CONSIDERATIONS OF NUMBERS USED IN TASKS FOR PROMOTING MULTIPLICATIVE REASONING IN STUDENTS WITH LEARNING DIFFICULTIES IN MATHEMATICS

by

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Considerations of Numbers Used in Tasks for Promoting Multiplicative Reasoning in Students with Learning Difficulties in Mathematics

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ABSTRACT

This study explored the impact of numbers used in instructional tasks on the construction and generalization of multiplicative reasoning by fourth grade students designated as having learning difficulties or disabilities in mathematics (SLDs). In particular, this study addressed the following research questions: (1) In what ways do SLDs' conception of number as a composite unit afford or constrain transition to multiplicative reasoning? (2) Which specific numbers, used in instructional and/or assessment tasks, may support or interfere with SLDs' progression from additive reasoning to multiplicative Double Counting (mDC)? Results suggested that in early participatory stages, using numbers with multiples familiar to the students, such as 2 and 5, promoted multiplicative solution paths (e.g., counting by fives while simultaneously keeping track of how many fives they have counted). This use of familiar numbers allowed for students' reflection on their multiplicative thinking. Introduction of more difficult numbers—any number for which the child was yet to master multiples—tended to limit the multiplicative thinking and move students back to more known (additive) solution paths. In later participatory stages, the introduction of more difficult numbers promoted the progression within mDC.

The form and content of this abstract are approved. I recommend its publication.

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INTRODUCTION

Focusing on students labeled as having learning difficulties and disabilities (SLD) in grade four, this study addressed the problem of how a teacher can purposely choose specific numbers (symbols of size, or magnitude) for tasks used to foster student acquisition and generalization of multiplicative reasoning. In particular, the impact that numbers chosen for tasks have on students' progressions through multiplicative schemes[1] was examined. Central to this research is the understanding of what schemes constitute progression in multiplicative reasoning (Tzur et al., 2013), how students labeled with learning difficulties construct those schemes, and what key difficulties these students may encounter. This study sought to understand the impact numbers have on SLDs' multiplicative reasoning in order to design more effective classroom situations for those students. This first chapter of the dissertation further elaborates on the research problem.

Context of the Research Problem

In order to be successful and to become proficient in mathematics, children must construct multiplicative reasoning (Hackenberg, 2010; Mulligan, 2011; Norton, Boyce, Ulrich, & Phillips, 2015; Schwartz, 1991; Simon, 2006; Steffe, 1992; Tzur, 2007). The failure of current pedagogical methods to promote learning for SLDs in general (McDermott, 1993), learning for SLDs in mathematics (Tzur, 2013), and specifically SLDs learning how to reason multiplicatively (Grobecker, 1997; Tzur, Xin, Si, Kenney, & Guebert, 2010) is an issue of concern.

To illustrate the achievement gaps between SLDs and normally achieving peers

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[1] The definition of schemes and these particular schemes will be further elaborated in Chapter II.

(NAPs), Table 1 shows a summary and comparison of the performance of SLDs and NAPs on the National Assessment of Educational Progress (NAEP) Long-Term Trend Mathematics Assessment (National Center for Education Statistics, 2014). The Long-Term Trend Assessment was designed to measure students' knowledge of basic facts, computational ability, and the ability to apply mathematics to daily living. To link overall student performance to multiplicative reasoning, Table 1 also includes a summary of the multiplicative knowledge and skill assessed at each of the levels.

Table 1

2012 NAEP Long-Term Mathematics Trend Results

	% SLD Age 9 at performance level	%NAP Age 9 at performance level	%SLD Age 13 at performance level	%NAP Age 13 at performance level	Multiplicative Knowledge and Skill Assessed at this Level
Performance Level 250 *Numerical Operations and Beginning Problem Solving*	23	49	49	89	• Initial under-standing of the four basic operations • Multiply to find the product of two digit numbers
Performance Level 300 *Moderately Complex Procedures and Reasoning*			9	37	• Compute with decimals, simple fractions and common percents • Calculate areas of rectangles • Solve simple linear equations • Find averages • Operate with signed numbers, exponents, and square roots

The results suggested an overall performance gap between students with and without disabilities. At age 9, only 23% of SLDs met the criteria for performance level 250, and at age 13, after four years of instruction, only 49% of SLDs were above level 250 as compared to 89% of NAPs. It is also noteworthy that performance level 300, which requires application of multiplicative reasoning to contextualized situations and operations on fractions, decimals, percentages, signed numbers, exponents, and square roots, was reached by only 9% of 13 year old SLDs and only 37% of NAPs. This may suggest gaps in the type of multiplicative reasoning required for higher-level mathematics.

Similarly, data from the main NAEP collected in 2013 (National Center for Education Statistics, 2014) suggested gaps between SLDs and NAPs as well as an overall drop in performance from grade four to grade eight. Table 2 includes the difference in percent proficient between SLDs and NAPs and lists the assessment frameworks related to number sense and multiplicative reasoning for those proficient and advanced scores.

Table 2

2013 NAEP 4th and 8th Grade Mathematics Proficient and Advanced

	% SLD	% NAP	Proficient and Advanced Assessment Frameworks For Number Properties and Operations Related to Multiplicative Thinking
At or above Proficient Grade 4	18	45	• Multiply whole numbers • Compare unit fractions in context • Recognize the results of cutting and folding paper • Use ratio to describe situations in context • Whole number division
At or above Proficient Grade 8	8	39	• Solve problems involving prime numbers • Solve an algebraic inequality • Determine area of figure, give side length and perimeter • Given a ratio solve a problem

To address and diminish the aforementioned failure to promote mathematics teaching that supports SLDs (and NAPs) in mathematics performance, it seems important to better understand the key ideas in the construction of multiplicative reasoning, how SLDs develop those ideas, and what makes those ideas difficult to learn. Understanding these three things might be key in supporting instruction tailored to the SLDs' needs and promote their learning of those key ideas.

Multiplicative reasoning is one key understanding that all children must develop in order to be successful and progress in multiplication, division, fraction concepts, and algebraic reasoning (Schwartz, 1991; Steffe, 1994; Steffe & Cobb, 1994; Thompson & Saldanha, 2003). To further situate the learning of multiplicative reasoning in the field, this chapter describes the limitations of two current national efforts to improve learning outcomes for SLDs. Then, a closer look at and distinction between multiplicative reasoning and additive reasoning is provided in order to point out the need to better understand what makes multiplicative reasoning difficult and thus contribute to closing the achievement gaps for SLDs.

Standards and Multiplicative Reasoning

One effort to raise achievement for all students, including SLDs, has been the adoption and implementation of standards at the national, state, and district levels over several decades. Most standards documents are written to clarify goals for students' learning and outcomes (endpoints for learning) expected at each grade level. Although clarification of goals and intended learning is a key component of effective instruction, it is only one element of effective instruction (Tzur, 2010). Standards, by design, provide the target (or end point) for students' learning. Standards are not written to articulate the thinking processes that underlie accomplishment of those endpoints and are not intended to help teachers understand the conceptions that a child brings to a learning situation (Ball, Thames, & Phelps, 2008; Hill, Sleep, Lewis, & Ball, 2007). Furthermore, standards do not guarantee that teachers know how to design effective tasks to promote the specified learning, each of which are important elements of effective teaching (Tzur, 2013).

Although implementing standards has been one attempt to ensure that students are academically successful (Common Core State Standards Initiative, 2010), results from state assessments have not shown such success. For instance, results from the 2013 Mathematics Transitional Colorado Assessment Plan (Colorado Department of Education, 2014) indicate that only 20% of 4th graders identified as having a specific learning disability scored proficient or advanced and that only 7% of 8th graders identified as having a specific learning disability scored proficient or advanced (4th graders have been assessed using standards based assessments in Colorado since 2005, 8th graders since 1999). A focus on student outcomes measured by standards-based assessment does not seem to be resulting in strong student achievement for SLDs.

Common Core Standards. In 2010, The Common Core State Standards (CCSS) were introduced with the goal of improving the teaching of core concepts and procedures so that children would be more likely to master important mathematics. The authors of the CCSS (2010) suggested that the standards are written to provide a clear understanding of what students are expected to learn. That is, the CCSS were an attempt to redefine goals for students' learning and expected outcomes. For example, the CCSS (CCSS.Math.Content.4.OA.A.2) state that children are to, "Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison." Although clear in stating what children should be able to do, this standard seems limited, for example, in providing information for teachers about what exactly *is* the difference between multiplicative comparison and additive comparison. Most importantly, this standard illustrates that the CCSS were not written in a way that provides information about what children need to understand in order to transition from additive to multiplicative thinking.

As written, this standard has the *potential* to help teachers understand the goals for grade level performance, but the aforementioned student performance results suggest that the potential has not yet been met. Teachers need more than standards to support effective mathematics teaching (Hill, Rowan, & Ball, 2005), they need models for teaching. One such model, a constructivist based, student-adaptive teaching system in mathematics (Tzur, 2010, 2013), requires that mathematics teachers teach with a high level of understanding of students' ways of thinking and how those ways of thinking are constructed. In this model, teachers plan instructional episodes being mindful of a Teaching Triad (Tzur, 2010), which includes attention to children's current conceptions, the intended mathematics for the instructional episode, and the tasks that are selected to support the children's construction of the intended mathematics. This model requires an understanding of mathematics, in this case multiplicative reasoning, that is not explicated in current standards documents.

Response to Intervention

Another effort to improve the learning outcomes for all children, particularly SLDs, is the adoption by many state departments of education of a Response to Intervention (RtI) model. An RtI model is intended to improve learning outcomes by monitoring the impact of various instructional interventions on student learning. In Colorado's response to intervention manual (Colorado Department of Education, 2008), the RtI model is suggested because "a comprehensive system of tiered interventions is essential for addressing the full range of student needs, and [because] student results improve when ongoing academic and behavioral performance data inform instructional decisions" (p. 22).

The RtI model recommends an increase in the frequency with which teachers collect data to determine student responsiveness to (presumably high quality) instruction and to inform instructional decisions. One common method of collecting data on student growth and progress in an RtI system is via curriculum-based measurement (CBM). CBMs refer to assessments designed to assess student progress with problem types specified in the regular curriculum (Fuchs, Fuchs, Hamlet, & Stecker, 1991). Multiplication CBMs are most often one- to three-minute timed tests that are scored by either correct answers or digits correct (CBMs are discussed in more detail in Chapter II). However, there has been limited data to suggest that increasing the frequency of this type of assessment has improved student achievement (Stecker, Fuchs, & Fuchs, 2005). At issue is whether the frequent collection of this type of data supports instructional decisions that are likely to support student construction of important mathematics (in this case a shift to multiplicative thinking). Teachers who are not clear on children's current conceptions, the intended mathematics for the instructional episode, and tasks that support the children's construction of the intended mathematics may actually be designing instruction that contributes to the lack of student responsiveness to the intervention (Tzur, 2013).

In the next section, the meaning of multiplicative reasoning is clarified in a way not articulated in the CCSS or in RtI frameworks. Clarity on what multiplicative reasoning *is* should be foundational knowledge for teachers who are charged with designing both initial and intervention instruction for all students and especially for SLDs in mathematics.

Distinguishing Multiplicative Reasoning

Researchers (Clark & Kamii, 1996; Greer, 1992; Mulligan & Mitchelmore, 1997; Park & Nunes, 2001; Schwartz, 1991; Steffe, 1992) have defined multiplicative reasoning as a complex process that differs from additive reasoning. Yet, conflicting understandings and definitions of multiplicative reasoning are prevalent and contribute to the problem of understanding how SLDs construct schemes for multiplicative reasoning. A particularly confusing depiction found in most textbooks (Duncan et al., 1991; Manfre, Moser, Lobato, & Morrow, 1992) is of multiplication as repeated addition.

A possible root of confusing multiplicative reasoning with repeated addition may be that calculating answers to multiplication problems can be done using the operation of repeated addition. Although addition can, indeed, be used to this end, the reasoning that underlies multiplicative operations on/with units differs markedly from additive operations (Blote, Lieffering, & Ouwehand, 2006; Clark & Kamii, 1996; Schwartz, 1991; Steffe, 1991a; Steffe & Cobb, 1994; Zhang et al., 2009). The challenge is to distinguish between those two ways of reasoning while acknowledging that repeated addition yields the same answer as multiplication.

Distinguishing between additive and multiplicative reasoning. To distinguish between additive and multiplicative reasoning, it seems useful to first identify the source concept required for both, namely, number as an abstract composite unit (Steffe, 1992). This core construct in children's mathematical development refers to one's recognition of a collection of items as a single quantity, an object in and of itself that is composed of smaller items (Steffe, 1992). For example, the child would be able to conceive of 6 as a unit of its own right and operate on/with it while not losing sight and not being distracted by the units of 1 (or 2, or 3), of which it has been composed.

An observable indicator of a child's understanding of number as a composite unit is her or his use of a counting-on strategy for adding two numbers (Steffe, 1991a; Steffe & von Glasersfeld, 1985; Tzur et al., 2010). For example, if a child produces a collection of 7 items and is then given a collection of 4 more items, the child who understands numbers as composite units would find the total not by counting from 1 but by counting on from the first known collection: “7; 8, 9, 10, 11.” When a child uses such a strategy spontaneously and independently, it indicates that, for the child, at least the first addend (here, 7) symbolizes an abstract unit composed of seven 1s that she does not have to recount in order to find its sum when four more are added. In contrast, a child who does not yet have number as a composite unit will recount each and every item in both addends to find the total in spite of having just established the numerosity of the first (e.g., “1, 2, 3, 4, 5, 6, 7; 8, 9, 10, 11”). The difference between these two strategies a child uses for adding two numbers is important to this research study because, to reason multiplicatively, number must be established as a composite unit (Steffe, 1992; Tzur et al., 2010).

