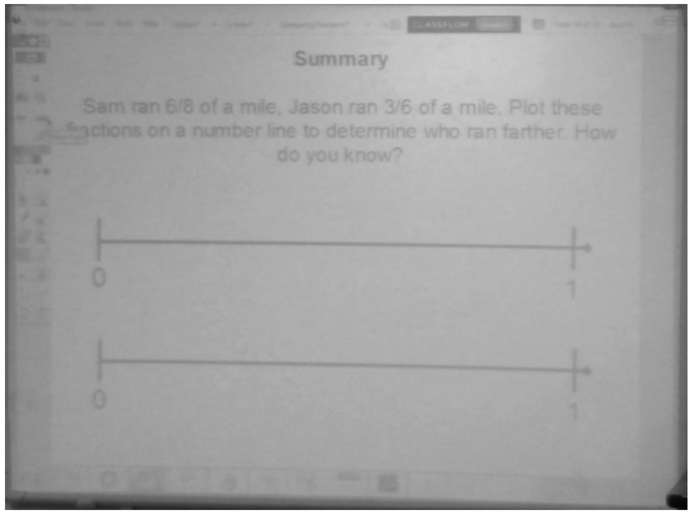

Figure 4.3. Charlie's end of class activity of finding non-unit fractions on a number line.

I provided this background to demonstrate that, at this point, Charlie's practice seems guided predominately by his FOM. For him, and I inferred he expected similarly for students, the logical process of plotting fractions requires first completing the production of unit fractions, then of non-unit fractions, and finally placing marks for the latter on the number line. In his FOM, the points on the number line also signify their distance from a starting point. Accordingly, expecting this thinking process on his students' part seems rooted in Charlie's FOM stance on how he believes the mathematics will be learned and thus how he has taught it. In this process, (every)one should first partition the whole in order to find the unit fraction; then, one uses that unit fraction and repeat it however many times the non-unit fraction requires. For him, I infer this logical process would repeat, apparently non-problematically, when comparing the other non-unit fractions. This reasoning on Charlie's part seems to also have driven what he would be noticing in students' work. At this point, there was no evidence from Charlie that he could consider what a learner's meaning for unit fractions might be. He continued to move forward with the lesson regardless of the students' struggle with the content.

At the beginning of the mid-interview of this data set (April, 2016), Charlie shared his thoughts on how the first lesson went. Excerpt 1 presents a portion of that interview.

Excerpt 1: Charlie's summary of observation-one (AOP 1, mid-interview, date: April, 2016).

00:35 Charlie: Um, so we spent quite a bit more time discussing unit fractions, finding those original two examples on the number line. Um, I was hoping they would be able, I guess, um, to maybe, more or less, to estimate them a little easier, but they really wanted them [the unit fraction marks for $\frac{1}{3}$ and $\frac{1}{8}$] to be super accurate on there. Which, took a long time; but I didn't want to stop them, because we had spent all of that time when we were doing the original French Fry activity talking about how there is only one size unit fraction that will fit in here X number of times. So, I sort of let it get a little drawn out. I didn't want to stop and say actually you know what, close enough is good right now. Um, so that took a little longer than anticipated.

1:19 Um, when I released them to their seats to work on the worksheet, there was some confusion about it. So, this worksheet had a few points on a number line. And they were fine when there were less points than the fraction they were looking for. So, by that I mean if they were looking for $\frac{2}{3}$ they knew that there needed to be 3 points somewhere in there in order for them to find this fraction. But when it got to something like, find $\frac{2}{6}$, there was a lot of confusion because they would say, well there's only four points here. So, they weren't quite seeing that they had to actually partition the whole themselves and then iterate that unit as many number of times as the numerator asked them to. So, I got a lot of questions about that.

So, I was going around clearing some of that up a bit. But I feel like once I went around to those students who were having the confusion finding some of these points that weren't necessarily labeled for them it cleared up, I guess.

2:26 RC3: Did anything surprise you yesterday [in Lesson 1/Observation 1]?

2:30 Charlie: Um, I am, sure, I am sure it did. That is why I am thinking about it right now. I am just going to open up what we did yesterday real-quick.

2:48 RC3: Oh yeah, absolutely.

2:50 Charlie: I saved all of my work, so I could go back. Um, I guess we had been working on finding some equivalent fractions, and I was hoping that they would [have] noticed that a little more. So, by that I mean our summary question, it said Jason ran $\frac{3}{6}$ of a mile. I was hoping right away that the students would be able to see that $\frac{3}{6}$ is equivalent to $\frac{1}{2}$. And they could put that line right in the middle there. But they struggled with that. Um, so I took some time to remind them, well remember we can multiply or divide any fraction by one, it is staying the same. And so, letting them know $\frac{4}{4}$ is equivalent to 1. The fraction is staying the same we are basically getting different numbers to represent the same thing. Um, so I think they had forgotten about that a little bit. When we got down to this part (points to the lower number line of the summary problem where students were to plot $\frac{3}{6}$), I sort of wrapped it up a little quicker than we did up here [referring to the thirty minutes of class when students found $\frac{1}{3}$ and $\frac{1}{8}$ on a number line]. Rather than taking all that time to find those eighths on the number line, I said, well, can we find an equivalent fraction and then move from there to help them.

I see in Excerpt 1 two points of importance in explaining my interpretation of Charlie's stance on learning and teaching fractions before the first manifestation of juxtaposing of thinking began to appear. First, I infer he seemed to recognize students having some difficulty with the learning goals he set for the lesson, while not alluding to possible conceptual sources for these difficulties. Instead he turned his focus on possible distractors in the lesson that might have led to students' confusion (see line in 1:19). He also stressed that students have forgotten what he had already taught them (see lines 00:35 and 2:50). Charlie did not appear to contemplate the plausible source for such forgetting. Rather, he noticed it and his responses suggested attempts to convey the 'forgotten' mathematics by reminding students what they have done before. Accordingly, the second point pertains to his recognition of the students' struggles. Charlie turned to pointing out/helping students with what he thinks they missed "seeing." I infer from this action that Charlie's underlying perspective is perception-based (PBP)—the students will come to see what he (his FOM) sees (see lines 1:19 and 2:50).

These two points suggest an interesting combination of a PBP, which, from my analysis, Charlie seemed to rely on, with operating from his FOM. He recognized that students struggled, but did not seem to have an alternative instructional move to "fall back on" when noticing this. Instead, Charlie reminded/pointed out the intended learning to students, using exploration activities for discovery of the mathematics along with his explanations. Upon completion, Charlie then expected that the students would see the mathematics that he so clearly saw (understood).

Both PBP and FOM have been identified, separately, in previous work (Simon et al. 2000, Steffe, 1992; 1995). However, my inferences from Charlie's actions suggest a new lens, which combines both PBP and FOM. Charlie set up a lesson in which the students would interact with the mathematics, here plotting fractions on the number line. Therefore, Charlie expected this would create the learning for the students via their discovery and active perception (a PBP move) of what he could see in the presented materials/actions. When students did struggle, and he noticed this, Charlie pointed out the learning or further explained it as he did with defining unit and non-unit fractions. However, Charlie seemed unable to work from the students' understanding/struggle. Instead, I infer he worked from his FOM, which drove how he seemed to expect the math could also be learned by students. At this point, he did not go through a more in depth inquiry into why the students behaved the way they did. The mathematical understanding seemed, for Charlie, to be out there and something students would gain from him leading them via actions he seemed to believe, from his point of view (FOM), would get to the mathematics. This way of working with students, seen through the combined lens of PBP and FOM, will be contrasted below with Charlie's focus on students' thinking as he will juxtapose it with his own.

Charlie concluded the mid-interview by discussing his goals and sequence for the next lesson (observation 2). His goal remained essentially the same – plotting non-unit fractions on a number line. The class would first review some work from the previous lesson (observation 1) by discussing how to find non-unit fractions on a number line, and then work through an example problem as a whole group that was identical to one of those on their worksheet from the previous lesson (plotting $\frac{3}{6}$). The students would then work in small groups to plot on a number line fractional cards, which included both unit and non-unit fractions.

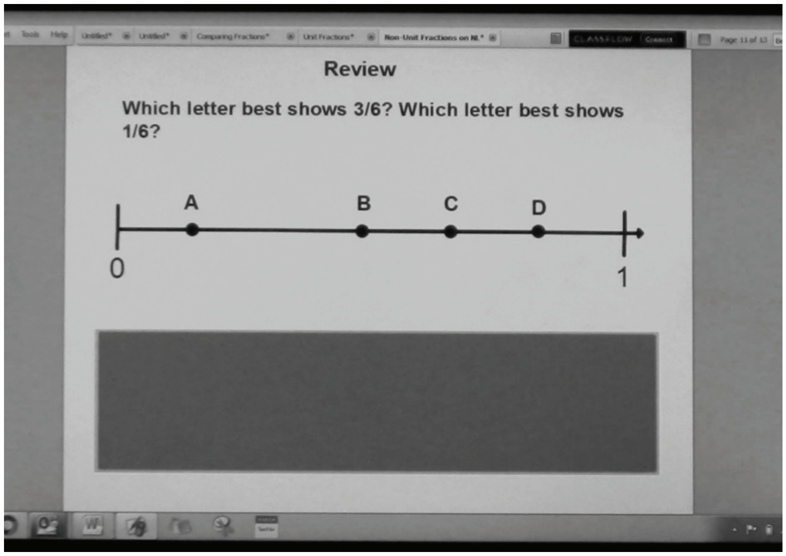

Figure 4.4. Charlie's review of the previous day's lesson, finding non-unit fractions on a number line with multiple choices.

Observation 2: Charlie's continued use of first-order model and perception-based perspective. The lesson (observation 2) began pretty much in the way Charlie had discussed it in his mid-interview. To begin the lesson, Charlie led a whole class review on how to find a non-unit fraction on a number line. For this, he led a class discussion reminding students of the process for finding non-unit fractions (as implied by his FOM): first determine the unit fraction, then find that unit fraction on the number line, and finally repeat it the number of times indicated by the numerator to find where the point for the non-unit fraction would fall. For the next part of the lesson he moved to an example problem from the worksheet, where multiple points were displayed on the number line that did not necessarily line up with the non-unit fraction students were trying to find (four lettered points when the question was asking about sixths, see Figure 4.4). He also revealed a number line just below the number line with the lettered points that was already divided into sixths for students.

He asked for volunteers to come up and mark where $\frac{3}{6}$ and $\frac{1}{6}$ would be on the number line. He stated to the class that everyone would then be asked to determine what letter those marks represented. He called on Monica as his first volunteer to come up and show $\frac{3}{6}$ and $\frac{1}{6}$ on the lower number line. It is then, when Monica came up to the board, where the rather lengthy Excerpt 2 begins – and shows Charlie's FOM-based exchanges with Monica.

Excerpt 2: Charlie's volunteer of the marking $\frac{3}{6}$ and $\frac{1}{6}$ (AOP 1, observation-two, date: April, 2016).

10:42 Charlie: So, I want you on that bottom number line to label $\frac{3}{6}$ and then when you have done that, we are going to see which point best represents $\frac{3}{6}$.

10:49 Monica: (Labels $\frac{3}{6}$ above the mark that is actually $\frac{2}{6}$.)

11:03 Charlie: Monica can you tell me why that point there is $\frac{3}{6}$?

11:10 Monica: (Inaudible, points to the 0, then the $\frac{1}{6}$ tick mark, then the $\frac{2}{6}$ tick mark.)

11:14 Charlie: Monica can you point to $\frac{1}{6}$ again for me real quick please?

11:18 Monica: (Points, wrongly, to the tick mark at 0.)

11:21 Charlie: Monica, keep your finger on $\frac{1}{6}$.

11:23 Monica: (Puts finger on $\frac{1}{6}$ tick mark then quickly moves it back to the 0 tick mark.)

11:24 Charlie: Ok, I want you to move your finger down a little bit, down, down, down.

11:25 Monica: (Moves finger to the right towards $\frac{1}{6}$ not down.)

11:26 Charlie: Uhh, towards the ground.

11:27 Monica: (Moves finger towards the number 0, which is below the tick mark she pointed to as $\frac{1}{6}$.)

11:29 Charlie: K, what does that number say right there?

11:30 Monica: (Answers, inaudible – likely saying "zero")

11:31 Charlie: (Repeats her answer.) Zero. That is the very start of our number line. That's where we start out. We don't have anything there yet. (Walks over to a different number line in class and points to the 0 mark.) That's where our number line begins. When we begin our number line we begin at zero. We start at zero and we move onward forever and ever. (Makes a forward movement with his arm towards the 1 of the number line.) So Monica if this point is zero, can it also be $\frac{1}{6}$?

11:56 Monica: No.

11:57 Charlie: No. Monica, do me a favor, I want you to count by $\frac{1}{6}$ and I want you to label that number line for me please.

12:05 Monica: (Labels the number line putting $\frac{1}{6}$ at the actual $\frac{1}{6}$ tick mark not at zero. Then puts $\frac{2}{6}$ next to where she originally put $\frac{3}{6}$ and continued to label, correctly, each tick mark by sixths until reaching the end of the number line, which she labeled as $\frac{6}{6}$.)

12:32 Charlie: Monica I noticed that you originally numbered, labeled this point here on the number line $\frac{3}{6}$ (points to the tick mark that Monica originally labeled as $\frac{3}{6}$ which was actually $\frac{2}{6}$.) But when I asked you to label it counting by units of $\frac{1}{6}$ you changed it to $\frac{2}{6}$. Monica can you take your hands and show me on that number line where the first $\frac{1}{6}$ piece is?

12:50 Monica: (Uses two hands and brackets the piece between $\frac{1}{6}$ and $\frac{2}{6}$ then points directly to $\frac{1}{6}$.)

12:55 Charlie: Can you use your other hand to show me where it starts?

12:57 Monica: (Uses two hands and brackets the piece between $\frac{1}{6}$ and $\frac{2}{6}$. See Figure 5.)

12:59 Charlie: Is that the first one?

13:00 Monica: (Uses two hands to bracket the piece between 0 and $\frac{1}{6}$.)

13:02 Charlie: Ok, so there is $\frac{1}{6}$. Monica, can you show me $\frac{2}{6}$ now?

13:04 Monica: (Uses her hand to first bracket the piece between $\frac{2}{6}$ and $\frac{3}{6}$ then moves them to bracket the piece between $\frac{1}{6}$ and $\frac{2}{6}$.)

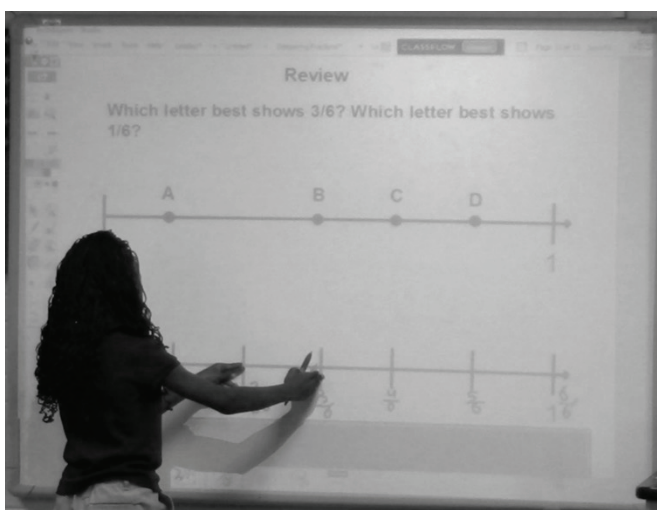

Figure 4.5. Monica's initial identification of $\frac{1}{6}$ on the number line.

13:08 Charlie: K, so that is our second sixth. And can you show me your third $\frac{1}{6}$ piece now Monica?

13:12 Monica: (Uses hand to bracket the piece between the $\frac{2}{6}$ and $\frac{3}{6}$ tick marks.)

13:13 Charlie: K. Monica do you see why this point here that you originally labeled $\frac{3}{6}$ is in fact $\frac{2}{6}$?

13:23 Monica: Yes.

13:24 Charlie: Okay.

Excerpt 2 shows the difficulty Monica had when trying to plot sixths on a number line that was already partitioned for her and Charlie's interactions with her to help her understand and label the tick marks correctly. Monica seemed to understand that, beginning at the zero tick mark, she could count three of them, which would then equate to $\frac{3}{6}$ because she had counted three (already created) tick marks on the number line, hence the tick mark at 0 for Monica was $\frac{1}{6}$. Charlie's exchanges with Monica seemed to reflect what I inferred to be his frustration that Monica did not understand what he clearly could see – the tick mark at zero does not indicate a measurement of any fraction. This frustration seemed to lead him to attempt actively directing her gaze to the correct answer (see lines 11:24, 11:29, and 11:31). Such a response seems compatible with how other teachers, identified as operating from a PBP, have responded (Simon et al., 2000) – attempting to directly show the correct answer through active guidance about what the students needed to do. That is, when frustration grew for the teacher, he reverted to directly leading students' gaze to what his FOM showed. My analysis adds to those previous studies the emphasis on how the teacher's FOM seemed to underlie his frustration and the resulting attempt to provide the content roots for his PBP approach to promoting students' learning.

This combination of Charlie's PBP and FOM is evidenced in the exchanges with Monica. Monica seemed to pick up that Charlie was indicating to her she did not have the correct answer (see line 11:56). However, she did not seem to understand what was wrong, as indicated by her continued struggle to identify $\frac{1}{6}$. Charlie's response to this was to direct her to the correct answer, through directing her activity and questioning about what, his mathematics suggests, is a clear, correct response (see lines 11:31, 11:57, 12:32, 12:55, 12:59, 13:02, 13:08, and 13:13). He indicated no intention to try to determine why Monica thinks the way she does. Rather, his reaction indicated giving way to his FOM-based goal of directly showing to both the correct answer, as he understands the mathematics.

In the post-interview, RC2 and RC3 questioned about this instance focusing mostly on Monica. Up until this point, through the pre-interview, observation 1, mid-interview, and observation 2, Charlie had repeatedly suggested that students will come to understand plotting non-unit fractions on a number line by a process of steps: find the unit fraction, determine the size of that unit fraction on the number line, and repeat that unit fraction however many times the numerator in the non-unit fraction requires (Excerpt 1, 1:19). This procedural description indicated Charlie's FOM of plotting non-unit fractions. When students struggled, he had them go through the steps and provided more assistance on how to do so with his questioning (Excerpt 2, 12:32-end). Throughout the lessons, he had tried to fit students to this process of understanding by having them repeat the process (or parts of the process) when they do not understand or make a mistake. He appeared to struggle to think about the sources for students' mathematical struggles with the procedure of finding non-unit fractions on a number line, likely due to his own thinking of the mathematics. However, I infer that for him there was no contrast between the way he understood the mathematics and the way the students understood the mathematics. Whenever there was a student who appeared to struggle with the task at hand, Charlie's instructional response was to guide ("show") her to the correct answer the way he understood it (FOM). This is about to be, for the first time, challenged through the interviewer's questioning.

Charlie's first observed juxtaposition of thinking. In Excerpt 3 (below), RC2 pushed the thinking of Charlie by asking him to state what, in Charlie's recall, actually happened in the lesson. Building off of Charlie's response, RC2 continued probing why, from Charlie's point of view, would that student (Monica) have done what she did based on the way she understood the mathematics to be – not what Charlie wanted her mathematics to be. It is in Excerpt 3 that data show how Charlie, for the first (observed) time, considered a student's thinking apart from his own (FOM), which seemed to be brought about by juxtaposing Monica's thinking with his. To support such consideration, RC2 detailed the situation.

Excerpt 3: Charlie's juxtaposition of thinking (AOP 1, post-Interview, date: April 2016).

16:43 RC2: [Point] B was at $\frac{3}{6}$ there was no, there was no point at 2 [$\frac{2}{6}$] and there was not [a] Point at D. That is what I remember. But for the sake of that my memory is wrong, so basically, she had a line and, uh assuming the $\frac{2}{3}$ and $\frac{3}{3}$ are not here, this was $\frac{1}{6}$ and this was [Point] A and this was [Point] B and this was [Point] C and [Point] D (draws on number line where each of the points were, Point A at $\frac{1}{6}$, Point B at $\frac{3}{6}$, and Points C and D further on the number line but not aligned with any sixth). And [Point] A was the $\frac{1}{6}$ that was written here, right. And [Point] B was $\frac{3}{6}$. And she said, you asked where would you put $\frac{1}{6}$ and she pointed to this one

(Point B) and the question I have is, do you have any way of thinking about, kind of out loud, what could have brought up her putting $\frac{1}{6}$ here (points to $\frac{3}{6}$)?

17:25 Charlie: (Pause for 6 seconds.)

17:31 RC2: From your thinking, I guess that the first question I should have asked but I forgot to ask it first, is at the time did you have any thinking? I assume you have not thought, so you are now thinking about the question, but at the time you did not.

17:44 Charlie: Yeah, so, um, you bringing it up is making me sort of jog my memory, so I am asking her to find $\frac{1}{6}$ and she points at [Point] B which is, sort of, clearly at the half way point. Umm. (Pause.) I'm, I'm trying to…

18:06 RC2: That's ok.

18:07 Charlie: Think about why she would have identified that as $\frac{1}{6}$. If she would have said anything was $\frac{1}{6}$, I would imagine it to be [Point] A for the reason that I said earlier, it is the first point that we come across. It is one of what she might believe is six parts. Umm, yeah, I am not entirely sure why she would have identified this as…

18:29 RC2: So, that's absolutely fair. So, let's take it to the next step, so the next thing that she did that you went into the explanation and I think that what Jodi brought again and other people discussed, you have this one and then we take one (points to the entire whole and then drawn number line referring to $\frac{1}{6}$) and you asked her and she put it there (points to the zero), so in what way, what you were doing in a response to a child saying this is $\frac{1}{6}$ (points to $\frac{3}{6}$ and Point B) what were you trying to accomplish with that part of your lesson?

18:54 Charlie: I mean, I guess sort of helping her to clarify not only my explanation but maybe also her thinking. Um, because sitting right here, if I can't quite determine why she might have thought this was $\frac{1}{6}$ (points to $\frac{3}{6}$). I don't know what her thinking was either as to why should thought this might be $\frac{1}{6}$. Um, so just going back and reiterating ok, if we are looking for $\frac{1}{6}$, what's the first thing we need to do? So, I guess asking some of those initial probing questions, to see if they are confident in order for what they need to do to be successful in the task. Um, at that point that is sort of where I was going back to. And I can't recall off hand what exactly she said at that point in time. Um, had she identified this as $\frac{1}{6}$ (points to $\frac{3}{6}$) then my questioning would have been something along the lines of, alright well, if we are looking for $\frac{1}{6}$ what's the first thing that we need to do? And I don't know if that is actually what was said but, I would have liked for her to have said, well first we need to split or partition this, what we call whole up into 6 equal pieces. So, and then, after we did that, then I would…

Excerpt 3 provides an initial insight into the manifestation of Juxtaposition of Thinking I attribute to Charlie regarding his ability to extend his thinking beyond his FOM and into consideration of the student's thinking. Charlie's (FOM) understanding of non-unit fractions is based on the ability to first partition the whole into an equal number of parts, and then take one of those parts and repeat it in order to find the point on a number line for a given fraction. Up until this point, Charlie has attempted, through his instruction, to fit students to his experience of knowing and learning by "getting the students to see" the mathematics the same way he does through his questioning and guided activities (see Excerpts 1 and 2). At this point in time (Excerpt 3), however, I infer that Charlie seemed to begin putting together, for himself, two distinct experiences; one of recalling the students' ways of operating and the other of reprocessing his own way of thinking (see lines in 18:54).