Although additive and multiplicative reasoning both require number as a composite unit, multiplicative reasoning requires children to operate on those units differently than in additive reasoning. Once children have constructed the understanding of number as a composite unit, they can begin to think multiplicatively in the sense of coordinating those units in a many-to-one correspondence. Researchers (Steffe & Cobb, 1994; Steffe, von Glasersfeld, Richards, & Cobb, 1983) have suggested that the child who is beginning to think multiplicatively can begin to “unite” the results of counting and meaningfully quantify a compilation[2] of several composite units. For example, the child may now be able to begin thinking about five cubes put together in a tower as one unit (of single cubes); the student then shows that she or he can conceive of a compilation of 3

[2] An aggregate of composite units will be referred to as a compilation (Tzur et al., 2013). An aggregate of individual units will be referred to as a collection.

such towers as another composite unit (of towers). Most importantly, the child is capable of coordinating the count of 1s in each tower with the accrual of 1s distributed across each tower, as in "1-tower-is-5-cubes, 2-are-10, and 3-are-15." The ability to conduct this double count, which yields a third type of unit (total of single cubes) different than the two units coordinated during counting (cubes per tower, towers), marks the commencement of multiplicative thinking.

The latter way of thinking is also the hallmark difference between additive and multiplicative reasoning. In reasoning additively, a child puts together, or joins, like units (e.g., 4 apples + 5 apples = 9 *apples*; 2 cubes + 2 cubes + 2 cubes = 6 *cubes*). That is, additive reasoning preserve the units on which one is operating (Schwartz, 1991): the units in each part added and the units in the sum (composed whole) are identical. In contrast, multiplicative reasoning entails change in the units, or as Schwartz (1991) suggested, in multiplicative situations units are transformed. When distributing 5 *apples-per-case* (apples per case is the unit) over the composite unit of 4 *cases* (case is the unit), the result is 20 *apples* (a different unit). To be engaged in multiplicative thinking, children need to consider the transformation of, operate on (distribute), and coordinate those three different types of units: a unit rate (e.g., 5 apples per case), number as a unit (compilation) of composite units (e.g., 4 cases), and a total number of single units (e.g., 20 apples in all 4 cases). The child needs to consider the latter number (20) as both a composite unit in its own right and as a unit that contains four units of five 1s each.

Figure 1 is a graphic representation of what the simultaneous coordination of units in this example might look like:

Number of Cases (Composite Unit)	1	2	3	4
		Contains the first case and the second case	Contains the first, second, and third cases	Contains the first, second, third, and fourth cases
Apples per case (Unit Rate)	5	5	5	5
Total apples	5	10	15	20

Figure 1. An illustration of multiplicative reasoning with a compilation of composite units.

The previous example illustrates that to reason multiplicatively children must be engaged in thinking about and coordinating three different types of units. In order to understand the total number of apples in 4 cases, a child must understand 4 as a composite unit, that is, that the 4 (cases) contain the first, second, and third cases. The child must also understand that the composite unit to be distributed, 5, represents the number of apples in *each* of the 4 cases. This is a unit rate, which remains constant throughout the distribution operation. Then, the child must coordinate the composite units and the unit rate (by the operation of distributing composite units given by the rate over each of the "1s" given by the number of composite units, such as 4 cases) to arrive at a total number of apples. When applied to pure numerical operations, the same argument holds (in place of "repeated addition"): 4 units of 5-units-of-1 each equals twenty 1s, which can be found by, say, the first unit is 5, the second unit is 10, etc. In all, multiplicative thinking requires a coordinated distribution that transforms units rather than a joining operation that preserves units. The next section discusses how prevalent resource materials may have impeded the construction of multiplicative reasoning by treating multiplication as repeated addition.

Hindering aspects of current teaching of multiplication. Typically, teaching of multiplication and division in schools has been hindered by a lack of clarity about how multiplicative and additive reasoning differ. Teacher support materials typically cast multiplication as repeated addition and provide tasks or experiences that do not support the construction of the coordination of units explained previously. For example, one teacher resource suggested that, "We want to teach students that they can *figure out the answer* to multiplication examples in two ways: Count to get the answer, or add to get the answer. If the students have already developed a level of proficiency with simple addition, it is *easier and quicker* to get multiplication answers by adding" (Tucker, Singleton, & Weaver, 2013, p. 46). Another teacher's manual suggested, "One of the first concepts to develop with manipulatives and questions is repeated addition. Use models to help the students *interchange repeated addition and multiplication sentences*. Then, have the students explore the order property to see that the order of the factors does not change *the answer*" (Manfre et al., 1992, p. 106A). In these examples, the manuals seem to suggest teaching tasks that do not necessarily support children's construction of the intended mathematics of multiplicative thinking.

When teachers are advised to treat multiplication as repeated addition (unit preserving operation), learning experiences in which teachers and students are engaged do not seem to support the construction of the coordinated distribution of units. This approach may limit children's ability to not only perform the operation of multiplication (Mulligan, 2011) but also to solve problems involving multiplication, division, rational numbers, and algebra (Hackenberg, 2010; Hackenberg & Tillema, 2009; McClintock, Tzur, Xin, & Si, 2012; Tzur & Hunt, 2015).

Although teaching multiplication as repeated addition may hinder mathematical progress for all students, it may have an even greater impact on SLDs. Groebecker (1997) found that teaching multiplication as repeated addition was particularly disruptive to SLDs, as they continued to use lower level strategies for longer periods than their normally achieving peers (NAPs). The next section discusses the importance of instructional tasks, and the numbers involved in the task, for constructing multiplicative reasoning.

Models for the development of multiplicative reasoning. Researchers have developed models for promoting children's learning to reason multiplicatively (Sophian & Madrid, 2003; Steffe, 1992, 1994; Steffe & Cobb, 1994). Building on these early works, Tzur et al. (2013) proposed a developmental framework of schemes used by children to reason multiplicatively. These models will be discussed in detail in Chapter II; in summary, they postulate *a developmental pathway* consisting of at least six schemes of multiplicative reasoning distinguishable based on the units on which a child operates. This dissertation study will focus on children's construction of the first scheme in that pathway, termed Multiplicative Double Counting (mDC). A child who is able to operate using mDC is able to recognize and anticipate that a composite unit, for example 12, can be made up of another composite unit (4 groups), each of which is made up of another composite unit (3 units of 1 per group) (Tzur et al., 2013).

Whereas those models provide a basis for making sense of children's construction of particular multiplicative schemes, less is understood about how SLDs construct understandings of multiplicative reasoning (Evans, 2007). Research has suggested that SLDs rely on counting based strategies longer than NAPs (Geary & Hoard, 2003). In this regard, Tzur, Xin, Si, Kenny and Guebert (2010) explained that SLDs may rely on counting strategies because they lack the requisite development of composite unit. SLDs also tend to demonstrate fewer strategies than NAPs when solving problems that involve multiplicative situations (Zhang et al., 2009).

The Impact of Numbers Used in Tasks on Multiplicative Reasoning

Research on number in multiplication tends to be focused on the relative difficulty of the recall of certain multiplication facts (Domahs, Delazer, & Nuerk, 2006; Ruch, 1932; Verguts & Fias, 2005). Researchers have suggested that the size of the numbers used in multiplication problems increases the difficulty of multiplication problems (problem-size effect) (J. I. D. Campbell & Graham, 1985; Groen & Parkman, 1972; Polich & Schwartz, 1974). According to problem-size effect, problems like 2x3 are easier to recall than problems like 6x8. These studies have not focused on how numbers impact the construction of multiplicative thinking and how schemes for such reasoning develop gradually when it comes to the numbers that constitute each composite unit and/or the compilation of those units. This lacuna was addressed by this dissertation study.

Problem Statement

Little research has been conducted and published about how a teacher considers the use of numbers in tasks to help SLDs overcome the conceptual leaps involved in solving mathematical tasks in general and in progressing from additive reasoning to mDC in particular. This dissertation study was designed to address this lacuna: how to improve SLDs' multiplicative reasoning by paying close attention to ways in which different-size numbers used for the unit-rate and/or number of composite units in compilations promote, or hinder, the acquisition and/or generalization of multiplicative schemes. In particular, the following research questions were explored:

1. In what ways do SLDs' conception of number as a composite unit afford, or constrain, transition to multiplicative reasoning?

2. Which specific numbers, used in instructional and/or assessment tasks, may support or interfere with SLDs' progression from additive reasoning to multiplicative Double Counting (mDC)?

CHAPTER II

CONCEPTUAL FRAMEWORK

In order to be successful and to become proficient in mathematics, children must construct multiplicative reasoning (Mulligan, 2011; Norton et al., 2015; Schwartz, 1991; Simon, 2006; Steffe, 1992; Tzur, 2007). The failure of current pedagogical methods to promote learning for students with learning difficulties or disabilities (SLDs) in general (McDermott, 1993), learning for SLDs in mathematics (Tzur, 2013), and specifically SLDs learning how to reason multiplicatively (Grobecker, 1997; Tzur et al., 2010) is an issue of concern.

To address and diminish the aforementioned failure to promote mathematics teaching that supports SLDs (and NAPs) in performing in mathematics, it seems important to better understand key ideas in the construction of multiplicative reasoning, how SLDs develop those ideas, what makes those ideas difficult to learn, and hence better understand how instruction might be tailored to the SLDs' needs and promote their learning of those key ideas.

In order to explore the problem of how to increase the learning of multiplicative reasoning in SLDs, a conceptual framework is provided to explicate the theoretical perspectives in the literature on both learning in general, and multiplicative reasoning in particular. The conceptual framework for this dissertation will be built on two core ideas, (a) a constructivist perspective on learning as a cognitive change in anticipation, and (b) an understanding of multiplicative reasoning: how it develops, how it is different from fact recall, and how number may influence multiplicative reasoning. Each of these components of the dissertation study framework is elaborated on below.

A Constructivist Perspective on Learning

Constructivism is a theory of learning that ascribes learning to a cognitive change that occurs when a problem presents a challenge to a currently held scheme (Piaget, 1986b, 1986c; von Glasersfeld, 1989, 1995). In order to explain this theory of learning, the perspectives of constructivist researchers will be discussed and the following central concepts in constructivism will be defined: schemes, accommodation, assimilation, perturbation and equilibration, reflection and abstraction, reflection on activity effect relationships (Ref*AER), and participatory and anticipatory stages in learning a new scheme.

Schemes. Piaget (1986b) described schemes both as currently held knowledge structures and as the organizing activity of those structures that exist in the mind of the learner. He suggested that schemes arise from physical and mental activities of the learner and that the goals of the learner (whether conscious or unconscious) help the learner organize that activity (Piaget, 1986c).

Von Glasersfeld (1995) built on Piaget's work and theorized that schemes were conceptual structures consisting of goal-directed operations used by people to reason in, say, mathematical situations. He defined a scheme as a single conceptual structure comprised of three elements, (a) a recognition of a certain situation that triggers a goal, (b) a specific activity associated with and carried out to accomplish that goal, and (c) the expectation that an activity produces a certain result. According to von Glasersfeld (1995) and Piaget (2002), schemes are actively constructed by the learner and are not a "picture" of reality. Rather, they arise from and organize the active subject's physical or mental activity (von Glasersfeld, 1995).

Dubinsky (1991) drew on Piaget's definition of scheme and described the construction of schemes. He described the process beginning with actions, which can then turn into interiorized processes. The interiorized processes may then become encapsulated into objects, which become schema (i.e., networks of interconnected schemes).

An example of a scheme that is important for this dissertation is that of additive reasoning. As mentioned in chapter one, schemes for additive reasoning require that a child understand number as a composite unit. An observable indicator of a child's understanding of number as a composite unit is her or his use of a counting-on strategy for adding two quantities (Tzur et al., 2010). For example, if a child produces a collection of seven items, and is then given a collection of four more items, the child would find the total not counting from one, but by counting on from the first known group: "Seven; eight, nine, ten, eleven." Such a strategy indicates that, for the child, at least the first addend (here, 7) symbolizes an abstract unit composed of seven 1s that she does not have to recount when acting toward the goal of figuring out the sum when four more items are added. In contrast, a child who does not yet have number as a composite unit will recount each and every item in both addends to find the total in spite of having just established the numerosity of the first (e.g., "One, two, three, four, five, six, seven; eight, nine, ten, eleven"). The difference is important to this research study because, to reason multiplicatively that is, to operate on/with numbers, number as the material for such operations must be established as a composite unit (Steffe, 1992; Tzur et al., 2010).

Assimilation. Assimilation refers to the recognition and interpretation of a new experience by already existing sensorimotor or conceptual structures (von Glasersfeld, 1995). Both Piaget (1976) and Von Glasersfeld (1995) hypothesized that no behavior constitutes an absolute beginning. All behaviors are grafted on previous schemes (Piaget, 1986c; von Glasersfeld, 1995). Understanding that knowledge is constructed from prior schemes is significant because it counters behavior-focused theories that might suggest that ideas can be given to children. In the example of counting-all and counting-on strategies above, researchers (Tzur & Lambert, 2011) found that methods for directly teaching the latter by instructing the child the behaviors that mark counting-on most often do not promote the intended change. Instead, teaching methods would have to bring forth and orient a child's reflection on goal-directed activities of counting-all as a means to transform it into the counting-on strategy (while constructing the first addend as a composite unit).

Piaget (1970) theorized that knowledge is constructed by the learner. He said, "Knowledge results from continuous construction, since in each act of understanding some degree of invention is involved; in development, the passage from one stage to the next is always characterized by the formation of new structures which did not exist before, either in the external world or in the subject's mind" (p. 77). Piaget described assimilation as the mental act of interpreting new information by bringing forth and adjusting to currently existing cognitive structures.

Von Glasersfeld (1995) suggested that an understanding of the three elements of scheme is essential to understanding the constructive processes of assimilation and accommodation. In order for children to recognize a situation, he said, they must have assimilated that situation into some prior experience and have grouped the current situation with a situation previously experienced. He defined assimilation as treating new material as *an instance of something known.*

In the previous example of additive reasoning, an example of assimilation is that the child might not have previously combined a group of 7 cubes and 4 cubes, but they may have combined other groups (e.g., 5+3) and have had experience with both the number 7 and the number 4. A child who has the scheme of 7 as a composite unit can assimilate combining 7 and 4 into a previously known situation of combination and count on when combining the two groups of cubes. Similarly, the child may then generalize this assimilation process by applying counting-on to, say, 8+5, 17+6, 109+3, etc. However, often as children learn they encounter situations where their activity does not produce results that they can assimilate into previously known situations. In these cases, the learner experiences a perturbation, which may lead to accommodation (transformation) of existing schemes.

Perturbations and accommodations. As previously stated, the third element of von Glasersfeld's definition of schemes was that the learner has the expectation that an activity will produce a certain result (von Glasersfeld, 1995).When a situation does not yield the expected result , the learner may experience a perturbation. To resolve that perturbation, the learner may make an accommodation in her/his previous schemes that were involved in assimilating the situation in the first place. For example, the child who has an additive reasoning scheme as described above, might not yet have a multiplicative scheme. When asked to get 6 towers with 4 cubes in each tower, and then asked how many cubes in all, this child might expect that, to figure out how many he or she should count, "6: 7, 8, 9 10." A perturbation occurs for the child when he or she notices that he or she is holding more than ten cubes (e.g., by counting all of them one-by-one). This perturbation might prompt the child to go back and recount the cubes and notice there are 24 cubes. In this instance, the child has resolved this specific perturbation, but is yet to construct the operation on a compilation of equal-size composite units (i.e., into a multiplicative scheme).

Equilibration. Equilibration refers to the dynamic mental process of elimination of perturbations (Piaget, 1985, 1986a; von Glasersfeld, 1995). Von Glasersfeld (1995) characterized cognitive development as expanding equilibration and an increase in the range of perturbations the organism is able to eliminate. Piaget (1970) referred to knowledge as a system of transformations that become progressively adequate. Equilibration allows the learner to eliminate perturbations and transform current schemes to more adequate schemes.

In describing equilibration, Piaget (1986c) stressed that assimilation and accommodation are regulated from within the child's mental system. The mental "regulator" is the child's goal, which provides direction and motivation of equilibrating to eliminate perturbations, organizing schemes and subsystems, and constructing totalities. The role of regulation was initiated by encountering a perturbation and by positive and negative feedback experienced by the child. As noted in the previous example the child who recognized that 6 groups of 4 cubes did not yield ten cubes but rather 24 had a single experience. If the child were to continue having experiences with finding the total number of cubes based on towers of a certain number, the child's goal might shift to find the total of cubes in a compilation of composite units by operating on those units, and the 1s that constitute each, simultaneously.

Reflection: A Mechanism for Cognitive Change. Researchers (Dubinsky, 1991; Piaget, 1985; Simon, Tzur, Heinz, & Kinzel, 2004; Tall, McGowen, & DeMarois, 2000; Tzur & Simon, 2004; von Glasersfeld, 1995) have further discussed how a learner reaches a point of re-equilibration via the mental process of reflection and reflective abstraction. Piaget suggested that the mental action associated with reciprocal assimilation is reflective abstraction. According to Piaget (1985), reflective abstraction consists of two components. The first is the act of projecting something from a lower level of thinking onto a higher level of thinking. The second is the cognitive reorganization of what is being projected. The cognitive reconstruction and reorganization addresses how new knowledge is constructed. Piaget (1985) suggested that equilibration requires construction and that is what moves the child beyond his or her initial equilibrium.

Von Glasersfeld built on Piaget's notions of reflection and suggested his own definition of reflection as the notice the mind took of its operations, reasoning, and manner (von Glasersfeld, 1995). He thus suggested that abstraction was the instance when ideas from particular beings became general representations and applicable across contexts. His example of abstraction was a child who only knew apples as red fruits. As the child became introduced to apples of other colors, she was able to use her experiences to abstract that apples could be of many colors. This shift, from only knowing an object as a particular, to being able to abstract their nature to a larger class of objects, suggests a shift into more abstract thinking.

Reflection on activity-effect relationships. Drawing on Piaget and von Glasersfeld's constructivist work, Simon, Tzur, Heinz, & Kinzel (2004) introduced a further elaboration of reflective abstraction, which they termed Reflection on Activity-Effect Relationships (Ref*AER). This reflection on activity-effect relationships model was intended to explain how learners construct conceptions beyond those they currently know in a way that can help theorize effective teaching for conceptual understanding. Simon, Tzur, Heinz and Kinzel (2004) suggested that the concept development process of Ref*AER includes three phases. First, the learner sets a goal. This goal might or might not be conscious to the learner and it might not be the goal set by a teacher for the learner's learning. For example, when a child is given a task to bring the teacher a tower of 5 cubes, the child's goal might be to follow the directions and count the number of cubes in the tower. The teacher's goal might be to orient the child's attention on the different units in a multiplication situation.

The second phase of the development of a concept is that the learner uses a current conception or mental activity to try and meet the goal. In the cubes and towers situation, the student might use his or her conception of counting to count out 5 cubes and attach them to one another to create a tower. This activity is available to the child using his or her current schemes of action as well as of mental operations on number as a composite unit.

In the third phase, the learner attends to the results of that activity and then makes note of whether or not that activity reached the desired state. In the towers example, the student is able to decide whether the results of her activity fit the sought after, correct size tower. Engaging in these three phases provides an explanation of how children assimilate new relationships or understandings into their current schemes.

As children participate in mental activity, they may reflect on and link anew their activity and the results (effects) of that activity. Tzur (2011) categorized the reflection as either Type-I or Type-II. A Type-I reflection refers to the conscious or unconscious comparison between the child's internal goal and the effects of the child's mental actions. For example, a child is engaged in a Type-I reflection when he or she tries to determine the total number of cubes in 6 towers of 4. If, as in the example above, she anticipated the outcome to be 10 cubes and the effect of her counting one-by-one was 24, this comparison between anticipated and actual effect underlies the perturbation she may experience as well as her noticing of that gap and thus attempt to resolve it.

Type-II reflection refers to reflection on the effects of activity across experiences. For example, Type-II reflection happens when the same child engages in multiple towers problems. For the first problems the child kept track of the 6 towers of 4 by counting all of the models (1,2,3,4…22, 23, 24). For some other problems, the child determined the total number of cubes in 3 towers of 7 by representing one group of 7 with one finger, a second group of 7 with another finger (while keeping track that the two groups of 7 made a total of 14), and then counted 15, 16, 17, 18, 19, 20, 21. This child might engage in Type-II reflection if they are able to generalize some comparisons across situations. Unlike Type-I reflection, which Tzur (2011) suggested is constantly and automatically carried out by the brain, Type-II reflections can occur for some students independently, but more often happen when supported by the teacher, tasks, or interactions with peers.

The Ref*AER theory offered an explanation of learning because it explicated what changes when children learn (current schemes), what mental processes may bring about such changes (Ref*AER), and how we know it has changed (children are able to operate mathematically in ways that were previously inaccessible). While the Ref*AER provided a strong theoretical framework for understanding learning, there are times when children appear to have reflected on activity and assimilated that activity into a new scheme or concept but then “lost” that concept or idea. For example, a child who counts on from 8 to add 5 more may count-all to add 9 and 4. Furthermore, the child may add 8+5 via counting-on and sometime later revert back to adding these very same numbers via counting-all (Tzur & Lambert, 2011). The next section will introduce the concepts of Participatory and Anticipatory stages in learning (Tzur & Simon, 2004) to provide an explanatory framework for the apparent “learning loss” that seems to occur when children are developing new mathematical concepts.

Participatory and anticipatory learning. Tzur and Simon (2004) further elaborated on ref*AER by distinguishing two stages in the construction of a new mathematical scheme, called participatory and anticipatory. Tzur (2007) provided the following definition for the participatory stage of learning:

> The participatory (first) stage in forming a new mathematical conception characterizes a mathematical understanding that depends on being prompted for the activity at issue. At this stage the learner forms a provisional anticipation of activity-effect relationship. Such anticipation includes the capability to reason why the effects follow the activity. Yet, it is available to the learner only when prompted for the activity (p. 277).

The participatory stage suggests that there is a difference in what the child can do with the support of an adult (or another knowledgeable learner) and without such support. The child is able to anticipate results of his or her activity when prompted for the activity that enables doing so, but does not spontaneously anticipate the effect. A child in the participatory stage of multiplicative thinking requires prompts and questions to correctly determine the total number of cubes. One example of a teacher prompt is, "Can you use my hands to help you keep track of the number of towers as you count the total number of cubes in each tower?" In this case, the child can think about the different units required in the task (number of towers, number of cubes per tower, and total number of cubes), but he or she requires support from the teacher via prompting to keep track of the added items.

Tzur (2007) provided the following definition for the more conceptually advanced, anticipatory stage. This definition suggested a stage whereby the learner can independently anticipate the effects of his or her learning.

> The anticipatory (second) stage in forming a new conception characterizes a mathematical understanding where the learner can independently call up and utilize an anticipated activity-effect relationship proper for solving a given problem situation. At this stage the learner has added an explicitly reasoned relationship – between the newly formed activity-effect relationship and an array of situations – that was not abstracted in the participatory stage (p. 278).

A child in the anticipatory stage would not require a prompt or support from the teacher to determine the total number of cubes in 3 towers of 7. The child would be able to independently figure out a way to determine the total number of cubes.

Tzur and Simon's (2004) two-stage hypothetical process for constructing a new scheme, participatory and anticipatory, is helpful in understanding situations in which children are able to use a strategy or seem to have an understanding one day or in one context but are unable to replicate that understanding in a different time/context. They referred to this as the *Next Day Phenomenon*. Specifically, participatory and anticipatory stages of learning are helpful in understanding SLDs ability to reason multiplicatively because inconsistent use of strategies and reliance on immature strategies have been noted as common for SLDs (Geary & Hoard, 2003; Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Zhang, Xin, & Si, 2011). It is also important for this study because the number that the child is asked to operate on may impact his or her ability to anticipate the activity-effect relationship. The examples provided above (e.g., using counting-on for 8+5 while reverting to counting-all in adding 9+5) illustrate the potential contribution of the stage distinction to the study of children's construction of multiplicative reasoning. For example, a child may use a coordinated count when the unit-rate is 2 or 5 while reverting to counting-by-1s when the "same task" asks for another unit-rate, such as 3 (or 7, or 13).

Folding back. Pirie and Kieren's (1994) description of "folding back" provided further insight into children's construction of mathematical understanding. Folding back referred to learning as a non-linear process whereby a learner who is faced with a problem that is not immediately known may revert to a previous level of understanding in order to access the new problem. For example, students who were able to solve problems with towers of 2 or 5 using a multiplicative Double Count were observed "folding back" to counting all or counting on when presented with towers of 3 or 4 cubes.

In summary, a constructivist learning theory provides answers to three important questions about learning, (a) What changes when learning occurs (schemes, and specifically the child's anticipation of the activity-effect relationship); (b) How does it change? (Schemes change as the child experiences perturbations via processes of assimilation and accommodation, and then equilibration via Ref*AER); and (c) What is evidence of this change? (The child's explicit anticipation of AER when the numbers in a given task have changed). In this dissertation study, the ways in which numbers used in tasks impact how children construct multiplicative reasoning were examined. In this constructivist framework for learning, it was hypothesized that children would construct and develop multiplicative schemes through a process of reflective abstraction (specifically, Ref*AER as explained above) and that changes in number may influence how children progress. Because this research was focused on multiplication, it is essential to review the literature on the development of multiplicative reasoning, including: a definition of multiplicative reasoning, defining what makes multiplication difficult, defining how the learning of multiplication will be framed, and examining the impact of number on the construction of multiplication.

Multiplicative Reasoning

As described in chapter 1, researchers have distinguished between multiplicative and additive reasoning (Greer, 1992; Hackenberg, 2007; Norton et al., 2015; Piaget, Grize, Szeminska, & Bang, 1977; Schwartz, 1991; Steffe, 1992; Steffe & Cobb, 1994; Vergnaud, 1983, 1988). Piaget, Grize and Szeminska (1977) suggested that multiplication requires a different type of thinking and understanding than additive reasoning. They particularly cautioned against reducing the complexity of multiplicative operations to that of addition (e.g., 'repeated' addition) because the operations are different.

Building on previous work, Steffe (1992) suggested that multiplicative situations require operations on/with composite units, and that for a situation to be multiplicative, it is necessary at least to coordinate two composite units in such a way that items of one of the composite units are distributed over the elements of the other composite unit. Steffe and Cobb (1994) further suggested that a child must be able to anticipate the results of a simultaneous coordination of two counting schemes, counting by ones and counting by composite units. They referred to such coordination as two-for-one or many-for-one operations. For example, a child must be able to keep track of one unit of five 1s (for a total of 5), and another unit of five 1s (now a compilation of two units of five) is ten, and then a third unit of five 1s (now a compilation of 3 units of five) is fifteen, in order to be thinking multiplicatively—a five-for-one scheme coordinated with a composite unit of three applied to the compilation of composite units. Such coordination underlies the child's simultaneous count of how many units of 1 and how many units of 5 are in the set to count all singletons (15) in the entire compilation.