Charlie used his understanding of the mathematics (his FOM) to determine what his students would understand. He also seemed to assume that students, through his instruction, would come to understand the intended mathematical concept (here plotting unit fractions and non-unit fractions on a number line) the same way that he did. His FOM logic, and thus he expected also for theirs, would proceed through first partitioning the whole into equal parts, and then iterate one of those parts to determine a point on the number line. Charlie's FOM seemed to underlie both how he thought about the mathematics and how he assimilated and interpreted student actions within class as to what he had hoped would happen. Furthermore, his FOM seemed to also underlie his choice of actions in response to students' struggles.

However, RC2's detailing of the students' work seemed to foster Charlie's juxtaposing of his FOM with students' responses, by encouraging Charlie to consider very specific student actions that did occur. Specifically, Charlie's juxtaposition was fostered when asked why Monica pointed to $\frac{2}{6}$ as $\frac{1}{6}$. This probing by RC2 (see lines 16:43 and 18:29) seemed to foster in Charlie a contrast between what Charlie thought happened based on his FOM to what really happened. Subsequently, I inferred that for Charlie this seemed to create a juxtaposition of thinking to which his honest response was, "I don't know" (silenced pause, 17:25, 18:07, and 18:54). Charlie seemed unable to describe how the student thought about the mathematics outside of what he would do, that is, outside his FOM (18:07 and 18:54). This "I don't know" statement is significant, because it indicates a novel distinction in Charlie's thinking – between his own and another person's thinking that, at this point, he could not specify.

The researcher's questioning prompted Charlie, seemingly for the first time, by emphasizing "tell me how you think" and "tell me how she thinks." This juxtaposition my inference into Charlie's thinking (not a claim of his awareness), where he seemed to move from "here is the correct math, and if a child does not have it, I need to provide it through actively showing the correct one" to possibly considering "here is the correct math, and if a child does not get it, I should ponder why." That is, Charlie could juxtapose his way of thinking with a student's way of thinking, which although was yet to be specified for him – seemed to him different in nature.

Manifestation 1: Juxtaposition of thinking summary. In the data presented in this section I presented inferences of moments in which Charlie demonstrated, in response to researcher questioning, what seemed to be his start to distinguish students' reasoning from his own FOM (distinction that does not include attribution of awareness on his part). That is, he began to move from a thinking of fitting students to the right answer, to thinking that students may understand differently, while pondering what might that understanding be. Charlie's juxtaposition seemed to emerge when RC2 oriented him to think of his students' thinking in ways particular to what the students did or say. Consequently, Charlie seemed to juxtapose his and his students' thinking, by contrasting what he expected would happen in the enacted lesson and what actually happened—specifically being questioned about Monica and why, for her, her answers were what they were. At this point, Charlie's inability to answer these questions indicated plausible lack of an SOM (e.g., saying, "I don't know" in Excerpt 3). While specifying students' thinking would be important, Charlie's recognition of the need to specify it supports the claim he began to distinguish his own FOM from students' understandings. This change from Charlie answering what students should know to "I don't know what the students are thinking" serves as evidence for me to answer the first question of this study. This juxtaposition of thinking, between a viewpoint of only FOM towards a viewpoint that the experience of the student may be different than the teacher's FOM, could serve as a basis for later development of a recognition that a student has unique ways of operating, that is, a potential for creating a hypothetical model of a students' thinking (SOM). My inference of Charlie's Juxtaposition of Thinking as part of his shift towards separating his own mathematical understanding from his students' mathematics addresses the second research question of this study.

Manifestation 2: Cogitation

To depict cogitation as a manifestation of a shift towards SOM, this section includes analysis of four AOP data excerpts. To recall, two of the key components of operating with an SOM are the observer's (here, the classroom teacher) ability to create a hypothetical model of a student's knowledge based on an inference from the student's behaviors, and to separate this (teacher's view) understanding from that which the student may understand. This section outlines how the ability to think deeply about students' mathematical reasoning shifted for the teachers. That is, teachers' cogitation regarding students' mathematics and the hypothetical models they could create became more enhanced, which serves as an indication of beginning to create hypothetical models of students' thinking (SOMs). The manifestation of Cogitation is illustrated in the following excerpts in the sense of how each participant both created hypothetical models of their students' mathematics as well as separated these inferences from her own understanding.

Charlie's cogitation. Charlie's cogitation took place during his AOP 2, just over a year after AOP 1. Charlie's AOP 2 consisted of an I-O-I-O-I. For AOP 2, Charlie taught a lesson on fractions to a small group of five students. For this lesson (observation 1), Charlie had two goals for students that he stated throughout AOP 2. First, he intended for students to explain unit fractions as they are related to the whole. For example, he wanted students to conceive of $\frac{1}{7}$ as $\frac{1}{7}$ because the whole is 7 times as much as the $\frac{1}{7}$ piece. In other words, $\frac{1}{7}$ repeated 7 times will arrive at $\frac{7}{7}$ or one whole. Second, he intended for students to understand that an improper fraction is composed of repeating a unit fraction a certain number of times. For example, $\frac{9}{8}$ is composed of $\frac{1}{8}$ repeated 9 times and can be written as $9 \times \frac{1}{8} = \frac{9}{8}$.

It should be noted that while these two understandings reflect Charlie's FOM, they include a focus on units and operations a person may be using to reason about fractions. Such a stance was fostered throughout the PD work with teachers in the larger project. The question was to what extent he would consider those units and operations apart from how he was using them.

During the lesson (observation 1), Charlie had students take a cube and then called that cube $\frac{1}{7}$. He then had students make the whole and describe why what they made would be the whole. A plausible response would be that a tower made of 7 cubes would be the whole, because it would represent $\frac{1}{7}$ iterated seven times, which is $\frac{7}{7}$ and therefore one whole. Charlie then brought in a real-world context and asked students to think about the following problem: If I am selling pizzas and each of you (5 students) bought $\frac{1}{9}$ of my pizza, how much would you buy altogether? The group agreed it would be $\frac{5}{9}$ and Charlie wrote for the group $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{5}{9}$

as a way of demonstrating the students' thinking within the group. Excerpt 4 begins right after Charlie wrote the additive equation on the board.

Excerpt 4: A response of Charlie's student to repeating $\frac{1}{9}$ five times (AOP 2, observation-one, date: May, 2017).

14:29 Craig: Wait, I think there is another way to write this quicker. [Referring to Charlie's demonstration of $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{5}{9}$.]

14:32 Charlie: I would love it if you could write, right below that (points to $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{5}{9}$), what the quicker way is.

14:38 Craig: Let me first look over it. (Pauses, then writes $\frac{2}{9}+\frac{3}{9}=\frac{5}{9}$.)

15:01 Leon: Ohhh, that is fast. Or you could do 4 + 1, $\frac{4}{9}+\frac{1}{9}$.

15:05 Charlie: So we can take the $\frac{5}{9}$, just like what Craig did, and break it up a bunch of ways, right? We can say its $\frac{3}{9}$ and $\frac{2}{9}$. We can say its, what else did you say Leon?

15:14 Leon: $\frac{4}{9}$ and $\frac{1}{9}$.

15:16 Charlie: $\frac{4}{9}$ and $\frac{1}{9}$. Is there any other way we could think about that? Maybe write an equation that gives us $\frac{5}{9}$ but without using addition?

15:26 Craig: Multiplication.

15:28 Charlie: Craig, what would that look like?

15:30 Craig: Hmmm, let me think about it.

15:35 Charlie: How many times did each of you come and buy one of my $\frac{1}{9}$ pieces?

15:38 Group: One.

15:40 Craig: 5 times one…

15:43 Charlie: Keep going.

15:44 Craig: 5 times 1 equals 5.

15:47 Charlie: 5 times 1 equals 5 but remember we are working with fractions of $\frac{1}{9}$.

15:51 Craig: $\frac{5}{9}$ times $\frac{1}{9}$ equals $\frac{5}{9}$.

15:56 Charlie: So be careful there, how many of you? (Points to each person in the group.)

15:58 Craig: Five.

15:59 Charlie: Five, and you each bought $\frac{1}{9}$ so what would it be?

16:05 Craig: Oh.

16:06 Daisy: One times one…One times five, is five, right?

16:09 Charlie: Hmmhmm (nods). So Craig, keep going.

16:11 Craig: $\frac{1}{9}$ times 5 e…equals $\frac{5}{9}$.

16:15 Charlie: Perfect. Once Daisy is done would you grab the marker for me.

16:23 Perfect. So Craig when you originally said that there's a little faster way to write this, I was thinking you had caught on to this [5 x $\frac{1}{9} = \frac{5}{9}$] but I love that you are able to see that $\frac{5}{9}$ can be broken down or decomposed. Remember our math terminology decompose, we can break it down into $\frac{2}{9}$ and $\frac{3}{9}$. So we call these non-unit fractions, those are made of our unit fraction $\frac{1}{9}$. So Craig good job seeing that buddy and I also love that you were able to hit on the multiplication equation as well.

Charlie's work in Excerpt 4 contrasts with his work in Excerpt 3. Thinking back to Excerpt 3 (above, within Juxtaposition of Thinking manifestation) when working with Monica who did not arrive at the correct answer of $\frac{3}{6}$, Charlie worked with her on repeating the procedural process to "lead" Monica to find the fraction. Later, when asked about the interaction with Monica, Charlie struggled to describe Monica's way of thinking as different from his own. He seemed unable to even contemplate how the actions that Monica demonstrated could help him infer into her thinking mathematically. He seemed only able to discuss the mathematics of Monica through his FOM, as a process of finding a non-unit fraction on a number line as he also thought of the mathematics.

In contrast, Excerpt 4 indicated a moment in class where Charlie responded differently to a student (Craig) who, like with Monica, did not seem to accomplish the goal of the understanding for the lesson. Charlie expected Craig would bring one type of multiplicative understanding (5 times $\frac{1}{9}$ is $\frac{5}{9}$, see lines in 16:23), but instead Craig brought a different way of additive thinking ($\frac{2}{9}$ plus $\frac{3}{9}$ is $\frac{5}{9}$, see lines in 14:38). Charlie's reaction during the lesson was to follow Craig's suggestion and build off of it. This is important to take note of when moving to the next Excerpt 5, when Charlie is asked about his interaction with Craig.

In Excerpt 5 (just below), which took place at the mid-interview of the AOP 2, I inquired into Charlie's understanding of Craig's mathematical thinking, and reasons Craig might have answered the way he did. Excerpt 5 depicts Charlie's cogitation, a manifestation of a shift in his thinking regarding mathematical reasoning of students.

Excerpt 5: Charlie's cogitation (AOP 2, mid-interview, date: May 2017).

14:41 RC1: (Describing Craig and Charlie's interaction during the lesson.) He [Craig] said $\frac{5}{9}$ times $\frac{1}{9}$ is $\frac{5}{9}$. And you said let's be careful here. Can you talk a little bit about what you, what you understood Craig to be thinking at that point?

14:52 Charlie: Yeah, so in my mind he was thinking either one of two things; so he knew that he had to multiply there, needed to be a 5 in there somewhere, because he knew he was moving from $\frac{1}{9}$ to $\frac{5}{9}$. He knew there was a change in that numerator and we had just got done talking about how, so I'm just trying to guess, maybe, what was going on in his head. So he knew that that 5 had to be brought in at this point, that's where he got the 5 from $\frac{5}{9}$ and we just got done having a conversation about how we don't change the denominator when we're working with same unit fractions. So in my mind, I am just thinking, well, maybe he had that in his head initially, 'Oh I need to have that 9 in there' to make that same uniformity like when we are adding. So we didn't necessarily draw the distinction between, okay, well, when we're adding we keep the denominator the same however the conversation wasn't had when were multiplying this is really $\frac{5}{1}$ we're multiplying straight across in this case now our denominator could potentially change when were multiplying. That conversation wasn't had, so I feel like he was bringing in his knowledge of how to operate on fractions additively to doing the same thing now operating on them multiplicatively and I feel like he just confused himself a little bit at that point.

16:31 RC1: So, any ideas why he didn't say $\frac{5}{9}$ times $\frac{1}{9}$ would be would be $\frac{6}{9}$?

16:40 Charlie: Why $\frac{5}{9}$ times $\frac{1}{9}$ would be $\frac{6}{9}$?

16:43 RC1: Yeah.

16:44 Charlie: Hmm, so he has, in my mind, he was able to switch from operating additively which we had just done, to operating multiplicatively. So he was adhering to the fact that this was multiplication. So, he was able to bring in 1 times 5, he knew was 5, so he knew that our operations had changed at that point.

17:11 RC1: And why not, another contrast, why not $\frac{5}{81}$?

17:22 Charlie: Which is, which is what it should have been (laughs).

17:24 RC1: Right.

17:25 Charlie: Yeah, so that's why I am thinking…

17:26 RC1: So he's bringing in, and then, that's why I'm asking the question, because I want to get an idea and he might not of, this is like, we are reflecting now – right? So why not if he is thinking $\frac{5}{9}$ times $\frac{1}{9}$ and he's having the switch from this [referring to $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{5}{9}$], where he knows he's not supposed to add the numerator…denominator here, why not, I'm just trying to get what you think he's thinking about. Why not $\frac{5}{81}$? I don't know that I've ever said that fraction before out loud. I don't know that I've ever thought of it.

18:01 Charlie: I don't think I have either, we now, we've got that out of the way.

18:05 RC1: Cool.

18:06 Charlie: So I'll go back to keeping, keeping the uniformity throughout, when we're adding fractions, keeping that denominator's the same; I think he was bringing that piece over into this problem. However, he knew that somewhere along the

> line the sign had changed; and I feel like he was only, since we mentioned before we are only operating on the numerator at this point, 6 plus 4 equals 10 is how they said it and then we went back six, six what's $\frac{6}{11}$ plus $\frac{4}{11}$ equals $\frac{10}{11}$ making sure they're unitizing it. I feel like he knew he had to do something with the numerator, because something with the numerator had been done over here they added it. So I feel like he multiplied it and then he said oh when we added, the denominator remained the same so I clearly must just need to put that over 9. That's the thing, I can't really think of why he brought in $\frac{5}{9}$ as opposed to just 5 and why he didn't say $\frac{5}{81}$.

Excerpt 5 indicates the Cogitation manifestation in Charlie's explanation of a child's response. In thinking back to Excerpts 1-3 (Juxtaposition of Thinking manifestation), Charlie originally could identify a different way of thinking in Monica but not to determine why Monica would, for example, point to $\frac{2}{6}$ and think of it as $\frac{1}{6}$. In Excerpt 5, Charlie is asked regarding why Craig may think of $\frac{5}{9}$ times $\frac{1}{9}$ as $\frac{5}{9}$ instead of saying 5 times $\frac{1}{9}$ is $\frac{5}{9}$ (what Charlie wanted Craig to say). Up until this point when Charlie was asked regarding reasons for a student's understanding of the mathematics (which may have not been correct), he would respond with "they forgot," or "I don't know," or "they don't yet understand the process which I am trying to teach them" (Excerpts 1-3).

In Excerpt 5, however, Charlie could engage in an elaborated description of his students' mathematical understanding. At issue here is not the correctness of his inferences; rather, it is the first recorded evidence he began to infer Craig's mathematics and how that may have led Craig to describe the mathematics behind 5 times $\frac{1}{9}$ equating to $\frac{5}{9}$. In fact, Charlie could contemplate not just one but a few reasons why Craig might have thought the way that he did (see lines in 14:52). These included Charlie contemplating that Craig was attempting to bring what he knew additively into a multiplicative situation. According to Charlie, Craig knew that, in fractions, additive situations require not to change the denominator when working with same unit fractions and that the move was going from $\frac{1}{9}$ to $\frac{5}{9}$ (see lines in 14:52 and 18:06). For Charlie, with what Craig already understood, Craig then attributed to 5 times $\frac{1}{9}$ but confused himself at that point. This, I inferred, shows that Charlie is beginning to contemplate, and describe, mathematical thinking as it relates to the student. Charlie is attempting to make connections between Craig's incorrect answer of $\frac{1}{9}$ x $\frac{5}{9} = \frac{5}{9}$ to what Craig understands – both about whole number multiplication and fraction addition.

In addition, Excerpt 5 highlighted Charlie is not depicting Craig's thinking as a process of finding the answer through a series of steps. That is, he did not answer my interview probing into Craig's reasoning with repeating back that if Craig did know he would understand the process of $\frac{1}{9}$ repeated 5 times is the same as 5 times $\frac{1}{9}$ (as he did with Monica and her ability to find and plot unit and non-unit fractions on a number line). Rather, Charlie discussed Craig's thinking in terms of what Craig could do (add fractions with like denominators) and how it might relate to the task at hand (multiply a whole number by a unit fraction, see lines in 14:52, 16:44, and 18:06). His descriptions regarding Craig's thinking are all based on how Craig might have interpreted the mathematics based on what Craig knows. This indicated to me that Charlie was cogitating about Craig's mathematics as separated from his own and able to contemplate the mathematical reasoning of another person, to which I therefore attribute an initial SOM. Next, I extend the depiction of the cogitation manifestation by turning to AOP data from the work with Sam.

Sam's cogitation. To contextualize the shift that Sam made towards cogitation, I am providing Excerpts 6a and 6b (below) from her AOP 1 (I-O-I). These excerpts provide a contrast against which to analyze the shift towards cogitation, which occurred in AOP 2 with Sam and is discussed later in this section. In Excerpts 6a and 6b, Sam is describing what she has noticed regarding her students' ability to solve multiplication and division problems.

Excerpt 6a: Sam's description of students solving multiplication and division (AOP 1, pre-interview-a, date: April, 2016).

3:54 Sam: From what I have seen, um, students have really taken on a lot of the strategies that we've been, um, practicing and uh, uh, breaking up multiplication problems into known facts and into equal groups or even arrays and then seeing groups in those. And, uh, and then I have some students who are still very, like, drawing the equal groups. Uh, you know, drawing out dots and circling them and that sort of thing.

Excerpt 6b: Sam's description of students solving multiplication and division (AOP 1, pre-interview-b, date: April, 2016).

0:00 Sam: For this set of kids, I've noticed [when students are solving a quotitive division problem] that if I, if it's like they kind of tend to say, they kind of tend to rely on, well I know how many to put in each group and I stop when I have 15. That sort of thing.

00:11 RC2: Okay.

00:12 Sam: But, then when you say, when you pose it in the sense that like we're going to have 5 equal groups or there's going to be 5 groups then they kind of have to think about how do I place, how am I [going to solve this], and I'm going to put one in each.

00:26 RC2: So tell me if it's a fair statement, because I'm trying to get a sense from what you are describing that you do think about these two as different, at least when it comes to the kids.

00:34 Sam: Yes.

Excerpts 6a and 6b provide an initial glimpse into how Sam thought about her students' reasoning when it came to solving multiplication and division problems. It should be noted that this differentiation of reasoning between students was not specifically probed by RC2's questioning. Sam brought forth her noticing how students could solve multiplication problems whether it be by known facts, equal groups, arrays, or drawing out dots and circling them (Excerpt 6a, see lines in 3:45). Importantly, she mentioned these are all strategies that the class has been taught. This is similar to how a PBP teacher would view the mathematics of their students, "I teach many different strategies and allow students to use whichever serves them best." In this instance, Sam's views of students seemed to reflect what she has shown the students to do in class – not a result of the students' assimilation.

When discussing division (see Excerpt 6b), Sam noticed (FOM) that the two types of division are different when it comes to the students. She also noticed that as a result, students may solve these differently; counting by groups or putting one in each group until there were no more to be distributed.

The first recorded instance of Sam's cogitation took place during her AOP 2, just over one year after AOP 1. For AOP 2 (I-O-I), Sam taught a lesson to her whole class on partitive division (givens are the number of groups and total amount, unknown is the amount per group, as discussed in Tzur et al., 2013). For this lesson, Sam suggested her goal was for students to connect multiplication and division and determine how they were related. For example, Sam described this as students would begin to connect that they can use their multiplication knowledge to help them determine how many to put in each group without reverting to putting the total amount into groups one by one.

Within Excerpt 7 (below), which took place in Sam's post-interview of AOP 2, Sam elaborated on what she observed during her lesson on partitive division. The focus of the class session was on the following task: "You have 42 cubes. You want to put the same number of cubes into 6 towers. How many cubes will be in each tower?" In the interview, Sam discussed differences among three students. Brittany solved the task by starting with 42 cubes and put one cube in each of the six groups until she had no more cubes to distribute. Rachel also started with single cubes, but then stated that she knew 7 times 5 was 35, so put five in seven groups and then distributed the remaining cubes by ones until she ran out of all cubes. (Note: Rachel worked with seven groups, not six as the problem suggested.) Finally, Dominic, drew out towers and cubes. He first drew out 7 towers with 6 cubes in each because he knew 7 times 6 was 42, and then, below that, he drew it out again with 6 towers and 7 cubes in each.

Excerpt 7: Sam's cogitation (AOP 2, post-interview, date: May 2017).

6:17 RC1: Okay, so how do you categorize all of that thinking? What do you think is, um, like mathematically, were they successful at the goal you wanted them… just, we will just use those three examples [referring to Brittany, Rachel, and Dominic].

6:34 Sam: Um. Yeah, so if I were to rank them in, like their understanding, I would rank Brittany and then Rachel and then Dominic. Um, Brittany, and even when we worked down here on the carpet, was having trouble thinking outside the box of ones. She pushed herself to do 3 in each [group] when it was supposed to be, when it was 6 in each [group] down here (pointing to the carpet). Um. So she's pushing herself to [even] think, "Oh I could put more into each group."

7:02 RC1: Instead of ones?

7:03 Sam: Instead of ones. But, she wasn't quite catching on to, like, and I don't think I did a good job of making it clear, how and why we were using anyway… So she was operating on ones. Um, and then, I would say that by thinking about the fact that I knew that [Rachel said she knew] 5 groups of 7 is 35. So I can start there and then figure how many I have left and divide those up, would be the next level of understanding to me. And then Dominic, um, you know, I see a lot of kids when we were doing this um, a while ago or I asked him to represent their thinking...

7:52 – 8:55: (Sam responds on phone call to student end of day departure question. Then, she resumes the discussion of the third child, Dominic.)

8:56 Sam: So, Dominic when he, he knew that he could multiply by 6 and that, that would get him [42] and then he stopped at 42 and he realized that he had 7 towers. And then I see a lot of kids stop there and say okay its 7. But I, then my question to them is okay you have, the way you represented it you have 6 in each and then you have 7 towers, but the information that we were given was that there were 6 towers. So kind of like, have them think about that. Where he then, that would normally be where I would like push them to think about, okay maybe I'm onto something but is what I'm doing really showing the units where they are, what, is it really, figurally, showing the correct units where they should be. And he got to that 7, 7 towers and then he flipped it and he knew, and I had never spoken to him about doing that or anything [and] he knew then to go back and do one tower of 7 was 7 and go up to 6.

Excerpt 7 indicates how Sam's cogitation regarding her students' reasoning shifted in the sense of her ability to make inferences. I see two main points as important. First, in contrast to the previous year (Excerpts 6a and 6b), I infer that Sam's assimilation of students' work included recognition of her students' mathematical reasoning outside of what she has taught them (her own FOM). In thinking about Excerpts 6a and 6b, Sam was initially able to describe what she observed students doing based on strategies she had taught to her students. This was illustrated by her descriptions of how students solved multiplication and division problems. She noticed students doing different things such as drawing all dots and counting them or repeatedly adding until reaching the total. In Excerpt 7, Sam does not relate the strategies she has observed to something that she has taught the students. In fact, she stated she recognized Dominic reasoning in a way that she has never discussed with him before (see lines in 8:56).