Hackenberg (2007) suggested that the simplest form of multiplicative reasoning involves the coordination of two levels of units. The example above, of 5-for-1 coordinated with 3 composite units, illustrates such a level of unit coordination. She also asserted that students who have multiplicative understandings know ahead of time, or can anticipate, that there are units contained within units. For example, they can anticipate that two "5s" will be ten and that another five will be 15 (i.e., for the child, the ten 1s are contained within the fifteen 1s). This is significant because it shows that the children are able to operate on composite units in an anticipatory way.

The operations of addition and multiplication have a different impact on the units used in the problems. Additive situations require putting together or joining like units. When the like units are put together the units are preserved (Schwartz, 1996), that is to say there is no change in the nature of the units. For example, we can add 4 apples to 5 apples to result in 9 apples; or repeatedly add 5 apples and 5 apples and 5 apples to result in 15 apples. In this joining situation, the units in each part and the units in the whole do not change (which explains why repeated addition belong in the category of additive reasoning). Joining situations like this do not require that children operate on different types of composite units, as do multiplicative situations.

In contrast when operating multiplicatively units are transformed not preserved (Schwartz, 1991). For example, when multiplying 4 cases (case is the unit) of 5 apples per case (apples-per-case is the unit), the result is apples (a unit that differs from the other two that generated it). To be engaged in multiplicative thinking, children need to consider the transformation of units. As they are considering the transformation of units they must, at the same time, operate on and coordinate those three different types of unit: number as a composite unit (e.g., 4 cases), a unit rate (e.g., 5 apples per case), and a total number of single units (e.g., figure out how many apples are there in all 4 cases).

In summary, the operation of multiplication transforms the composite units used in the problem. Beyond a transformation of composite units, researchers have further categorized multiplicative structures. The next section describes these structures and specifies which are most relevant to this research.

Multiplicative structures. Vergnaud (1983, 1988) identified three subtypes of multiplicative structures. He suggested that these structures describe three sets of problem types that require multiplicative thinking, (a) isomorphism of measures, (b) product of measures, and (c) multiple proportion other than product. These subtypes allow an analysis of the elements of multiplicative thinking problem types.

Isomorphism of measures, or simple proportion, refers to multiplication and division problems that have a direct proportion structure (Vergnaud, 1983). Greer (1992) provided examples of multiplicative situations that fall within the category of isomorphism of measures. His examples included situations of: (a) equal groups (e.g., 3 children have 4 oranges each); (b) equal measures (e.g., 3 children each have 4.2 liters of juice); (c) rate (e.g., a boat moves at 4.2 meters-per-second, how far does it move in 3.3 seconds); (d) measure conversion (e.g., an inch is about 2.54 cm., so about how long is 3.1 inches in cm?); (e) multiplicative comparison (e.g., iron is .88 times as heavy as copper, if a piece of copper weighs 4 kg, how much does a piece of iron the same size weigh), and (f) part/whole (e.g., 3/5 of the students passed the test. If there are 80 students, how many passed the test). Each of Greer's examples represents situations that are directly proportional and therefore require a consistent unit rate—the type of multiplicative structure that will be presented to the students in this study. This subtype of multiplication problems is most relevant to the research questions posed for this study. The students who will be participating in this project will be thinking about situations that require equal groups.

Product of measures, the second of Vergnaud's categories, refers to situations where "two measure spaces, M1 and M2 are mapped onto a third, M3" (Vergnaud, 1983, p. 134). In product of measures situations, there are at least three variables. An example of the situations that have three variables are areas of rectangular rooms, where one variable is the length, one variable is the width, and the product of those variables creates a third variable, area. These problems differ from the isomorphism of measures problems where there were composite units, a unit rate, and a total number of single units; however, they still represent a situation where units are transformed. Students in this study will not be solving product of measures problems. Yet, it is important to understand how this type of problem is similar to and different from isomorphism of measures problems so that one can be clear about the types of multiplication problems they are presenting to students.

Vergnaud's (1983) third category, multiple proportion, refers to a problem structure that has variables with meaning that cannot be reduced to the product of others. An example of a multiple proportion problem is a Cartesian product problem. A simple example is if a child has 3 shirts and 4 pairs of pants how many different combinations of shirts and pants (outfits) can she make? Again, this type of problem involves the transformation of units via distribution of items of one composite unit over items of another composite unit, but it does not necessarily require the understanding of units, units per group, and total units.

As suggested in the initial definition of multiplicative reasoning as being different from additive reasoning, all of the models included in Vergnaud's (1983, 1988) conception illustrate that multiplication structures are different from additive structures because each of the three problem types are grounded in the notion of ratio rather than repeated addition. Vergnaud stressed the importance of understanding and categorizing the field of multiplication in order to understand how and which parts of that field children understand. In the next section, the focus shifts to a review of the literature on how multiplicative reasoning (within Vergnaud's first category) may develop.

The Construction of Multiplicative Reasoning

Researchers have observed prerequisite understandings to the construction of multiplicative reasoning (Steffe, 1992; Steffe & Cobb, 1994). The ability to assess and understand these prerequisites is essential in designing instruction that builds on and fosters intended transformations in a child's current schemes.

Number sequences. Steffe (1992, 1994) and Steffe and Cobb (1994) have suggested that the development of an iterative composite unit is a prerequisite to a child's ability to understand multiplicative situations. In a multiplicative situation, the child must coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other composite unit, so in order to coordinate composite units, the child must have constructed those composite units.

Steffe (1988) developed three categories of number schemes that describe how a child comes to construct a composite unit. According to Steffe (1991a), children's understanding of number can be categorized into three number sequence schemes, (a) the initial number sequence (INS), (b) the tacitly nested number sequence (TNS), and (c) the explicitly nested number sequence (ENS). Each of which are described below.

A child who is operating within the INS scheme will not be able to operate multiplicatively because, while they do have number as a composite unit, they cannot yet operate multiplicatively on number as a composite unit. For example, if a child with the INS is asked to figure out how many cubes are in a covered compilation of 3 towers of 4, they will be unable to complete the task. Because a child with INS is not yet able to coordinate composite units, they will not be able to determine the total number of cubes in 3 towers of 4 without counting all[3] of the (seen) cubes by 1s.

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[3] The term counting all is used in this research to refer to the first strategy used by children to add two numbers. In this dissertation this definition is expanded to include a strategy used for adding three or more numbers, while the core ideas of (a) starting from one and counting by ones is maintained from the original meaning of counting all.

In contrast, a child who has constructed and is operating with the TNS can reason multiplicatively while physically *enacting,* (not yet mentally anticipating) the situation. The child with TNS would be able to determine how many cubes in a covered arrangement of 3 towers of 4 by re-presenting those towers; perhaps by using fingers to double count and say, “1 tower of 3 has 4, 2 towers of 3 have 5, 6, 7, 8, and 3 towers of 4 have 9, 10, 11, 12.

Finally, a child who has constructed and is operating with the ENS can anticipate (mentally- without physical representations) effects of goal-directed operations on composite units and is therefore able to reason multiplicatively. Anticipation here refers to the potential iteration of a composite unit a given number of times (i.e., the child’s ENS implies an iterable composite unit). Steffe (1994) used an example of a child with the ENS who is able to create four rows of three with 12 blocks. When asked how many more rows of three she could make if she had 27 blocks altogether, she was able to think about two unit types, the total number of blocks and the number of rows at the same time. This type of operation indicates three levels of unit-coordination (a compilation of rows with more rows, each row is a composite unit).

In this section, the importance of the development of an iterable composite unit to the construction of multiplicative reasoning was clarified. In sum, it is essential for children to have constructed composite units and the ability to operate on those composite units (anticipate their iteration) in order to reason multiplicatively. In the next section, many-to-one correspondence is introduced as a way that children may begin to use their composite units in the construction of multiplicative reasoning.

Many-to-one correspondence. Multiplicative reasoning requires the formation of a many-to-one relationship (Blote et al., 2006; Clark & Kamii, 1996; Nantais & Herscovics, 1990; Steffe, 1992). Clark and Kamii (1996) suggested two primary ways that multiplicative reasoning differs from additive reasoning. According to their research, multiplicative reasoning requires a child to understand operations not required in addition, namely, many-to-one correspondence.

Sophian and Madrid (2003) described the many-to-one correspondence as being a key for transition from additive to multiplicative relations. As an initial multiplicative scheme, they described a many-to-one correspondence as several objects being associated with a single object in a many-to-one grouping, where the objects of the first grouping are not the same as the objects of the second grouping (e.g., three flowers to one vase, and then a few vases are considered). Sophian and Madrid argued that children must re-conceptualize additive relations in order to achieve multiplicative reasoning. They suggested that the cognitive change in the move from additive to initial multiplicative situations is that of iteration and the relationships are constructed mentally. They also suggested that it is cognitively difficult for children to coordinate the two dimensions of many-to-one problems. Other studies (Blote et al., 2006), however, suggested that young children can develop many-to-one counting.

In a study of four year olds in the Netherlands, Blote, Lieffering, and Ouwehand (2006) determined that children as young as four years old can develop many-to-one counting. Using an experimental design, both the experimental and control groups were presented with many-to-one tasks, but only the experimental group was taught how to make many-to-one correspondences. Interestingly, both the control and the experimental groups showed increased insights into many-to-one correspondence. The authors suggested that experiences with many-to-one tasks might be key to the construction of multiplicative reasoning. The specific way in which a many-to-one coordination of units will be fostered and studied in this dissertation is described in the section about the Please Go and Bring for Me game.

While previous research focused on many-to-one counting, Steffe (1992) differentiated a two-for-one units coordination from a many-for-one units coordination. In his experiment, he had a piece of red paper onto which the child placed six blue rectangles that filled the red paper. He then showed the child two orange squares that fit exactly onto one of the six blue rectangles and asked the child how many of the orange squares it would take to fill the red paper. The child in this experiment was able to figure it out by using counting two taps on the table for six successive fingers. Steffe suggested that this action showed that the child was able to distribute a unit of two over six units of one and therefore had a two-for-one unit coordination scheme.

In another experiment, Steffe (1992) asked the same student to make four rows of three blocks each. After the child established that there were 12 blocks, Steffe covered the 12 blocks and asked her to create two more rows of three blocks and put them under the cover. Without counting, the child was able to determine that, with the additional rows, there were then 18 blocks. Steffe determined that this solution indicated that the child had a many-for-one correspondence and a three-level unit coordination scheme.

Steffe's distinction between two-for-one and many-for-one correspondence is of particular importance for this research because it refers to a difference in how number may influence children's abilities to reason in a multiplicative situation. Yet, Steffe's distinction and work did not explicitly examine how different numbers might change children's thinking in situations of many-to-one correspondence, be it 2-for-1, 3-for-1, 6-for 1, 10-for-1, etc. Specifically, this dissertation study will focus on possible differences in the ways in which a child who can reason about, say, two rows of three blocks without counting the blocks, will be able to reason about six more rows of three blocks. Many-to-one correspondence is one scheme that is viewed as key to the construction of multiplicative reasoning; in the next section, another important scheme, splitting, is defined and explained.

Splitting. Splitting is a term used to describe children's actions that suggest early multiplicative thinking (Confrey, 1994; Empson & Turner, 2006). Confrey (1994) suggested that splitting, or repeated-dividing, is key to children's development of many-to-one correspondence. The splitting operation was viewed as an initial step in multiplicative reasoning and suggested that the understanding of splitting may develop before formal introduction of multiplication.[4]

Kieren (1994) viewed splitting as the informal basis for multiplicative reasoning. He observed that some students who were engaged in paper folding activities (where paper is folded to create splits in the original paper) began to predict how many pieces would be created before they made the next folds. For example, students could say that the number of folds tells you how many times you have to multiply by two (e.g., they said that if a paper is folded in two three times, it will have 2x2x2=8 parts once opened).

[4] There is also another definition of splitting that is currently used in fractions research. Steffe (2003) and Norton (2008) use splitting to refer to a simultaneous use of partitioning and iterating. An example of a splitting question in this context is, "If this strip is my bar and it is 5 times as large as your bar, can you show me what your bar will look like? Students who can envision the larger bar cut into 5 parts and then translate that into the size of their bar are said to have the scheme of splitting.

Kieren suggested that this anticipation of the number of pieces created by successive folds is an informal understanding of multiplicative relationships.

Confrey (1994) defined splitting as creating multiple versions of an original. This operation is focused on an anticipation of one-to-many relationship. She differentiated this from repeated addition by clarifying that in additive situations, the learner identifies a unit and then counts consecutive instances of that unit. In splitting, the focus is on a one-to-many action. For example, when splitting one cake, the first split results in two cakes, a second split results in 4 pieces of cake. There are no additions to the whole; rather there are more pieces because the larger pieces have been subdivided.

Empson and Turner (2006) confirmed the importance of splitting to develop multiplicative reasoning in paper folding experiments with first, third and fifth graders. They noted a distinct indicator of the development of multiplicative thinking as children anticipated the number of pieces that would be created by folding paper. For example, in the initial interview, 21 out of the 30 children incorrectly predicted that folding the paper in half three times would create six pieces. By engaging in experiences of folding the paper and then reflecting on the number of parts upon opening the folds, many of the students began to anticipate the results of their folds before they made them. At the end of the experiment, 20 out of 22 of the third and fifth graders were able to anticipate the number of pieces created via a multiplicative operation that coordinated the number of 1s in each composite unit (2 parts for 1 fold) and the number of folds. None of the ten first-graders made that shift in thinking.

Both many-to-one correspondence and splitting are two ways by which children may develop multiplicative thinking. Researchers have also described other ways in which children may come to reason multiplicatively without formal instruction on multiplication. Research on informal development of multiplicative thinking is reviewed in the next section.

Informal multiplicative reasoning. Researchers have suggested that core ideas about multiplication might be present in children without formal instruction. McCrink and Spelke (2010) conducted an experiment with five- to seven-year old children using videos of groups of objects that were presented in scalar ratios (doubling, quadrupling and a scale factor of 2.5). Based on their results, they concluded that, prior to formal instruction in multiplication, children might possess a core multiplication idea that allows them to recognize a proportional relationship in one group of objects and then apply that to different groups of objects. This "core multiplication" is independent of reliance on repeated addition and of formal schooling in multiplication and division. The inability to recognize proportional relationships, or a constant unit rate, may be a factor in students' inability to progress in multiplicative thinking and in more formalized multiplication (Mulligan & Mitchelmore, 1997).

Mulligan and Mitchelmore (1997) classified ways in which children solved multiplicative problems into three categories: (a) direct counting, (b) repeated addition, and (c) multiplication methods. The authors suggested that third grade students who focused on an additive rather than a multiplicative structure in solution strategies had difficulty developing more advanced multiplicative strategies. Similarly, in a study of the origins of multiplicative reasoning (additive vs. correspondence), Park and Nunes (2001) found that students taught with notion of correspondence rather than repeated addition made more progress in multiplicative reasoning.

The studies summarized above suggested that children who constructed informal multiplicative understandings were more successful in formal multiplication instruction and they confirmed a qualitative difference between additive and multiplicative thinking. Building on these earlier studies, recent research has elaborated on how children progress from additive to multiplicative thinking. In the next section, a model that specifically defines six schemes in the development of multiplicative reasoning is described.

Multiplicative Reasoning Schemes at the Focus of this Study

Tzur et al. (2013) postulated *a developmental pathway* consisting of at least six schemes of multiplicative reasoning distinguishable based on the units on which a child operates: Multiplicative Double Counting (mDC), Same Unit Coordination (SUC), Unit Differentiation and Selection (UDS), Mixed Unit Coordination (MUC), Quotitive Division (QD), and Partitive Division (PD). This dissertation study focuses on children's construction of the first scheme in that pathway, which was termed Multiplicative Double Counting (mDC). This choice is based on the rationale that mDC is the first time a child can be considered to have advanced from additive to multiplicative reasoning.

A child who is able to operate using mDC is able to recognize and anticipate that a composite unit, for example 12, can be made up of another composite unit (4 groups) each of which is made up of another composite unit (3 units of 1 per group) (Tzur et al., 2013). This child is able to use fingers or another re-presentation to determine that an additional tower of 3 would result in a new total of 13, 14, 15 cubes. This scheme implies a child has constructed and is operating with the ENS.

Using the Please Go and Bring for Me Game to Promote mDC. Woodward et al. (2009) described a game called Please Go and Bring Me (PGBM) that was designed to draw on students' available conceptions and promote movement toward mDC. PGBM is described below and is important because it was used as the primary task for children participating in this study.

To play PGBM children are presented with the task of creating towers out of individual linking cubes. Certain rules govern the creation of the towers. The first rule is that, initially, the number of towers and the number of cubes per tower can't be the same (e.g., one cannot ask for 3 towers of 3). The second rule is that all of the towers must be the same size. The third rule for the initial iteration of the game is that towers must be composed of two, five or ten cubes and the total number of towers cannot exceed six. Limiting the total number of towers to 6 supports children's initial double counting (which is most often conducted with finger counting) because they can use one full hand and one additional finger to count composite units in the compilation.

The game involves a "Sender" who is charged with instructing a "Bringer" to bring back a tower of *m* cubes. Initially the teacher demonstrates the play of the game and begins by placing a container of linking cubes some distance away from the children. The teacher then asks the students to "Please go and bring me" *m* number of cubes. For example, the teacher says, "Please go and bring me one tower of 5." The child then moves to the container of cubes, builds a tower of 5 cubes, and brings the tower back to the teacher. As the teacher interacts with the children during the game he or she asks a consistent series of questions that prompt the child to focus and reflect on the effects of activities on composite units (Q-1), unit rate (Q-2), and 1s (Q-3 & Q-4). The questions are: (a) How many towers did you bring? (b) How many cubes are in each tower? (c) How many cubes are in all the towers? And (d) How did you figure out this [total]?

As the children play the PGBM game, they are encouraged to verbalize and/or show their thinking. For example, to solve a problem and/or to explain how she got a solution, a child might say, "One tower is 5 cubes, two towers is 6, 7, 8, 9, 10 cubes, three towers is, 11, 12, 13, 14, 15 cubes" as they put up fingers to keep track of the accrual of towers and/or cubes. As children participate in the game, verbalize their thinking, and represent the accumulation of towers and cubes with their fingers they are able to notice the results of their tower building and gathering.

As the PGBM game progresses, new situations are introduced. Once students have constructed and are using mDC at an anticipatory stage, the teacher may ask the child if they have 3 towers of 4 cubes and get two more towers of cubes, how many towers do they have. This question will help orient students to the coordination of composite units within different compilations.

In this section a specific multiplicative reasoning scheme, multiplicative Double Counting (mDC) was described and a task found conducive to the construction of those schemes was introduced. The next section elaborates on research on multiplicative reasoning and multiplication fluency with students with learning disabilities to provide a comparison of the nature of the research and the implications of those findings.

Multiplicative Reasoning and Students with Learning Disabilities (SLDs)

There is limited research on the development of multiplicative reasoning for students with learning difficulties disabilities (Grobecker, 1997; Tzur et al., 2010; Xin et al., 2009; Zhang et al., 2009). Tzur, Xin, Si, Kenney and Guebert (2010) examined SLDs who were not demonstrating multiplicative reasoning and confirmed that, just as in the general population, children who identified with learning disabilities are not able to progress in multiplicative reasoning if they have not constructed number as a composite unit. The researchers observed that students who had constructed number as a composite unit were able to engage in the multiplicative Double Counting activities and make progress. The students who had not yet constructed number as a composite unit were only able to participate in the multiplicative reasoning game by using strategies of counting all 1s. The researchers concluded that the child's reasoning lacked a construct of number as an abstract, symbolized composite unit, which is required in order to operate multiplicatively. Xin, Tzur, Si, Zhang, Hord, Luo et al. (2009) and Zhang, Xin and Si (2011) observed that SLDs tend to use counting strategies longer than NAPs and suggested that moving children to more advanced strategies that require number as a composite unit supported the construction of multiplicative reasoning in SLDs.

Groebecker (1997) suggested that teaching repeated addition procedures is particularly disruptive to SLDs construction of multiplicative reasoning. In a three-month study of 84 7-13 year old SLD and non-SLDs, Groebecker found that SLDs were more likely to stay with lower level additive strategies than their non-SLD counterparts. Importantly, Groebecker noted that many of these children were able to generate answers consistent with GL on standardized tests but reasoning was at a lower level.

Zhang, Xin, Tzur, Hord, Si and Cetintas (2009) suggested that counting was initially the most prevalent strategy employed by SLDs and that SLDs tended to follow a similar pattern of development as NAPs, moving from counting by ones to double counting, repeated addition and then fact recall. The authors suggested that instructional interventions supporting mDC are especially effective for SLDs.

There is relatively limited research on how students with learning disabilities begin to reason multiplicatively In contrast, there is a heavy focus in research on recall of multiplication facts and fluency for SLDs.

Multiplication Fact Recall and Fluency for SLDs

There are distinct differences between the research on multiplicative reasoning and the research on getting answers to multiplication problems in terms of multiplication fact recall and computation. As described in the first sections of this chapter, the research on multiplicative reasoning seems to be focused on how children develop multiplicative schemes for thinking about and coming to operate on composite units. The multiplication fact recall and fluency research is focused primarily on speed, correctness, fluency, and automatic recall of multiplication facts (Bliss et al., 2010; Burns, 2005; Greene, 1999).

While multiplication fact recall and fluency are not the focus of this study, it seems important to include this research as a way to contrast the notion of multiplicative reasoning from the arithmetic operation of multiplying. At issue is the prevalent use of and focus on such recall and fluency when teaching SLDs, as opposed to focusing on the reasoning that underlies and gives meaning to these procedural aspects of knowing.

Several recent studies have examined the impact of flashcards, mnemonic devices, and other behavioral interventions on children's fluency in multiplication (Bliss et al., 2010; Burns, 2005; Greene, 1999; Irish, 2002). These studies focused on children with learning disabilities or difficulties in mathematics, and considered an increase in fact fluency to be essential for success in higher mathematics. The researchers used pre and post-test measures of number of digits correct in multiplication facts to measure children's fluency with multiplication facts and in all of the studies the children made gains in fluency on single digit multiplication. What was not emphasized in those studies, and is essential for this dissertation, is how children operate on the numbers involved in a multiplication problem.

Woodward (2006) compared an integrated strategy approach to teaching multiplication facts to a traditional drill approach for students with learning disabilities. He concluded that drill and integrated strategy approaches were comparable in automaticity but suggested that the long-term goal of number sense is enhanced by an integrated strategy approach. This finding is significant to this dissertation study because it suggests the need for methods that investigate student strategy (i.e., goal-directed activity). The following study provides an example of one such study that attempts to connect fluency to multi-digit multiplication.

Some fact fluency research is focused on connections to higher-level mathematics. Lin and Kubina (2005) examined the relationship between multiplication fact fluency and the ability to solve multi-digit multiplication problems. As in the previous studies, the focus of this study was on fluency as measured by number of digits correct on multiplication problems, not on the development and/or use of thinking strategies. The researchers concluded that children were more accurate than fluent (fast) on single digit facts, and that this lack of fluency might contribute to the lack of fluency with multi-digit multiplication problems. While this finding may suggest a correlation between fact fluency and solving difficult problems, the reliance on digits correct as a measure does not allow for an analysis of error patterns or an analysis of multiplicative reasoning.

In the reviewed fact fluency research, children were measured on their ability to recall and generate answers to single digit multiplication problems. In multiplicative reasoning research, the ways in which children operate on composite units was measured. Children who can only recall facts but who cannot operate on composite units seem not to have multiplicative reasoning, and are therefore less likely to progress in multiplication, division, fraction concepts, and algebraic reasoning (Steffe, 1994; Steffe & Cobb, 1994).

Curriculum Based Measurements. The most common method monitoring student progress in an RtI system is the curriculum-based measurement (CBM). CBM refers to assessments designed to assess student progress with problem types specified in the regular curriculum (Fuchs et al., 1991). In Mathematics, CBM's are commonly decontextualized computation problems that are scored by determining digits correct in each answer (Stecker et al., 2005). For example, if solving 9x7=58 on a CBM a student is given one point for the correct digit in the tens place (5) and would miss one point for the incorrect digit (8) in the ones place. CBMs have been noted for ease of administration, ease of scoring, and ease of use when graphing or showing student progress over time(Stecker et al., 2005). Critiques of the CBM have suggested that it has limited usefulness in supporting teacher's development of lessons (Fuchs et al., 1991), CBMs were designed as a measure of general math ability and are still being studied to determine what is measured (Thurber, Shinn, & Smolkowski, 2002), and there has been limited support that student achievement has improved (Stecker et al., 2005). Overall, CBMs are currently the most used measure of student progress in special education (Deno, 2003) and are designed to measure increases in fluency and correct answers over time. They are not measures of developments in multiplicative thinking.

The Impact of Number on Multiplicative Reasoning

Research on number in multiplication tends to be focused on the relative difficulty of the recall of certain multiplication facts (Domahs et al., 2006; Ruch, 1932; Verguts & Fias, 2005). Researchers have suggested that the size of the numbers used in multiplication problems (problem-size effect) increases the difficulty involved in solving those problems (J. I. D. Campbell & Graham, 1985; Groen & Parkman, 1972; Polich & Schwartz, 1974). According to problem-size effect, problems like 2x3 are easier to recall than problems like 6x8. The easier recall has been attributed to easier retrieval when the factors of the problem and the answers are relatively close (J. I. D. Campbell & Graham, 1985). Others (Domahs et al., 2006) suggested that factors other than size of number can interfere with fact recall. They determined that typical errors are close to the correct answers, but they are more often correct with the decade number and have errors in the units (i.e. 4x7=24 is a more likely incorrect answer than 4x7=34 because the correct answer has a 2 in the tens place.)

Two commonalities in the aforementioned research are particularly relevant to this dissertation study. First, all of these studies were focused on difficulties in recalling previously learned multiplication facts (not the acquisition of multiplicative reasoning), and they were all conducted with adults. Research that has compared fact recall between children and adults has suggested that there are notable differences in the relative difficulty of particular single digit multiplication facts between children and adults (LeFevre & Kulak, 1991; Ruch, 1932) and the types of errors that are made (LeFevre & Kulak, 1991). Ruch (1932) compared three studies that ranked the relative difficulty of each of the 100 basic multiplication facts. The participants from two of the studies were in third grade; the participants from the third study were between fourth and eighth grades. The relative placement of the multiplication problems by difficulty was similar for the studies with third grade participants, but the results of the third study suggested a different order of difficulty of the multiplication facts. Ruch concluded that children who are just beginning to learn multiplication face different difficulties in learning multiplication facts than the difficulties that older children or adults face in recalling facts.

In terms of number and multiplication, there have been studies that have attempted to rate multiplication facts by relative difficulty, but these studies are not focused on how number impacts the acquisition of multiplication or multiplicative thinking. This gap will be addressed by the proposed study.