Second, I infer that Sam seemed to have gained the ability to think about her students' reasoning based on the units and operations each of the students might have been using to think about and solve the problem. She inferred Brittany operated by putting one in each group; Rachel began with a group of 5 and then changed to operating on ones; Dominic used his multiplication understanding of 7 groups of 6 to determine what 6 groups of 7 would be. This description, in spite of whether the inferences are consistent with an expert's inferences or not, is something that up until this point Sam had not demonstrated. They suggest Sam was cogitating students' mathematical thinking in terms of the units and operations students are using (instead of referring to FOM strategies she taught to students to arrive at the correct answer).

Manifestation 2: Cogitation summary. In the three excerpts presented in this section both Charlie and Sam demonstrated an enhanced ability to contemplate their students' mathematical reasoning, which I inferred based on their observations and inferences into their students' understanding. This enhanced ability to contemplate students' mathematical reasoning (cogitation) serves as evidence in addressing the second research question of this study. This is indicated in each teacher's discussions of not only what their students did in class but also of hypothetical models of what these observations might mean for the student as separated from the teacher's own understanding or what they believe the mathematics to be. For Charlie, this is found in his description as to why Craig originally thought $\frac{5}{9}$ times $\frac{1}{9}$ was $\frac{5}{9}$. For Sam, this is found in her discussion of types of students' solutions and what each solution means in terms of the similarities and differences in thinking with units and operations. These changes in Charlie and Sam's explanations of their students' mathematical activity serves as evidence in addressing the first research question of this dissertation study.

Manifestation 3: Distinction

In this section I present four excerpts to depict the manifestation of Distinction. Key to SOM is the ability to make inferences about available schemes and evolving schemes a learner may have. To shift towards SOM, those making the inferences must be able to distinguish between their own mathematics and the mathematics of others. Manifestation 3, Distinction, captures this shift, as teachers begin to infer into their students' mathematical thinking through making distinctions in two ways: between one student's mathematical reasoning and another's mathematical reasoning, and between possible inferences of mathematical understanding within one student. Both way encompass teachers' distinctions between their own mathematics and the students' mathematics.

Charlie's distinction: Part-A. Excerpt 8 (below) is taken from Charlie's post-interview of AOP 2. It highlights Charlie's distinction regarding the mathematical reasoning of multiple students. If thinking back to the initial excerpts presented (Charlie's Juxtaposition of Thinking, Excerpts 1-3), Charlie was not able to describe his student's thinking apart from his. In Excerpt 8, Charlie discusses his inferences into mathematical reasoning of his students as he gleaned from a formative assessment students took earlier in the day. The assessment, which Charlie himself created, contained the following questions:

1. Complete the pattern: 4, 8, 12, ______, ______, _____, _____
2. If each pizza has 4 slices, how many slices would you have if you ordered 3 pizzas?
    a. What if you ordered 6 pizzas?
    b. How many ore slices will you get by ordering 6 pizzas as opposed to 3 pizzas?
    c. What fraction of a pizza is 1 slice?

Excerpt 8: Charlie's distinction part-A (AOP 2, post-interview, date: May 2017)

15:15 RC1: So, from this (points to the assessment worksheet), what did you, kind of see with the students in their reasoning?

15:23 Charlie: Um, so sort of a little all over the place. Again, I mean, it was sort of confirming to me that Craig sort of got it. Um, I did make some notes down here that I don't know how important it is, but when he was figuring out how many of those slices he would have if he ordered six pizzas [referring to question 2a], he did end up using a count-all strategy. I physically saw him counting each and every one of these and there are the pencil marks on them and he was one off; but he didn't have the ability to self-assess and see that this is actually four slices split up over six pizzas, which should have been…so he also used the wrong operation looking at it. How many slices will you get ordering six pizzas as opposed to three? [referring to question 2b.] He turned it into an addition problem as opposed to a difference problem. Which actually, everybody did. They all turned it into an addition problem as opposed to a difference problem um, which wasn't a huge concern at that point. Again, that was more for my own record-keeping but um, it confirmed that for the most part they do have skip counting [referring to work done for questions 1 and 2 together].

16:35: A couple of them got a little tripped up, if we looked at Noel she started skip counting by 8s [referring to question 1]; however, she did do so accurately and I'm curious whether or not she just missed the 8 between the 4 and the 12 or if for some reason she honed in on the 8 and started thinking that that was in some way the indicator of what she needed to continue skip counting by. Um…and I guess she was the only one that didn't have that.

17:06: And then moving into the double-counting for the most part they were all able to get the initial problem right as far as how many slices were in three pizzas [referring to question 2]. There was a bit of a struggle when it came to six pizzas and again Noel, she put 36 [referring to question 2a and its relation to the skip counting required also in question 1]. So, in my mind, that shows that she knew that she had six groups here that was going to be the sixth number that she said in her counting number sequence. Which I thought was great, even though she didn't continue the pattern correctly. She didn't continue counting by 4s. So, I put a little question mark for her there. I feel like had we gone back and taken the time to say remember what we were skip counting by, what comes after 12 if we continue this pattern, which is why I brought the cubes back in at the end again. For her and for Antonio who had initially gotten a little tripped up on this problem [referring to question 2 and 2a]. I thought having these (points to the cubes) in front of them would sort of solidify, oh, we were skip counting by fours that's why I asked how the pattern was changing every time. We added another pizza or tower in this case.

18:06: Um, Leon is operating multiplicatively here, which is great [referring to question 2]. He is able to, without, I am assuming, I asked him to show his work, so I'm thinking he might-of added this on separately. However, I wasn't watching him. However, it does look like he did go through with a pencil and count some of those individually but the fact that he's able to create the equation from it, I think is really supportive.

18:34: And then Antonio, sort of (sigh), my odd one out. Um, I think he originally got confused as, with pizza and slices [referring to question 2] at this point which is something that we saw yesterday and even today when he was talking about the individual unit fraction as opposed to the whole, he was getting tripped up on that comparison. Which it looks like I see here again today, instead of 6 pizzas I think he had attempted here to create 6 slices in those original 3 pizzas [referring to

> question 2a]. Even though some of them look like they're split up into eighths. But the fact that he came up with 30 um, throws me for a bit of a loop. He was able to do so successfully here 3 pizzas, 4 slices [referring to question 2]. He's able to accurately say 4, 8, 12 slices. Here I'm not totally sure what he did, unless he did end up splitting these into eighths…8, 16, 24 and then he wrote the 6 down here to help him remember that there are six slices per pizza and then added that 6 onto the 24 to get 30. I don't know but I think he definitely did something here to trip himself up and he's definitely not seeing the difference between the unit that we are operating on and the whole at least at this level.

This excerpt (8) highlights Charlie's ability to make distinctions in his students' mathematical reasoning. Two points seem of importance. First, he distinguished among four different students and the reasoning they have based on the work they have done: Craig using a count-all strategy (see lines in 18:34); Noel counting by 8s instead of 4s but doing so accurately (see lines in 16:35 and 17:06); Leon thinking multiplicatively with an addition equation, however seems to have counted individually (see lines in 18:06); and Antonio struggling to determine what to count by essentially getting confused with the pizza and the slices (see lines in 18:34).

In addition, Charlie is able to infer into reasoning of other students regardless if they have arrived at the correct answer, and compare that thinking to other students who also did not arrive at the correct answer. Charlie stated quite early on that all students turned 2b into an additive problem instead of a difference problem (see lines in 15:23). Yet, he was able to infer into the way that students skip counted in the problems leading up to that problem. His insight into Noel's reasoning shows he made distinctions in how she was counting and what her mathematical reasoning may have been as she was answering the questions: "So, in my mind, that shows that she knew that she had six groups here that was going to be the sixth number that she said in her counting number sequence. Which I thought was great, even though she didn't continue the pattern correctly (17:06).” Charlie seemed to have shifted from this previous way of thinking – from little to no description of student thinking to being able to think about his students’ mathematical reasoning and begin to compare how that thinking may differ from other students. Importantly, he is able to do this whether students got the right or wrong answer. Charlie is making distinctions of similarities and differences of the mathematical reasoning (SOM) from one student to another student in the class, which includes juxtaposing.

Second, based on the distinctions between students, Charlie seemed able to also distinguish between his own mathematics (FOM) and the mathematics of others (SOM). Charlie still struggled slightly, making a few statements like, “I am not sure” or “I don’t know.” Nevertheless, he attempted to make conjectures as to what the students were thinking. Charlie’s thought process while considering Antonio’s work and mistakes is an example of this: “Here I'm not totally sure what he did, unless he did end up splitting these into eighths…8, 16, 24 and then he wrote the 6 down here to help him remember that there are six slices per pizza and then added that 6 onto the 24 to get 30” (18:34). I infer that, at this point, Charlie is beginning to make distinctions about the mathematics that students may use as different from the way that he thinks about it. This is contrasted with his earlier work, where his own thinking seemed to serve as the main point of comparison. His struggle indicates he is pondering students’ thinking about the mathematics differently than he understands it even though he may not be fully able to determine what that difference is.

Charlie’s distinction: Part-B. Excerpt 9 (below) brings a different distinction made by Charlie. During his AOP 2 (observation 1), Charlie asked Antonio: “If you have $\frac{9}{8}$ and you take away one whole, what would be left?" To this, Antonio replied that he would have taken away $\frac{1}{8}$, to which, Charlie, during the observation, immediately moved Antonio to work with the cubes and explain his answer. Charlie told Antonio that each cube represented $\frac{1}{8}$ and asked Antonio to build the whole for him with the cubes.

In the mid-interview, I asked Charlie to talk a little bit about why, with Antonio, he made the instructional move to working with the cubes. It is here, where Charlie manifested making distinctions of mathematical reasoning *within a student*.

Excerpt 9: Charlie's distinction part-B (AOP 2, mid-interview, date: May 2017).

23:01 Charlie: So, in my mind, I wanted to see if he was saying I'm taking away, in his (Antonio's) mind, if he's thinking the whole is $\frac{1}{8}$ or if he was already able to do in his head, oh okay we just talked about one hole is a $\frac{8}{8}$. If I take that away from $\frac{9}{8}$ and I'm left with $\frac{1}{8}$. So, the answer he was giving me and the question that I was asking didn't necessarily lineup, but I was curious as to whether he knew that this $\frac{1}{8}$ that he said right off the bat, was what it was going to be left with once he took the whole out. Um, when the question I was really asking was if we were to take one whole away, how many eighths we would be taking away. So, I wanted to actually put something in front of him where actually he could build it and then take that whole away to see if he knew that one whole was $\frac{8}{8}$ and then he could say okay here is one whole; I'm left with this one piece, this one extra eighth; and see if he could then make the connection that it's one whole and $\frac{1}{8}$. So, in my mind, I was, by giving him the cubes, I was sort of differentiating whether he was thinking, knowing that he would be left with a $\frac{1}{8}$ piece after taking $\frac{8}{8}$ away or if when I said what would it look like to take one whole away from that if he was thinking $\frac{1}{8}$ was the same as one whole. So, if he was getting the whole and the unit fraction mixed up.

24:29 RC1: And you feel like, so then he went on to, you said how many wholes is this (holding up the towers of 8 cubes) and he said eight and he actually said it twice. So, do you feel like that was something that you, it, when you're trying to understand if it was one eighth or he already done it, do you feel like that he, how do you feel he actually understood it?

24:52 Charlie: So, the fact that I said what would, the fact that when I said what would it look like to take away one whole he said $\frac{8}{8}$ and then I said well how many wholes do we have here. The fact that said that he said eight again is showing me that he is seeing this unit fraction of $\frac{1}{8}$ (holds up one cube in right hand) as a whole. There is a disconnect there between the unit fraction and the whole. So, he is seeing this [$\frac{1}{8}$] as our whole (holds up one cube in right hand and puts it down) instead of this as our whole (holds up tower of 8 cubes in left hand and puts it down).