This chapter provided an overview of the learning theory, constructivism, that will be used to guide this study as well as an overview of multiplicative reasoning: how it develops, how it is different from fact recall, and how number may influence multiplicative reasoning. The research suggested that multiplicative reasoning is qualitatively different from additive reasoning and relies on a child's conception of number as a composite unit. The shifts from additive reasoning may begin before formal schooling in terms of the development of many-to-one correspondence and an informal notion of proportionality. There is a stark contrast in methods between the nature of the studies of the construction of multiplicative reasoning and studies of multiplication fact fluency and recall. Studies of the impact of number on multiplication are limited and tend to be focused on fact recall. Research was not found that focused on the impact of specific number on initial or early development of multiplicative reasoning. This gap was addressed in this research study.

CHAPTER III

METHODS

This Constructivist Teaching Experiment explored the impact of numbers used in instructional tasks on the construction and generalization of multiplicative reasoning by fourth grade students designated as having learning difficulties or disabilities in mathematics. While building on previous teaching experiments that have explored the phases that children move through as they develop multiplicative reasoning (as discussed in chapter two), this dissertation study explored the ways in which numbers impacted within-phase development. The models generated from this research provided an account of the ways in which numbers impacted the construction of multiplicative reasoning in Students with Learning Disabilities (SLDs). Such a study compelled a design that included teaching students as part of studying the impact of numbers used on their learning; thus the qualitative research strategy known as Constructivist Teaching Experiment (Cobb & Steffe, 1983; Steffe & Thompson, 2000) was employed.

The study aimed to answer the following research questions:

1. In what ways do SLDs' conception of number as a composite unit, afford or constrain transition to multiplicative reasoning?

2. Which specific numbers, used in instructional and/or assessment tasks, may support or interfere with SLDs' progression from additive reasoning to multiplicative Double Counting (mDC)?

While the tasks and shifts that children make when developing multiplicative reasoning have been studied (Tzur et al., 2013), this dissertation study contributed to further understanding how specific numbers impact the ways in which SLDs are able to operate when developing multiplicative reasoning. This chapter includes a description of the Constructivist Teaching Experiment (which includes two major sections, data collection and data analysis); a description of the participants; and sections on trustworthiness, limitations and delimitations.

Constructivist Teaching Experiment

This section describes and provides examples of key elements of the *Constructivist Teaching Experiment*, the overarching qualitative approach used in this study. Prior researchers have suggested qualitative methods are well suited to record and analyze why or how complex phenomena occur (Creswell, 2007; Miles & Huberman, 1994) and are also suggested when seeking to generate and explore hypotheses (Miles & Huberman, 1994).

The Constructivist Teaching Experiment differs from a traditional experiment in two important ways. First, an experiment in the traditional sense is associated with practices of the manipulation of variables in order to confirm or reject an initial hypothesis (D. T. Campbell & Stanley, 1963). The Constructivist Teaching Experiment is rooted in the rejection of the possibility to control and manipulate variables pertaining to student learning. This research method is called an experiment because the methodology involves continuous and responsive experimentation with the ways in which students' might be constructing knowledge (Steffe, Thompson, & von Glasersfeld, 2000). Secondly, the Constructivist Teaching Experiment employs the development and testing of hypotheses throughout a research study, rather than the design of an experiment to test an a priori, static hypothesis to be accepted or rejected in its initial form (Glass & Hopkins, 1995). This dissertation study required the flexibility inherent in the

Constructivist Teaching Experiment, because of the nature of the initial hypothesis. Indeed, it was initially hypothesized that using different numbers may promote or hinder the acquisition or generalization of multiplicative reasoning. However, it was expected that the responses and experiences of the participants in the study would lead to refinement and revision of the hypothesis throughout the study.

The Constructivist Teaching Experiment is a research methodology designed for formulating empirically-grounded models of students' learning and ways of reasoning; that is, it addresses research questions about how children construct particular knowledge and understandings (Cobb & Steffe, 1983; Steffe & Thompson, 2000). Cobb and Steffe (1983) proposed this methodology as an effective method for allowing researchers to build models of children's mathematical realities. Their rationale was that this methodology combines a theoretical framework that grounds the researchers' observations in past studied theory with direct interaction with children, which requires the researchers to be responsive to student thinking and understanding in the moment. Such a combination is the hallmark of data collection methods employed in a teaching experiment.

The theoretical elements of the teaching experiment along with the examples are organized into two sections: Data Collection and Data Analysis. The Data Collection section includes (a) a description of the roles of the researcher, (b) a description of the teaching episodes, (c) the goals of the teaching experiment, and (d) the use of records from the teaching and learning episodes. The Data Analysis section includes a description of methods used in Constructivist Teaching Experiments and a description of the evolving research hypotheses. Specifics about the teaching experiment used for this dissertation study follow each of the general descriptions.

Data collection. In a teaching experiment, data collection requires that the researcher assume two complementary roles: researcher and teacher. This *researcher*/teacher (the title given to the primary researcher in this dissertation) conducts or observes and co-plans a series of teaching episodes with one or a few students to test hypotheses about students' ways of operating and how these ways are expected to evolve as a result of the teaching (Steffe & Thompson, 2000). The data from the teaching episodes are collected using video recording that can be used for ongoing and retrospective analysis (Cobb & Steffe, 1983; Steffe, 1991b; Steffe & Thompson, 2000; Steffe et al., 2000). Researcher/teachers who take part in the episodes also take intensive field-notes to document their experiences, impressions, thoughts, and hypotheses about the students' mathematical thinking.

This teaching experiment also included a second researcher role. In this case, the intervention teacher, who engaged in most of the teaching in the teaching episodes, was referred to as the *teacher*/researcher. This teacher/researcher was engaged in more than just teaching during the episodes; she was an active participant in planning. Steffe (2000) suggested that the extended connection between the researcher/teacher, the teacher/researcher, and the student-participant allows the team to experience and study the constructive process (discussed in chapter two), as all are present as the child constructs new mathematical ideas. The teacher/researcher also participated in both the ongoing and retrospective analysis with the researcher/teacher.

Teaching episodes. Teaching episodes consisted of lessons conducted and/or observed by the researcher/teacher and the teacher/researcher. These episodes provided the data generated via the teaching experiment. Constructivist-based teaching employs particular design elements that allow the researcher to gain understanding of the mathematics of children (Cobb & Steffe, 1983). In this case, the teaching design was the Please Go and Bring for Me game (PGBM, as discussed in chapter 2). The teaching episodes were conducted with a group of four students who either worked in pairs or worked one-on-one with either the teacher/researcher or the researcher/teacher for 30-45 minutes, approximately twice per week, for 18 episodes. Initially, PGBM was played asking the students to construct towers of 2 or 5 cubes[5], and the number of towers was limited to 2, 3, 4, 5 or 6 towers. Students were observed answering the following questions: (a) How many towers did you bring? (b) How many cubes are in each tower? (c) How many cubes did you bring in all? And (d) how did you know? Within this context, the study focused on how children responded to the numbers used to play the game. Other numbers for the unit rate were introduced gradually, adding towers of 3 and 4 cubes followed by towers of 6, 7, 8, and 9 cubes each. The number of towers (composite unit) was also controlled, starting with up to six towers, then increasing the number (up to 12 towers). During the work, the researcher/teacher also assessed the extent to which students could solve realistic word problems, using numbers of each layer described here (both unit-rate and number of composite units) for which the child had either developed an anticipation or was participatory in the PGBM context.

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[5] 2 and 5 are initially chosen because they tend to be "easy for the child" multiples. 10 may also be an easy multiple but was not used in this sample.

Teaching episode goals. Cobb and Steffe (1983) outlined essential goals of constructivist-based teaching designs that were considered in the design of the teaching episodes used in this experiment. They suggested that such teaching is designed to: (a) learn how to communicate mathematically with children, (b) learn how to engage children in goal-directed mathematical activity, (c) learn the mathematics of the children one teaches, and (d) learn how to foster reflection and abstraction in the context of goal-directed mathematical activity. The following paragraphs explain each of these goals and connect them to the teaching episodes in this study.

The first goal, learning to communicate mathematically with children, required careful observation of students and careful planning. The researcher/teacher either observed the teacher/researcher interacting with children or interacted with the children. The numbers used for each PGBM task within the episodes were initially chosen based on preliminary hypotheses and were chosen for subsequent episodes based on student responses. The researcher/teacher and teacher/researcher worked to communicate with children in a way that allowed each child to express his or her current thinking. Each interaction led to more understanding about how number choice impacts how children respond to and understand the tasks. While it was not possible with this design to determine what might have happened if different numbers were chosen for each of the tasks within the episodes, or which numbers were the best choices at a given time, it was possible to record the ways in which children responded to different numbers and look for trends in those responses.

The second goal, learning how to engage children in goal-directed mathematical activity, required a major focus of the researcher-teacher to be placed on the changes in the children's anticipation. These changes in anticipation are postulated to occur via children's reflection on their activity of coordinating the number of towers, number of cubes per tower, and the resulting total. The teaching experiment methodology, in its inclusion of teaching based on hypotheses about the mathematics of the children, can provide data about how they progress from additive to multiplicative reasoning while intentionally fostering reflection and abstraction.

The third goal, learning the mathematics of the children who participated in the study, required the researcher/teacher and the teacher/researcher to develop good questions that guided each teaching episode observation. Steffe (2000) distinguished between *students' mathematics,* which he defined as student mathematical realities, and the *mathematics of students* that refers to observers' interpretation of students' mathematics. The latter is a model researchers construct about the children's mathematical models/realities via inferences into mental structures and operations made based on observable behaviors (actions, language). Questions that guided the observations and inferences about mathematics of students included: (a) What did it look like when children learned to reason multiplicatively, particularly when they used different numbers for the unit-rate and for the number of composite units? (b) What did the children say or do that suggested that they were engaging in multiplicative reasoning while operating on these numbers? And (c) what changed as the children developed multiplicative reasoning and were challenged with different numbers? These questions were central to understanding how the children were thinking and thus to articulating the mathematics of the children, as it was afforded (or hindered) by the numbers chosen for each task. As a result they were used in both the ongoing and retrospective analyses.

Chapter 4 includes examples of differences in student responses to numbers selected for PGBM tasks.

The fourth goal of a teaching experiment, fostering reflection and abstraction in the context of goal directed mathematical activity, was achieved as a result of learning to communicate with students (goal 1) and better understanding of students' mathematics (goal 2). In the context of Reflection on Activity-Effect Relationship (Ref*AER, see Chapter 2) framework, a major focus of the researcher/teacher was placed on the changes in the children's anticipation. These changes in anticipation were postulated to occur via children's reflection on the effects of their activity of coordinating the number of towers, number of cubes per tower, and the resulting total. The teaching experiment methodology, in its inclusion of teaching based on hypotheses about the mathematics of the children, provided a model of how they progressed from additive to multiplicative reasoning while intentionally fostering reflection and abstraction.

Record of the teaching and learning interactions. The teaching experiment data consisted of video records of the teaching and learning processes that ensued during teaching episodes (Steffe & Thompson, 2000). In this dissertation study the researcher/teacher (a) observed the teacher/researcher teaching the students, (b) taught the students, (c) video recorded each of 18 teaching episodes (176 PGBM problems across four students), (d) engaged in ongoing analysis (with the teacher/researcher) of the video recorded data after each teaching episode, and (e) retrospectively analyzed the video record with a team that included the researcher/teacher, the teacher/researcher, the advisor, and an additional doctoral student.

The video records and the collaborative reflection on the video supported the analysis of the student learning and supported the researcher/teacher and teacher/researcher team in planning for the ongoing teaching episodes. The video record of the teaching was essential to this Constructivist Teaching Experiment because it allowed the researcher/teacher and the teacher/researcher to fully interact with students in the moment without worrying about documenting participants' work. The video records provided a basis for re-observing teacher-learner interactions during ongoing and retrospective analysis. In the next section, methods for data analysis, hypothesis testing and generation, and model building are introduced and specific examples are provided.

Data analysis. This section includes a description of the data analysis process used in a Constructivist Teaching experiment as described by Steffe and Cobb (2000). Data was analyzed using a recursive process of hypothesis generation and testing, in both an ongoing phase and a retrospective phase.

As described in the data collection section of this chapter, each of the teaching episodes was observed by the researcher/teacher and video recorded. After each teaching episode the researcher/teacher and the teacher/researcher conducted the first phase of ongoing analysis, which included debriefing the lesson, focusing on what had happened with each of the students in that day's lesson, and discussing any points that might be related to an ongoing hypothesis. The team also noted student behaviors or responses that might not have been related to the initial hypothesis, but provided interesting insight into student thinking and thus could be developed into new hypotheses. Themes, or trends, that might have emerged that day were noted and compared or contrasted to trends that were noted in previous episodes. For example, one of the earliest trends that emerged across all of the students was that multiplicative double counting was evident for all students when given a unit rate of 2 or 5 in the PGBM problems. Following the initial, ongoing analysis, several iterations of retrospective analysis were conducted. These are described in the next section.

Retrospective analysis. Steffe, Thompson, and von Glasersfeld (2000) suggested that retrospective analysis be a part of a teaching experiment to provide the researcher/teacher with an additional opportunity to look at the historical evolution of past experiences with students. Such a historical focus can support noticing student behaviors over the course of the experiment that may have been missed in the initial analysis.

The retrospective analysis phase of this study began with the researcher/teacher reviewing and transcribing each of the episodes. She focused on student words, gestures, and actions that might support the trends noted in the initial rounds of ongoing data analysis. The researcher/teacher then compiled notes from each of the problems presented in the teaching episode in order to continue to monitor trends, to test current hypotheses, and to generate new hypotheses. For example, one trend, noted across students, was making errors in solving PGBM problems when the unit rate was greater than 5 the students. Looking further into that trend, it was noted that the inability to keep track of where to stop in one or both of their coordinated counts seemed to be a factor in getting wrong answers. These observations generated the hypothesis that students may have difficulty keeping track of their coordinated counts because the coordinated count requires a dual anticipation of stops and starts.