Excerpt 9 shows my inference into Charlie's ability to make distinctions of different mathematical reasoning within one student. Charlie goes back and forth between trying to determine if Antonio's understanding is that one whole is $\frac{8}{8}$ and when taken away from $\frac{9}{8}$ that there would be $\frac{1}{8}$ left, or if Antonio was thinking that $\frac{1}{8}$ was the whole (see lines in 23:01). Charlie is focused on inferring into Antonio's reasoning as a result of what Antonio attended to in answering Charlie's question (see lines in 24:52). Consequently, he determined that, for

Antonio, there is a disconnect between understanding the unit fraction and its relation to the whole.

I infer that Charlie separated the way he understands the mathematics (his FOM) from Antonio's. This is exampled by Charlie not attributing his process and reasoning to Antonio's thinking by stating "if he did have it" or "in my mind Antonio should…" in Excerpt 9 above. Charlie, instead, is indicating an SOM of Antonio and making distinctions within that SOM between two potential ways of reasoning based on his observations (understanding $\frac{1}{8}$ as something that would be left or something that would be taken away).

In summary of Excerpts 8 and 9, I see them as examples of how Charlie's shift towards SOM involved making more specific distinctions between his own understanding and the students understanding, student to student reasoning, and within student reasoning. These distinctions illustrated Charlie's growing ability to infer into a student's conceptions and how that might differ from his own understanding of the mathematics (FOM). Like Charlie, Sam's ability to make distinctions regarding her students' mathematical reasoning also seemed to shift.

Sam's distinction. Sam's first observed indication of distinction took place at the very beginning of her post-interview of her AOP 2 (Sam's Distinction will be analyzed using both Excerpt 7, above within Cogitation, and Excerpt 10, below). As described within the Cogitation manifestation, for this observation Sam taught a lesson on partitive division. She gave her students the problem of: "You have 42 cubes. You want to put the same number of cubes into 6 towers. How many cubes will be in each tower?" While students were working on the problem she moved about the class and observed their work, as well as asked some students questions. Interestingly, Sam independently brought her reflection on this experience right at the beginning of the post-interview and her observations during the lesson.

Excerpt 10: Sam's distinction (AOP 2, post-interview, date: May, 2017).

00:52 RC1: Talk to me a little bit about how you think the lesson went.

00:54 Sam: Um, well I didn't at all get to, like, what I had intended to, kind of, and um, well, I did, but I didn't. Like, it went in a weird way that I didn't expect. I kind of just went with it. And so, um I saw kids when I gave them, um, a chance to go back to their seats and work with the manipulatives or represent their thinking [regarding the problem of 42 cubes and putting them into 6 towers], um, I kind of, just working at the one table that I did, I saw three things happen. I saw one student, um working on, working with just, like you, one, working with ones. Like breaking up the pile of cubes into ones, the 42 cubes. And then, um, I saw another student start with 5 in each. I saw Chloe do that down here too or um, Diana. And then just add on what had then, then by ones, the remainder that they had. And then I saw Dominic who went back to drawing towers and, um, cubes in each, and he originally started with 6 in each because it was six towers but he drew out, he drew six and each and then he got to seven towers and a total of 42 and he knew that even though that got him to his answer that he had sort of like represented the units incorrectly and then went back and flipped it. So, when I spoke to the student who put 5 in each she said well I knew there would be at least 5 in each because I knew that 5 groups of 6, 5 groups of 7 were 35 is what she told me.

In thinking back about Excerpts 6a and 6b, Sam was able to recognize different solution strategies from her students based on what she had taught (her FOM). She seemed unable to use these observations as evidence for how students understood the mathematics different from what she had taught them. In this sense, I infer her observations of students were based on her FOM.

In Excerpts 7 and 10, however, it appears this mindset has shifted in that Sam is making distinctions among students' reasoning. Specifically, she is comparing the three students and seems to have gained ability to rank those students' strategies as indicating different levels of conceptualization – as opposed to all of the students having the correct answer therefore the same thinking, regardless of how they got there (Excerpt 7, see lines in 6:34 and 7:03). In other words, Sam begins to attribute what she observed students do differently to different types of thinking for each student and therefore different levels of mathematical understanding that were pertinent to them. Sam was able to discuss why Brittany, Rachel, and Dominic would be ranked from highest to lowest understanding.

In addition, Sam seems to begin to generalize other students' reasoning and categorize them based on these three exemplars of conceptualizations (Brittany, Rachel and Dominic). This is indicated in Sam's comparison of Diana to Rachel (Excerpt 10, see lines in 00:53). This generalization of an SOM to other student responses can be considered her initial creation of what could become epistemic subjects (Piaget, 1966; Ulrich et al., 2014). This serves as evidence Sam is further making distinctions of students mathematical reasoning beyond just the three initially discussed students.

Manifestation 3: Distinction summary. In the four excerpts discussed in this section (7-10), both Charlie and Sam displayed an improved capacity to make distinctions in their students' mathematical reasoning. I infer Charlie and Sam to have manifested an improved capacity to make distinctions in their students mathematical reasoning, which serves as evidence in addressing the second research question of this study. Charlie indicated he was able to distinguish between mathematical reasoning from one student to another and also to contemplate mathematical reasoning a student may be having between two different types of thinking. Sam's distinctions indicated she was moving away from "the students do these strategies as a way of what was taught to them" to "they do these strategies because of the mathematics that is available to them and that is an indication of their level of understanding." Critically, Sam's distinctions allowed her to rank student understandings and begin to place/categorize other students within this ranking. Hence, both teachers' distinctions allowed them to create hypothetical models of what their observations might mean in terms of the student's mathematical reasoning as separated from their own understanding or what they believe the mathematics to be. The hypothetical models described by teachers indicate a change in their ability to explain their students' mathematical reasoning outside their FOM, and thus serve as evidence in addressing the first research question of this study.

Manifestation 4: Mindfulness

In order to create instructional environments where teachers can infer into the SOMs of students, instruction needs to facilitate interactions where the students are able to structure their own activity based on what they know – as opposed to interactions where the activity is structured based on what the teacher's FOM entails students should do and demonstrate correctly. In other words, a teacher who operates with an SOM and FOM encourages students to bring their reasoning to learning situations and then to structure their learning accordingly. This is in contrast to a teacher who only operates from an FOM and creates an activity and engages students in it regardless of what students can assimilate. The manifestation of mindfulness indicates this inclination in teaching. A shift in mindfulness seem to gravitate from a mindset of, "*if students do this then I will do th*at," to beginning to incorporate, "*I wonder what this student* *is thinking and how can I use instructional time to infer and facilitate learning based on my inferences*." With both Charlie and Sam, their mindfulness around the goal and intended plan of teaching seemed to undergo such a shift. In place of thinking, "my students will understand the math as I do, so if they do anything other than that I will assume the students did not get it and I will reteach," they seemed to think, "I need to figure out what the students do know, so that I can then facilitate a plan for teaching that can accommodate that learning." Below, I provide further data and analysis of Charlie's and Sam's mindfulness.

Charlie's mindfulness. In Excerpt 11 (below), Charlie described a learning situation regarding AOP 2 (observation 1), and how he would have liked the lesson to have gone differently than it did. In the lesson, Charlie was working to help students understand $\frac{9}{8}$ could also be thought of as 1 and $\frac{1}{8}$ and as $\frac{1}{8}$ repeated nine times. Charlie did this by having a student repeat the unit fraction of $\frac{1}{8}$ in Fraction Bars (Kaput Center, University of Massachusetts, 2016) on the board for all the students in the group to see. While repeating the $\frac{1}{8}$ piece, the students stated how many of the $\frac{1}{8}$ pieces are represented. For example, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, and so on until reaching $\frac{8}{8}$ at which point the students also said, "or one whole". Charlie then stopped the repetition of the $\frac{1}{8}$ pieces and asked the group of students: "What if we were to repeat that [$\frac{1}{8}$] one more time? What if we didn't stop? What fraction would we be left with at that point?" Students responded to Charlie with $\frac{1}{9}$ instead of $\frac{9}{8}$.

In the AOP 2 mid-interview, after observation 1, Charlie discussed how he would rather not have stopped at one whole or $\frac{8}{8}$ and just kept going to see what the students would say. I inquired into Charlie's reasoning about this erroneous response given by students.

Excerpt 11: Charlie's mindfulness (AOP 2, mid-interview, date: May, 2017).

20:10 RC1: How would that have changed the situation? [If the student would have kept repeating $\frac{1}{8}$ past $\frac{8}{8}$ rather than stopping at $\frac{8}{8}$.]

20:12 Charlie: Um, so, they [the students] started saying, like, I can't even remember what it said, but I feel like it was something along the lines of, like, a ninth, 1 and $\frac{1}{9}$. So, I feel, like, had they kept going with what we had been originally doing, $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}$, counting by ninths up until $\frac{8}{9}$ or up until, I'm sorry, $\frac{8}{8}$, I was curious to see what would come right after $\frac{8}{8}$. If they would have kept going with that counting scheme $\frac{9}{8}$ or if they would have said to themselves $\frac{8}{8}$, ok that's a whole, there's something else after that, that's an extra $\frac{1}{8}$ so that's 1 and $\frac{1}{8}$. I was just curious to see how they would count that piece afterwards. And I feel like the visual up there [fraction bar on the board] would have been a little helpful especially for someone like Antonio and I believe Daisy who even started to change it into 1, like 1 and $\frac{1}{9}$. I feel like she changed it to ninths because we now have 9,

21:15 RC1: Yeah, she said $\frac{1}{9}$, yeah…

21:16 Charlie: In her mind there were now 9 pieces up there. She had that one extra piece so in her mind that was one of those nine pieces that were now up there on the board. So, I feel, like, that's where she got $\frac{1}{9}$ but I feel like it would have been helpful, I also wanted to go back and figure out when we first pulled it out [the $\frac{1}{8}$ piece from the whole bar], ask the question what is this piece? To see if they could tell me $\frac{1}{8}$, and then from there I wanted to label and repeat it. I tried to do

> that afterwards with Craig but it didn't work out, going back a labeling. But I feel, like, had we had that $\frac{1}{8}$ and talked about repeating it once they got to $\frac{8}{8}$ it would have been a little easier for them to say oh $\frac{8}{8}$ I'm just adding on one more $\frac{1}{8}$ piece and I have an extra 8th, $\frac{9}{8}$. And then the idea was to work backwards from there using that visual which never got up on the board (laugh).

Excerpt 11 was selected to highlight my inference of Charlie's enhanced state of being aware of his teaching moves within a lesson. In thinking about Excerpts 1-3 under the Juxtaposition of Thinking manifestation, Charlie's way of reasoning through students who had difficulty understanding was to repeat the process/steps of how a unit fraction would be found. In Excerpt 11, however, Charlie seems to provide his own lens of analysis on his instruction and how it should have gone in order to elicit more of what the student brings to the learning experience. Specifically, Charlie stated he would have liked it to go differently by not stopping at $\frac{1}{8}$ (see lines in 20:12) and also have students initially diagnose and label the $\frac{1}{8}$ piece (see lines in 21:16).

This sharing of the desire for the instruction to have gone differently, and his description of what he conjectures this may have brought for students' thinking, alludes to his growing propensity (hence the term mindfulness) for bringing forth what the students know to a learning situation (see lines in 20:12). This mindfulness seems to underlie his questioning what might have happened had he continued with the action of repeating the unit fraction of $\frac{1}{8}$ past the whole. Charlie's mindfulness around his teaching and instructional moves showed he no longer wanted the students to simply repeat the process of counting unit fractions. Instead, he wanted to provide activities that allow him to understand what the students could reason and would say (see lines in

20:12 and 21:16). With this intended instructional shift of allowing students to bring what they know to learning situations, Charlie can have more opportunity to determine the SOM of his students. Sam's mindfulness, while seemingly not as developed as Charlie's (I explain this comparison after the excerpts), also indicates a shift in the way she intends her instructional activities to play out.

Sam's mindfulness. The following two excerpts (12 and 13) serve as examples of Sam's mindfulness. Excerpts 12 and 13 took place during Sam's AOP 2 within the post-interview. In this post-interview, Sam discussed how she thought the lesson went and different solutions she observed students using when trying to solve the task, "If 42 cubes are placed equally into 6 towers, how many cubes in each tower?" (lesson described in the Cogitation and Distinction manifestations). During parts of the interview, Sam seemed to struggle with how to facilitate learning situations in which students could bring what they knew as opposed to Sam telling them how to do mathematics based on the way she understood it.

Excerpt 12: Sam's mindfulness (AOP 2, post-interview, date: May, 2017).

4:13 Sam: Okay, ehhhhhhh, it's, like, really hard because you are just seeing what's in front of you. And then you have to make a game-time decision. And so, like, I'm good with that when I have a thorough understanding of the content and what I'm doing, but, like, when I have this lesson where I'm, like, okay it's going to go like this. I hope it goes like this because if it doesn't and then I, so I, with content that I'm still a little rurrururu (makes and uneasy sound), like I have an idea but then once it goes places, my brain goes, like, crossfire. So, go ahead (looks at RC1 for more questions regarding the lesson).

Later in the interview, Sam brings again this conundrum she is experiencing when talking about her interactions with one specific student.

Excerpt 13: Sam's mindfulness (AOP 2, post-interview, date: May, 2017).

10:59 Sam: And I said what kind of, what did you do up here then that's where my brain like doesn't know how to, like, clearly, like, probe them to think about the connection that they made. And I said, okay, I saw you do some math that, represents your thinking and in a way that we have been working with for a long time now. So, can you, kind of, tell me what kind of math you were doing to help you figure it out? And he said multiplication.

I consider data in Excerpts 12 and 13 to indicate that Sam's mindfulness was not as developed at Charlie's, in that she was not able to specify how she would have preferred the lesson to change. Rather, she seemed to be just developing an idea that her instruction should shift from a mindset of, "provide strategies so that students can pick one" to a mindset of, "let the students do what comes to them and then work with them to facilitate a more advanced understanding." Said in another way, I infer she is aware that the way she went about her instruction needed to change in order for students to bring their own mathematical reasoning to the activity. However, I also infer that Sam was not sure how to change it, as indicated in her constant struggle with determining "game-time decisions" and admittance of it being really hard (Excerpt 12, see lines in 4:13).

Manifestation 4: Mindfulness summary. In Sam's and Charlie's shift towards SOM they manifested growing mindfulness about their teaching practices. My inference on the manifested mindfulness of the teachers' instructional practices serves as evidence in addressing the second research question of this study. Both teachers seemed to move away from trying to direct students to the mathematics that they themselves understood, that is, their FOM. Instead, Sam and Charlie were attempting to understand the students' thinking from the students' point of view, so they could cater their instruction to that thinking. Charlie's mindfulness seemed more advanced than Sam's. He was able to not only reflect on his moves but also describe how he might have changed them to gain better insight. For Sam at this point, it was still just a separation between what she should do and what she can do with description of how it is hard for her.

In Summary

In this chapter, I analyzed data of teachers' shift in explanations of their students' mathematical behaviors, from being based primarily on the teachers' mathematical knowing (FOM) to also attributing mathematical knowing that differs from that of the teacher (SOM). Within the analysis, I focused on teachers' ability to describe the mathematical reasoning of their students as different from their own and as compared to other students within the class. This focus yielded my distinction of four manifestations:

Manifestation 1 – Juxtaposition of Thinking: Teachers experienced a contrasting effect when comparing their own FOM mathematical experience with a student's different mathematical experience.

Manifestation 2 – Cogitation: As teachers shifted towards SOM they were able to think more deeply about their students' reasoning.

Manifestation 3 – Distinction: A shift in teachers' inclination to describe their students' mathematical reasoning not based on their own understanding but based also on the mathematical behaviors and explanations of their students was rooted in a greater ability to make distinctions in their students' mathematical understanding. A teacher was able to describe similarities and differences in students' mathematical reasoning and compare those similarities and differences to the teacher's own understanding, to the understanding of other students, and even to different mathematical understandings of one student.

Manifestation 4 – Mindfulness: A shift towards SOM included teachers showing thought in how instruction should change from "I tell the math" to "I look for opportunities to understand my students' thinking so I can build on those understandings." Teachers began to indicate propensity for instructional moves they could make in order to bring what I would consider students' assimilation as a way to interpret SOMs of their students.

CHAPTER V

DISCUSSION

This dissertation study addressed the following questions:

1. What changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift away from mostly relying on their first-order models (FOMs) to teach mathematics?
2. What may be manifested in elementary mathematics teachers' work and explanations, as they shift from using only first order models towards differentiating between their first order model and students' mathematical reasoning?

I collected data on two participants who underwent professional development geared towards Student-Adaptive Pedagogy (Steffe, 1990; Tzur, 2013). These data consisted of video recordings of the teachers' participation in professional development (workshops, summer institutes, Buddy-Pair sessions) and in Accounts of Practice Sets (AOPs, see Chapter III, Methodology).

In Chapter IV, the data analysis yielded four manifestations that emerged as teachers shifted in their ability to explain their students' mathematical behaviors based not only/mainly on their own mathematics (FOM) but also on the mathematics students bring to the learning situation (FOM and SOM). Those four manifestations are:

Manifestation 1: Juxtaposition of Thinking. Teachers begin to realize that students' sense of learning situations are different from the teacher's experience or what the teacher assumes the experience to be for the students. This manifestation, in turn, seemed to allow the teacher to interpret a student's understanding from the lens of both an FOM and SOM.

Manifestation 2: Cogitation. Teachers begin to indicate an ability to deeply think about their students' mathematical reasoning as separated from their FOM.

Manifestation 3: Distinction. Teachers begin to indicate an ability to make distinctions about their students' mathematical reasoning. This ability includes two main distinctions: (a) students' distinguished from other students', and (b) possible distinctions made within a student. Both distinctions include the teacher's separation of her own mathematical understanding from the learners' mathematics.

Manifestation 4: Mindfulness. Teachers begin to indicate an ability to consider and reflect on their mathematical instruction, including the intention to foster learners' construction and advancement of the learners' own mathematical reasoning and mental operations (not the teacher's). In other words, teachers shift towards a mindset that allows instruction to build on and promote students' assimilation of tasks that are accessible to the students and possibly lead to the intended learning.

In this final chapter, I discuss key contributions of my study, its implications for practice and future research, and its limitations.

Key Contributions of this Study

By examining how teachers' shift towards SOMs of their students' mathematical thinking, this study aimed to create a foundation for understanding, and fostering, teacher learning to identify student existing mathematical understandings and promote further advances in students' mathematics. In this section, I examine five main implications for research regarding teachers' shift towards SOM. First, I consider existing research on Perception Based Perspective (PBP) and FOM and the important link between the two that can be derived from my study. Second, I explain the importance of the four manifestations for fostering a shift in teachers' perspective of learning and knowing towards Conception Based Perspective (CBP) (Simon et al., 2000) and use of Student-Adaptive Pedagogy (Steffe, 1990; Tzur, 2013). Third, I elaborate on how the four manifestations distinguish facets of what has previously been termed SOM. Fourth, I discuss how the findings of this study may be similar to and different from the levels of SOM researchers have identified (Ulrich et al., 2014). Fifth, and finally, I discuss how a shift towards SOM can expand the work on teacher noticing (Jacobs et al., 2010; Mason, 1998, 2008).

Linking perception based perspective and first-order model in instruction. Perspectives on knowing and learning (detailed in Chapter I) include what teachers do in the classroom and also the way they think about their practice, motivations behind the methods they use, and the intentions that drive their instructional moves (Simon et al., 2000). Each perspective affords or constrains a teacher's ability to create an SOM of her students. PBP is a belief that learning mathematics involves some sort of interaction or hands-on activity for students to "see" the mathematics. Therefore, mathematics is learned by promoting students' active discovery of the mathematical concepts and usually concludes with an explanation of the understanding to make sure students "saw it". With such a perspective, what the students see and discover is largely based on the teacher's understanding of the mathematics and how the teacher's FOM shapes how the mathematics should be seen and discovered. Data analysis I presented in this dissertation study support further relating PBP with FOM.

Before shifting to SOM, teachers in this dissertation study held characteristics of PBP and used their FOM to create situations for students to actively discover the mathematics. In addition, it seemed the teachers' FOM formed a lens through which they considered students' success with the mathematics (the students are successful if they demonstrate using the mathematics the way a teacher sees it). For example, Charlie in AOP1 was using his FOM – mathematics he had learned through the larger project – to make sense of his students' mathematics and instruct further learning. That is, it seemed his interpretation of students' mathematics was based on observable activities he experienced and wanted his students to experience the same way. Hence, when Monica struggled to identify $\frac{1}{6}$ on the number line, Charlie guided her through the process of finding $\frac{1}{6}$ and then measuring that amount from zero to mark the correct place. Of importance here, is that this PBP, which was coordinated with Charlie's FOM, seemed to obstruct Charlie's ability to create an SOM for Monica. This is significant, because it points in the direction of combining PBP with FOM. Specifically, FOM seems to be a part of the assimilatory schemes a teacher with PBP on knowing and learning uses to assess what students know and design and enact instruction. Thus, the shift towards SOM may provide a precursor to changing one's stance on knowing and learning.

Teachers' change in perspective on learning and knowing. The four manifestations discussed in this dissertation study further explain how an individual teacher may transition into a Conception Based Perspective (Simon et al., 2000) and thus become able to utilize Student-Adaptive Pedagogy (Steffe, 1990; Tzur, 2013). As stated in Chapter I, a teacher who holds a CBP and uses Student-Adaptive Pedagogy must have the ability to create SOMs of her students. If thinking about a shift from FOM alone to FOM coupled with SOM as precursors in changing perspectives, then these four manifestations may serve in teacher educators' work to identify a potential beginning of a change in perspectives. That is, the four manifestations can guide researchers and teacher educators in having an initial idea of what may constitute moving from one perspective to another. The data in this study depict what a teacher goes through when shifting from FOM alone to FOM coupled with SOM.

Facets of second-order models. SOM is a model of someone else's mathematical reality, based on inferences an observer makes of a student's mathematical understanding from observed behaviors (Steffe, 1995, 2000; Steffe & Thompson, 2000; Thompson, 2000; Ulrich et al., 2014). The four manifestations distinguished in this dissertation study may serve as facets of this definition and overall SOM. This can, in turn, provide a platform for future work on elements of an SOM. Specifically, the four manifestations may provide insight into milestones on the path of moving from FOM alone to FOM coupled with SOM.

Comparisons between a teacher's and researcher's shift towards SOM. To date, no research focused on how a teacher may shift towards practice in which FOM is coupled with SOM. In referring back to Chapter I, research does exist on different levels of an SOM researchers might have and use (Ulrich et al., 2014): Emerging SOM, Developed SOM, and Elaborated SOM. At the Emerging level, researchers have insight into a student's mathematical thinking as separated from their own mathematical understanding, but are not yet able to make instructional adaptations as a result of the SOM – as it is still being constructed. At the Developed level, a researcher can anticipate and plan interactions with students based on the SOM. Finally, at the Elaborated level, the researcher is able to determine a viable SOM of the student and situate this with other SOMs in a class to create several epistemic subjects that drive design and enactment of mathematical instruction. Such epistemic subjects can then be used to anticipate and plan for instruction that serves to orient students' assimilation and reorganization of what students know into new (for the student) mathematics.

Based on this dissertation study, I can point out how a shift in teachers towards SOM as represented by the four manifestations may be similar to or differ from that of the researcher. This comparison between teachers and researchers is important because it may help to lead to future research on how mathematics educators can promote teachers' development of SOM. Moreover, it applies the very notion of SOM to researchers of the need to distinguish their understanding of SOM from how teachers understand SOM (in a way, a researcher's third order model of the teachers' evolving SOM). Following, I further discuss these similarities and differences.