From this first round of retrospective analysis, the researcher/teacher identified segments of video from each of the episodes that supported or challenged the initial and ongoing hypotheses. These segments were viewed by a larger group, comprised of the faculty advisor (Tzur), the researcher/teacher (Risley), and the teacher/ researcher (Hodkowski). This group viewed and reviewed the segments several times, again noting each student's words, motions, and/or gestures that might provide insight into his or her learning. Multiple viewings and this team analysis resulted in the development of new hypotheses, and supported the development of claims about student learning. The evolving research hypothesis and claims are discussed in detail in the next sections.

Evolving research hypotheses. A Constructivist Teaching Experiment both generates and tests hypotheses (Steffe & Thompson, 2000). The evolving research hypotheses are included in both data collection and data analysis, because changes in the hypotheses impact the nature and actions of the teaching episodes. The researcher does not begin the experiment with a fixed hypothesis; rather, hypotheses are continuously tested and revised as the study is conducted (Steffe & Thompson, 2000). The evolving hypotheses are important because, while the researcher/teacher brings some understanding of the general ways in which children construct mathematics, she must be focused on the child's actual mathematics. This keeps the researcher/teacher focused on what is happening in the teaching episode, being specifically attuned to children's ways of solving the tasks and what could constitute underlying schemes for those ways.

The research hypothesis that the ways in which SLDs were able to engage in and generalize multiplicative reasoning would differ based on the types of numbers in the problems presented by the researcher/teacher was challenged and revised during the course of the study. New hypotheses emerged and were tested during the course of the teaching experiment. Steffe and Thompson (2000) suggested that the expectation of evolving hypotheses supports researchers in focusing on what actually happens as children develop mathematical ideas and have changes in their anticipations. Whenever the children did things not anticipated by the researcher/teacher, those episodes were used as instances of hypothesis generation and, subsequently, testing. For example, when one student was not able to figuratively re-present and double count covered towers of 5 (the initial hypothesis was that towers of 2 and 5 would be easy for students to re-present and double count), the research team had to develop a new hypothesis about why 5 did not support mDC in this case.

The initial (and temporary) research hypotheses for this dissertation study were based on the work of others (discussed in chapter 2) who explained how children develop multiplicative reasoning (Confrey, 1994; Kieren, 1994; Sophian & Madrid, 2003; Steffe, 1992; Steffe & Cobb, 1994; Tzur et al., 2013; Tzur et al., 2010; Vergnaud, 1983; Xin, Si, & Tzur, 2006). In particular, a central hypothesis suggested that SLDs would construct and generalize multiplicative reasoning as a result of their reflection on the numbers presented in PGBM, but that there might be shifts in their understanding based on the numbers they were coordinating. For example, we predicted that a student who had constructed number as composite unit would show some evidence of a change in his or her scheme in terms of anticipation to include mDC when he or she was operating on and coordinating numbers of 2 or 5. This student might require prompting, or revert (Tzur & Lambert, 2011) to a non-multiplicative scheme, when asked to operate on groups of 3, 4 or 7. This initial hypothesis was expanded as data was collected that suggested that it made little difference if the composite unit (number of towers) was 2, 3, 4, or 5 *if* the unit rate was 2 or 5. When the unit rate changed to 3 or 4, the child would revert to counting 1s. This indicated that a child's progression from additive reasoning to multiplicative reasoning is not accomplished in one-fell-swoop but rather in steps that depend on the numbers used in a problem (the specific focus of Research Question #2). The next section includes a description of model building, the specific ways that children's mathematics are explained in a Constructivist Teaching Experiment.

Model building. Model building refers to the ways in which the researcher is able to provide specific and concrete explanations of children's mathematical activity (Steffe & Thompson, 2000), both observable and mental (inferred). Cobb and Steffe (1983) suggested that a goal of the Constructivist Teaching Experiment is to produce models that specify students' knowledge as they develop coordinated schemes of actions and operations. For example, Tzur, Xin, Si, Kenney, and Guebert (2010) built a model that suggested that children were unable to move to multiplicative reasoning if they lacked number as a composite unit. In this case, the researchers understood the schemes of action and operation specific to multiplicative reasoning: they observed and interacted with individual children participating in the teaching experiment and then determined that a lack of number as composite unit inhibited the move to multiplicative reasoning. This model of learning was useful in accounting for other children's mathematical development.

The model building from this experiment relied on the core concepts of the conceptual framework discussed in chapter 2, including: assimilation, accommodation, Ref*AER, and participatory and anticipatory stages in the construction of a new scheme.

The core concepts of the conceptual framework combined with the data collected supported substantial claims about the thinking of the children with whom they had interacted (Steffe & Thompson, 2000). These substantiated claims provided the foundation of model building. Models address research questions about mathematics of children and, in particular for this study, the impact of numbers used on their multiplicative schemes.

In order to develop this conceptual model of children's mathematics, the researcher/teacher moved through four phases. First, an empirical account of the teaching and learning that transpired was created. Second, the model was reformulated using a theoretical foundation. Third, an analysis of how children make progress (or not) was conducted. Finally, an analysis of the teaching episodes that supported the construction of more sophisticated schemes (Cobb & Steffe, 1983) was conducted. The resulting model was grounded in both theory and the empirical evidence from the cases observed in the teaching experiment. Throughout each of these phases the data was monitored to ensure that there was enough information to provide an account of the teaching and learning that adequately described what changed as a student learned and how those changes were documented. It was also essential that the connection between the teaching episodes and the children's progress toward the construction of more sophisticated schemes be articulated.

Tzur et al. (2010) provided an example of model building in their study of students' progression in multiplication that supported the model building design in this study. Particularly, they applied the theoretical model of participatory and anticipatory stages in the construction of a new scheme (Tzur & Simon, 2004). This model informed their analysis and model building of how and why students with learning disabilities were or were not progressing in multiplication. Their study reported that the starting point for 3 out of 12 students was at a 1st grade equivalent. They suggested that these three students were eventually able to make relatively quick progress in multiplicative reasoning (11 teaching episodes) because of the attention given to the conceptual prerequisite of number as a composite unit. Specifically, they used data to demonstrate that a student who had moved to an anticipatory stage in the construction of number as a composite unit could make substantial progress in multiplicative schemes, whereas a student at the participatory stage needed more work before she could construct the mDC scheme.

Participants. Participants in this dissertation study were four fourth grade children from an urban elementary school in a western state in the United States who had all been identified as requiring intervention based on multiple school-based assessments and their performance on the state assessment in grade 3.

The sample for this study was a purposive criterion based sample (Onwuegbuzie & Collins, 2007). The three criteria for participation in this study were: (a) an understanding of number as a composite unit, (b) an identification of a learning difficulty or disability in mathematics, and (c) an assessment that not even the initial mDC scheme has been constructed by the child. The following paragraphs further describe each of the three criteria for participation in this study

Number as a composite unit. In order to participate in multiplicative tasks, a child must first have constructed the understanding of number as a composite unit. An observable indicator of a child's understanding of number as a composite unit is the child's ability to use a counting-on strategy when adding two known quantities (Tzur &

Lambert, 2011). For example, consider a situation in which a child is given a collection of items, establishes its numerosity as ‘seven,’ and then the collection is covered. The child is then given more items and establishes the additional collection as ‘four more items.’ If the child finds the total of items by counting-on from the first known number while keeping track of the added items, as in “seven; eight, nine, ten, eleven,” there is evidence that the child has constructed number as a composite unit. A child who does not yet have number as a composite unit will recount each and every item (from 1) to find the total even when the total of the first quantity has previously been determined by the child (e.g., saying, “one, two, three, four, five, six, seven; eight, nine, ten, eleven”). The difference is important to this research study because number as a composite unit is considered a conceptual pre-requisite for children to move to multiplicative reasoning (Tzur & Lambert, 2011). The children participating in this study were assessed by the teacher/researcher for composite unit understanding using an addition and subtraction word problem criterion test (Tzur et al., 2010).

Learning difficulty or disability. The second criterion for participation in this study was the identification of a learning difficulty or disability. In this study, one child had a current Individual Education Plan (IEP). All four of the students had been identified as needing Tier-2 or Tier-3 interventions in mathematics as part of a Response to Intervention (RtI) process. (In this school, such students are pulled out of class to work with a math interventionist.)

mDC. The third criterion was that the participating children had not yet constructed a scheme for mDC. This was assessed using problems from the Nurturing Multiplicative Reasoning in Students with Learning Disabilities in a Computerized

Conceptual-Modeling Environment (Xin et al., 2006) Software Readiness Survey. The children were unable to solve problems such as, “Miguel puts all of his marbles in 6 bags. In each bag there are 3 marbles. How many marbles does Miguel have?” and therefore met the criterion for inclusion in the study. A short description of each of the four students included in this study is provided in the next section. Detailed excerpts that illustrate the mathematics of each student will be included in Chapter 4.

Case 1 Dana. The data collected for Dana included 52 PGBM problems. Initially, Dana relied on counting on or doubling strategies to solve PGBM problems with a composite unit of 2, 3, 4, 5, and 6 and a unit rate of 2 and 5. She rarely (only one recorded incident) used counting all to figure out any of the problems involving towers and cubes. Dana ended the observations with the most “known facts,” meaning that there were several problems where she did not appear to engage in any mDC; rather, the answer was readily available.

Case 2 Devin. The data collected for Devin included 52 PGBM problems. Initially, Devin relied on counting on strategies to solve PGBM problems with a composite unit of 2, 3, 4, 5, and 6 and a unit rate of 2 and 5. Devin was the first student in the group to demonstrate a clear mDC using one hand to track towers and the other hand to track cubes in each tower. Throughout the 18 teaching episodes Devin anticipated and demonstrated a multiplicative Double Count but consistently struggled to keep track of the counts within his double count.

Case 3 Jake. The data collected for Jake included 28 PGBM problems. Initially, Jake relied on counting all starting from 1. Though Jake was able to anticipate in mDC in some instances, he often folded back to counting by ones when the number of towers and cubes per tower became more difficult for him.

Case 4 Luke. The data collected for Luke included 44 PGBM problems. Initially, Luke relied on counting all starting from one. He expressed the most hesitance when figuring out the problems when covers for the cubes were introduced. He was also quite interesting because his re-presentations were different for different unit rates. Whereas the other three students tended to adopt one way to re-present and track the accumulation of towers and cubes, Luke had five distinct ways that he kept track of towers and cubes.

While each of these students was a unique case, and each progressed toward mDC in somewhat different ways, there were themes and trends that emerged from each of these cases. Excerpts from each case that pertain directly to the research questions will be included in Chapter 4.

Trustworthiness of data and findings. The next section describes issues of trustworthiness and considerations of reliability and validity in this study. It begins with a discussion of key elements of validity and reliability and connects those to the particular approaches used in this study.

Validity and reliability. Threats to internal and external validity are an important consideration in any research (Onwuegbuzie & Leech, 2007) Researchers must be concerned with both the internal validity (credibility, dependability, truth value) (Onwuegbuzie & Leech, 2007) as well as the external validity, which in this study refers to the transferability of the findings to similar contexts (Miles & Huberman, 1994).

In this study, validity was supported by a prolonged engagement with the study participants. The students were observed approximately two times per week for a three-month period for a total of 18 observations. The four students participating in the study were observed for a total of 170 PGBM problems. The extended observation period and then number of problems observed allowed for multiple opportunities to observe the students solving the same problems across episodes.

Validity was also supported by the ongoing and retrospective nature of the data analysis. Ongoing data analysis was conducted with the teacher/researcher and the researcher/teacher. It was retrospectively analyzed by the researcher/teacher and a larger group that included the researcher/teacher, the teacher/researcher, the advisor, and another doctoral student. It was then analyzed again by the researcher/teacher as described earlier in this chapter.

One threat to validity is inherent in the Constructivist Teaching Experiment. The aim of the research is to gain insight into the mathematics of the child (Steffe, 2000), and the only venue that we have is the child's external responses, including words, actions, and lack of actions, gestures or expressions. Because the data is based on inferences, it is especially important to include precise transcripts and recounts of the student work that provide the reader with a rich picture of the child's responses beyond just her or his words.

Reliability refers to the level of confidence that a future researcher, using the same procedures, would arrive at the same results. In teaching experiment research, reliability is supported in two primary ways. First, the data is collected using video-records and filed notes. Video allows the researcher to revisit the episodes as new questions are generated or as new hypotheses need to be tested. The video records are transcribed with special attention to student gestures and movements as well as to words.

Second, the data are reviewed and coded several times, which allows for checks of inter-coder reliability.

In this study, both the researcher/teacher and the teacher/researcher took field notes and video-recorded each of the 18 episodes. The episodes were analyzed first by the researcher/teacher and the teacher/researcher team. They were transcribed by the researcher/teacher, then retrospectively analyzed by the larger team that included the researcher/teacher, the teacher/researcher, the advisor, and another doctoral student; this doctoral student paid particular attention to viewing and re-viewing to catch the smallest of hand movements, gestures, and/or pauses that might provide insight into the mathematics of the child. Following the team retrospective analysis, the video records were viewed a third time by the researcher/teacher, and each PGBM task within each episode was categorized by numbers used for composite unit and for unit rate. These multiple opportunities for different groups to view and review the data support the reliability of the findings.

Limitations and delimitations. Limitations refer to the conditions or situations that were out of the control of the researcher that may present a weakness in the study (Creswell, 2005). One limitation was the number of students selected for an intervention group. Based on school scheduling, only one group of four fourth grade students was available for the intervention class. Another limitation was the available dates for observations. Because the student participants were involved in school-wide assessments and other beginning-of-the-year activities in August and September, they did not begin the intervention classes until early October, which meant the data collection could not start at the beginning of the fourth grade year. Additionally, three out of the four students showed strong growth in mDC by early December and were beginning to work on schemes other than mDC. Data that pertained specifically to the shift from additive to multiplicative Double Counting were limited to one student by the end of the data collection period.