Teachers in this study seemed to develop a separation between the way they thought about the mathematics and the way their students thought about it. For example, Charlie's Juxtaposition of Thinking allowed him to shift towards inferring into the thinking of another student. As such, there seems to be no parallel between the manifestation of Juxtaposition of Thinking within teachers and any of the three levels for a researcher. Juxtaposition of Thinking involves an initial period in which the teacher begins to contrast her own FOM with a students' actual mathematical activity. This contrast can then allow for a separation from the teachers' own understanding of the mathematics and that of the student's. At the Emerging, developed and Elaborated, levels it is assumed such a contrast exists on researchers' part. That is, a researcher who falls into one of the three levels mentioned above, already established SOM as a way of thinking, even at Emerging Level.

Unlike the Juxtaposition of Thinking, the manifestation of Cogitation seems related to the first, Emerging level of SOM used by researchers. Teachers in this study shifted towards an increased ability to cogitate—contemplate and think deeply—about what their students' mathematical activity might be. This Cogitation allowed for teachers to further separate the students' thinking from their own thinking, and think more deeply about how students make sense of mathematics the teacher presents. Likewise, at the Emerging Level, researchers were able to initially describe the mathematics of students as separated from their own understanding albeit the SOM still being constructed or not quite accurate. Teachers who manifested Cogitation were beginning to describe the mathematics of students as separated/different from their own understanding as well as seemed to be still constructing or at times not quite sure regarding the

SOM they were describing. Therefore, the parallel between an Emerging Level for researchers and Cogitation for teachers is the initiation of describing the mathematics of a student outside of an FOM however the inference may not be accurate. Of course, a difference between teachers and researchers still seems important to note – teachers may be unable to articulate students' mathematics that has already been depicted by researchers at the Developed and/or Elaborated levels.

As for the manifestation of Distinction, I point out an important similarity between teachers and researchers. Distinction is a manifestation of teachers' ability to distinguish their students' mathematical reasoning from student student and within a student while separating the model from the teachers' own FOM. Distinction is likely what researchers may incorporate regularly at all three levels. Particularly in the Elaborated level, researchers use the Epistemic Subjects (Piaget, 1966) to group students and organize instruction based on inferences into students' assimilatory schemes. Sam's initial attempts to compare and group students seemed to be of similar nature. The difference being that, particularly at the Elaborated Level, the ability to create an epistemic subject is established and with Distinction this ability seems just to begin to surface.

Lastly, teachers in this dissertation study began to manifest Mindfulness, or an intention to facilitate instruction, which fostered students' construction of the intended mathematics by building on the students' distinct, available ways of reasoning. While evidence within the analysis showed teachers' contemplation and intention of this and fell short of acting (in practice), this seems most in line with that of the researchers' Developing and/or Elaborated levels. It is at the Developing Level where a researcher can anticipate, plan for, and enact interactions with students. Mindfulness captured teachers' contemplation about future and past student interactions with each of the participants of this study even though not yet enacting it.

Better understanding the similarities and differences between SOM in researchers and how teachers may develop SOM may help future researchers and teacher educators design professional development that can build on researchers' experiences as a way to promote learning in teachers. Specifically, researchers and teacher educators can focus on creating learning opportunities for teachers' shift towards SOM based on what the researchers themselves already have come to know (their FOM) and create SOMs of the teachers for the differences in what needs to be developed in the shift. Thus, the levels created by Ulrich et al. (2014) may serve as end goals of acquiring SOM and the four manifestations as potential milestones in how to get there.

Assimilation: Beyond teacher noticing. For a teacher to shift towards an SOM requires that she infer into student's assimilatory schemes and separate these from her own. Making such inferences into students' mathematical reasoning necessitate the teacher's noticing of what students are attending to (Jacobs et al., 2010; Mason, 1998, 2008). That is, teacher development of ability to notice what students are attending to seems like a good first start. However, as I explained in Chapter II, a shift to SOM requires the notion of assimilation.

A teacher who makes a shift towards SOM, and manifests Juxtaposition of Thinking, Cogitation, Distinction, and Mindfulness can better understand why students are attending to what they do. As a result, the teacher can both notice students' mathematical strategies and infer into conceptual roots of what students may attend to. An example of this is Charlie's Cogitation. In order for him to make inferences into Craig's thinking of $5 \times \frac{1}{9}$, Charlie first had to notice Craig's solutions for adding fractions with like denominators and whole number multiplication problems. With such noticing, Charlie could then infer that Craig was applying what he knew about whole number multiplication and adding like denominator fractions to create an invented strategy for 5 x $\frac{1}{9}$. In other words, teacher noticing seems to be necessary but insufficient for a shift towards SOM, as analyzed for each of the manifestations in Chapter IV. In this sense, this study can help expanding the work on teacher noticing by characterizing it as a basis for inferring into student mathematical activity, that is, into the conceptual roots of students' attention. Next, I discuss implications of this study for practice.

Implications for Practice

With teachers' shift towards SOM, transformation can be fostered in teaching practices, which may be better adapted to students' available mathematics. In Chapter II of this study, I provided an image of a mathematics classroom with three groups of students: those above grade level, those at grade level, and those below grade level. In that image, I outlined how students at each of these levels would learn mathematics based on a teacher who lacked SOM. I now return to those three images while linking them to a teacher who has manifested Juxtaposition of Thinking, Cogitation, Distinction, and Mindfulness. In doing so, I hope to describe how a shift towards SOM in a teacher can better pinpoint affordances and constraints in students' work and can, in turn, promote learning in all three of those groups.

Juxtaposition of Thinking allows a teacher to separate her own mathematics from the student's mathematics and thus begin to recognize differences in the way the student assimilates (thinks about and interprets) the learning experience. As a result of this separation, teachers can better pinpoint below, at, and above grade level students' likely assimilation of instructional materials (tasks, manipulatives, numbers used, etc.), and begin to better understand what the student does know and is still struggling with. Specifically, this can help teachers recognize exactly what students below grade level are struggling with, or when at grade level students know the math and are ready for more advanced mathematics, or determine how to advance above grade level students to higher-level understandings. For example, Charlie, when recognizing Monica's struggle, seemed to begin to question his inferences into how she thought about the mathematics of a unit fraction as a source of that with which she seemed to struggle. This contemplation opened the door for Charlie to recognize there was a difference between his and Monica's thinking of the mathematics. In turn, such a recognition may pave the way for adapting goals and activities for promoting Monica's progress.

Cogitation allows for a teacher to think more deeply about the inferences she has of her students' mathematical reasoning. Such deeper thinking can allow a teacher to infer into the mental actions of the student and begin to predict possible ways to help the student reorganize the existing mathematics to more advanced concepts. This ability seems to be at the heart of differentiating instruction to below, at, or above grade level students so each of them is making progress. For example, Charlie's inferences into Craig's additive fraction knowledge and understanding for the move towards multiplication of a whole number by a fraction, could allow him to better design instruction targeting the inferred conceptions and promote more advanced mathematics. Cogitation can thus yield for the teacher a model that improves designing instruction targeted to what students do know, as well as with what they struggle.

Distinction allows a teacher to focus on what students attend to and the conceptual roots of why that student attends to it, and thus be able to compare those inferences to the mathematics of others. Thus, the teacher can begin to create, in her mind, an epistemic subject – groups of students (below, at, or above grade levels) who are likely to assimilate tasks similarly. Sam, for example, began categorizing students as she began grouping Diana with Rachel. In making distinctions, the teacher can design and enact learning experiences that accommodate all three levels because those distinctions underlie selecting goals and tasks for students' learning to fit with what the teacher infers their existing understanding to be. As a result, distinction can support clarifying the model of students' thinking for the teacher.

Mindfulness allows a teacher to bring the inferences of students' mathematical thinking into the design and enactment of instruction tailored to students' conceptual level. As teachers begin to wonder and inherit a need to change their instruction, becoming mindful supports opportunities for students to bring what they know to the learning experience. Accordingly, all three levels of students benefit as opportunities are opened for them to bring and use what they know to initiate, sustain, and reflect on their goal-directed activities. This was illustrated with Sam, when she struggled with what to do next with Dominic. Sam admitted Dominic brought something she did not teach him. She seemed mindful of the need to further support his use of what he brought to advance his reasoning, but was unsure how to proceed. Her example of not knowing what to do is important in that it demonstrated she began to be mindful of the intent to use her understanding of the students' existing mathematical realities (SOM) as a basis for instruction. If all students have this opportunity available to them, whichever level they are at, the teacher would provide opportunity for the student to indicate what she or he does know. Subsequently, the teacher could use such indications of knowledge to facilitate reorganization of what is known into more advanced mathematical concepts.

In all, each of the four manifestations distinguished in this study can promote mathematics instructional practices for teachers and further lead to learning opportunities for students below, at, and above grade level. With each manifestation, a teacher further establishes a way to build an SOM of students' thinking, which can assist her in understanding why the students may be struggling or excelling. This can then promote learning targeted to each student's assimilation at each level. Next, I will turn to suggestions for future research.

Suggestions for Future Research

The findings of this study open new possibilities for future research when studying teachers' shift towards SOM. In this section, I propose three potential future research endeavors based on the results of this study.

First, while I distinguished four manifestations through this dissertation study, it is yet to be determined if each of the four manifestations will also occur in other teachers. Future research should involve inquiry into other teachers' shift towards SOM and to what extent they are similar to the two participants of this study. In such research, the four manifestations I distinguished could serve as a starting point and guide the inquiry further.

Second, this study did not include analysis on the sequential (or not) nature of the four manifestations. The reason, this study and my distinction of the four manifestations being an initial work. Nevertheless, I make conjectures about possible progression. Specifically, I conjecture that Juxtaposition of Thinking would manifest first, as it underlies a teacher's contrasting of FOM from students' reasoning. The other three manifestations, Cogitation, Distinction, Mindfulness, are expected to follow Juxtaposition of Thinking. Examining such a conjecture would further our understanding of how the four manifestations might be linked to one another.

Third, and most important in my opinion, is to focus future research on how these manifestations emerge. Now that the four main manifestations are identified, research can be conducted as to how each manifestation may be developing, and thus also possibly be promoted in teachers. By promoting more shifts, a teacher can make use of FOM and SOM with instruction, which can then potentially lead to their consistent practice of Student-Adaptive Pedagogy, and provide a basis for improving student learning and achievements in mathematics.

Limitations

This dissertation study has two main limitations: exploratory versus explanatory research findings, and the preliminary nature of the findings. I briefly discuss each of them.

Prior research on development of second-order model. To date, no previous research focused on how teachers may shift towards SOMs. This dissertation study was the first to study this phenomenon. Thus, a limitation of this study was that there was no previous research and theorization on which to build. Accordingly, this research should be considered an exploratory work, as opposed to an explanatory model. That is, the study provides a possible starting point for future research designed to explain the phenomenon, including the developmental nature (if any) of the four manifestations.

Preliminary nature of the findings. As a result of this study being the first of this nature and exploratory the findings are preliminary and therefore can serve as a limitation. It is unclear if similar findings would be conclusive if researching other participants that were part of the larger study. Future work should include comparisons to determine if similar manifestations occurred within other teachers in the larger study.

In Summary

My main focus in this study was twofold: (a) what changes can be noticed in elementary teachers' explanations of their students' mathematical activity as teachers shift away from relying on their FOM to teach mathematics, and (b) what may be manifested in their work while shifting towards differentiating between an FOM and students' mathematical reasoning. My qualitative study, which included constant comparative analysis of observations and interviews with two case study teachers, yielded four manifestations: Juxtaposition of Thinking, Cogitation, Distinction, and Mindfulness. I also found further links could be made between the constructs of Perception-Based Perspective (PBP) and First-Order Model (FOM), as well as potential future indications of how a teacher may shift towards a Conception-Based Perspective (CBP). In addition, I discussed how the four manifestations can relate between researchers' ability to create SOM and teachers' shift towards SOM. Based on this study, future work can focus on how a shift towards SOM may occur for elementary mathematics teachers, and on how such a shift may lead to CBP and, ultimately, to teachers' shift toward a Student-Adaptive Pedagogy.

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