Delimitations refer to the choices made by the researcher to bound the scope of the study (Creswell, 2005). This study was purposely designed to look at the impact of numbers on students' movement from an additive scheme to an mDC scheme. The study did not look at the impact of number on student construction of schemes beyond mDC. Data was collected from fourth grade students' participation in an intervention setting using the PGBM game. These fourth grade students were selected because they had received initial instruction in multiplication but did not make progress with that instruction. Data was not collected from regular classroom settings to analyze the type of instruction they received outside of the intervention group. The conceptual framework also delimited the scope of this study. Data analysis was focused on determining how students constructed mDC as a result of reflection on mental activity Ref*AER (Simon et al., 2004).

Summary of the Research Design and Methods

This study employed a Constructivist Teaching Experiment. The teaching experiment employed video records and field notes that chronicled teaching episodes conducted by the researcher/teacher, teacher/researcher team. Ongoing analysis of the data was conducted by the researcher/teacher following each of the teaching episodes. A larger team that included the researcher/teacher, the teacher/researcher, the faculty advisor, and a doctoral student conducted retrospective analysis. From this recursive process of hypothesis testing and generation on both an ongoing and retrospective basis three themes emerged. Chapter IV provides the results of this research.

CHAPTER IV

RESULTS

This study explored the impact of numbers used in instructional tasks on the construction and generalization of multiplicative reasoning by fourth grade students designated as having learning difficulties (SLDs) in mathematics. Specifically, the study addressed the following research questions:

1. In what ways do SLDs' conception of number as a composite unit afford or constrain transition to multiplicative reasoning?

2. Which specific numbers used in instructional and/or assessment tasks may support or interfere with SLDs' progression from additive reasoning to multiplicative Double Counting (mDC)?

The data and analysis presented in this chapter consist of excerpts taken from teaching episodes that were conducted and/or observed by the researcher/teacher and the teacher/researcher. In each teaching episode, the students worked with a version of the Please Go and Bring Me game (PGBM, as discussed in Chapter II). The 18 teaching episodes were conducted with a group of four students who worked in pairs or in one-on-one settings, with either the teacher/researcher or the researcher/teacher for 30-45 minutes, approximately twice per week. Teaching episodes were informed by the reflection on activity-effect relationship framework (Ref*AER), specifically seen in the focus on promoting the two types of reflection (mental comparisons) explained in Chapter 2.

This chapter is organized according to three themes revolving around the claims made to address the research questions, as follows:

Theme 1: Early in the construction of an mDC scheme, students seemed able to coordinate a multiplicative Double Count when given the numbers 2 and 5 for the unit rate and when given numbers of up to 5 composite units (CU). These students were not yet able to coordinate units in mDC tasks involving what, *for them*, seemed to be harder numbers (e.g., three and four). When faced with the more challenging numbers, students tended to fold back (Pirie & Kieren, 1994) to a lower-level strategy. This theme accentuates Claim 1: The construction of an mDC scheme is not a once-and-for-all event, but rather a process that requires attention to the numbers used in tasks to support the construction of multiplicative reasoning.

Theme 2: As students progressed in the construction of a multiplicative scheme, numbers for the unit rate and the CU that exceeded 5 (the number of fingers on one hand) encouraged the construction of a revised figural representation of the units required for a coordinated count. This theme further accentuates Claim 1 above.

Theme 3: As students progressed in the participatory stage of the mDC scheme, they no longer tended to revert to additive schemes when given harder numbers. However, the harder numbers chosen for the CU and unit rate impacted whether they could anticipate where to start and where to stop with each of their coordinated counts: the compilation of CUs, the unit rate, and the accrual of 1s. This theme accentuates Claim 2: The construction of an mDC scheme requires attention to both anticipation of where to start and stop a count for the CU, and anticipation of where to start and stop the accrual of each of the unit rates. This twofold anticipation seems unique to multiplicative thinking, and attention paid to the numbers used in tasks can support SLDs' reflection, particularly on keeping track to know where to stop the counts for CU and unit rates in an mDC scheme.

In order to connect these themes to the conceptual framework and the research questions, the analysis of data will include inferences about the child's available conception of number as a CU and the child's available operation, as well as compare and contrast how the numbers presented in the task influenced those operations. The summary analysis will also compare data that highlight shifts in the construction of multiplicative thinking and reflection Type I and II based on numbers presented to research on fact recall and memory based conceptions of multiplication for SLDs. The claims will be discussed further in Chapter V.

Theme 1: Folding Back with "Hard Numbers"

To illustrate the differences in student multiplicative thinking when presented with different numbers, this section includes analysis of seven data excerpts consisting of transcripts from observed lessons. In Excerpts 1 & 2, one of the students (Jake) worked with a peer on PGBM tasks constrained to a unit rate of two or five and up to five CUs. In Excerpt 3, another subject (Luke) worked with his teacher (Nina) on tasks constrained to a unit rate of five and less than five CUs. In these three excerpts, both Luke and Jake were inferred to be at a participatory stage of an mDC scheme, specifically demonstrating early evolution of multiplicative thinking in their solutions and explanations. In Excerpts 4 & 5, Jake played the role of a bringer while Nina played the sender, with tasks allowing both unit rates and number of CUs to be two, three, four, five, or six. In Excerpts 6 & 7, Luke solved tasks with unit rates of three or four and up to five CUs. In Excerpts 4 through 7, with the 'harder numbers,' both Jake and Luke no longer employed a multiplicative scheme and instead folded back to additive strategies to solve the tasks. The different numbers used in those tasks allow analysis of the impact of “easy” and ‘hard’ numbers on a student’s problem-solving strategy and the inferred scheme (or stage).

Excerpt 1 contains data from Jake’s first turn as a bringer after he had produced (from single cubes) and brought four towers of five cubes each (4T5) and properly responded to the first two questions (SS stands for Student-Sender).

Excerpt 1: Multiplicative scheme when finding the total number of cubes (student: Jake; task 4T5; date: October 7, 2014).

13:35 SS: How many cubes did you bring in all?

13:38 Jake: [Glances at the towers for one second] I brought … [Uses his left hand to tap five times on the palm of his right hand.] I brought 20 cubes altogether.

13:45 SS: How did you figure this out?

13:49 Jake: I figured this out by counting.

13:51 Nina: How did you count?

13:53 Jake: I counted by fives.

13:56 Nina: How did you know to stop counting?

13:58 Jake: Cause if you don’t, cause if you can’t …[Reaches with his left hand and brushes over the towers that the SS is holding.]

14:04 Nina: [Presents Jake with an example of counting by fives without keeping track of the total number of counts as a way to push his explanation of how he knew to stop.] I can count by fives too: 5, 10, 15, 20, 25, 30 ... [Jake joins in counting by fives up to 50, possibly to show Nina his facility with the fives counting sequence.] So how did you know to stop at 20?

14:19 Jake: [Looks at the towers that the student sender is holding.] It's … because that … [turns his head away from the towers for five seconds, then turns his gaze back on the towers.] It's because I only brought four towers.

14:35 Nina: So you knew to stop because...? How did you know you had counted the four towers, Jake? I agree that you only brought four towers. How did you know that 20 [cubes] was four towers?

14:50 Jake: I counted on my fingers.

14:52 Nina: Can you show me?

14:54 Jake: [Raises his right hand, then folds his index finger] Five; [folds his middle finger] 10, [folds ring finger] 15, [folds pinkie finger] 20! [Body language indicates that he is done with the task.]

Excerpt 1 provided a glimpse into Jake's mental operations as he solved a task with an easy number – the unit rate of five. Jake first reestablished the number of 1s that constituted each CU (five palm taps at 13:38). That is, he seemed to have created a figural, mental template of the size of every CU. Because counting by fives was within his capacity, his count of the accrual of 1s while raising a finger for each unit of 5 indicated a purposeful method of keeping track of the number of CUs so he would know to stop at four towers (14:54).

Jake's ability to perform such a purposeful action did not yet seem to support expressing how he did it. Rather, when prompted to explain how he knew to stop at 20, he initiated a shift to a different figural re-presentation. He used the fingers of his right hand to represent the compilation of CUs in the situation (towers) and his number sequence (by five) to re-present the accruing 1s (cubes). This seemed to assist his growing anticipation of the link between coordinated actions taken to figure out a progressive total and the effect of stopping the count of 1s when reaching the number of CUs given in the task (which fits his statement, “I brought four towers”). That is, Jake, when operating mentally on a unit rate that was a known counting sequence (fives) for him, could anticipate, initiate, and complete the coordinated, goal-directed mental activity involved in mDC.

It is quite possible that Jake has initiated and carried out goal-directed actions while not being aware of them, let alone the steps he took to monitor those actions. By asking Jake to explain, Nina thus attempted to orient his reflection on and awareness of his own purposeful actions (e.g., if, in fact, he would keep track of the accrual of CUs that constituted the entire compilation of 4 towers). To explain his strategy, Jake used fingers, which the teacher intended as a means to promote two critical reflections in constructing the mDC scheme. First, she focused Jake’s attention on a specific aspect of his coordinated counting—monitoring accrual of the CUs. Second, she simultaneously focused his attention on the key in his monitoring as a means for the goal of knowing when to stop (14:35).

In the following PGBM task, in which Jake played the bringer, he had to figure out and explain how many cubes are in five towers with two cubes each (5T2). He again illustrated that he could engage in mDC with a familiar unit rate (in this case, 2). This excerpt also provides a second instance where Nina pressed for Jake’s explanation of his solution, as his correct response (10) came after quietly nodding his head five times but not using his hands/fingers. Excerpt 2 shows he re-used the previous way of explaining.

Excerpt 2: Multiplicative scheme when finding the total number of cubes (student: Jake; task 5T2; date: October 2, 2014).

21:02 Jake: [Glances at the cubes and then looks to the posted prompts ("sentence starters") on the board.] I brought 10 cubes in all.

21:17 SS: How did you figure out this total?

21:19 Jake: I figured it out by [plays with the pencils in the pencil container] … I figured it out [begins tapping the fingers of his hands together] by counting with my hands [as explained in Chapter 3, showing solution strategies using fingers to represent each of the counts is a norm established early on in the PGBM game.]

21:30 Nina: Can you show me?

21:32 Jake: Like this … [raises his right hand, folds down his index finger] 2, [folds his middle finger] 4, [folds his ring finger] 6, [folds his pinkie] 8, [folds his thumb] 10.

21:38 Nina: So, again, each one of your fingers … this seems similar to something else you did. Each one of your fingers, you put them down: 2, 4, 6, 8, 10 [paraphrases Jake's utterances with corresponding motions]; so you stopped here [wiggles her thumb]; why?

21:58 Jake: It's because I only brought five towers.

Excerpt 2 provided further evidence to the evolving regularity in Jake's ways of operating as well as the teacher's involvement in that process. Freed from a mental focus needed to account for the accrual of 1s due to his facility with the sequence of multiples of two, he could initiate and complete a coordinated count of figural composite units (5 towers) through nodding his head (in his initial solution) and later through using fingers (21:32). His actions seemed purposeful in the sense of anticipating the need to coordinate two simultaneously accruing number sequences: CUs (towers, counting by 1s) with unit rate (cubes distributed over each of the towers, counting by 2s). Similarly, when prompted to explain his thinking, Jake used the fingers of his right hand to re-present each CU in the sequence of five towers while keeping track of accrual of 1s via his number sequence (by twos) to ten. Considering Excerpts 1 & 2 combined, Jake seemed to have established at least an enactive anticipation of mDC, and a way of acting by using his fingers as figural items standing for each CU (tower), as a means to accomplish his goal of finding the total number of items in a compilation of CUs with "easy" numbers.

Luke's demonstration of an enactive anticipation of mDC. The next excerpt shows data from a second student, Luke. In this, Luke's first episode as a bringer, he produces 3 towers with 5 cubes in each tower (3T5) for a student sender, is able to answer the task correctly (15 cubes in all), and is able to answer all of the sender's questions. Like Jake, Luke seemed to use mDC to solve these tasks, but was less able than Jake to articulate how he knew to stop his count.

Excerpt 3: Using a multiplicative scheme when finding the total number of cubes (student: Luke; task 3T5; date: October 7, 2014).

16:32 Luke: I brought 3 towers.

16:36 SS: Thank you. How many cubes did you bring in each tower?

16:39 Luke: Five (5).

16:44 SS: [The sender orients Luke toward the sentence frame posters- prompting him to answer in the convention of a complete sentence.] Nope, There are five cubes in each.

16:48 Luke: [Corrects the way he phrases the response.] There are five cubes in each tower.

16:54 SS: How many cubes did you bring in all?

16:55 Luke: I brought … [Looks toward the posted sentence prompt and reads the sentence starter for about 10 seconds, seemingly also calculating the total.] I brought 15 cubes in all.

17:06 SS: How did you figure this out?

17:09 Luke: I figured it out by counting. [He leans over toward the towers and points three times at the group of towers.]

17:13 SS: Counting by what?

17:15 Luke: Counting by 15 … [independently stops and shakes his head no indicating recognition of an error] I meant counting by fives.

17:18 Nina: Can you show me what you did?

17:20 Luke: Uh…[pauses, takes the towers from the SS and counts each cube out loud pointing at the individual cubes] 1, 2, 3, 4, 5, 6, 7, 8.

17:27 Nina: [Explains the reason for her questioning.] Because I saw you look away for a minute when you were… and you were counting just like that?

17:30 Luke: [Nods; indicating agreement.] I was trying to focus.

17:32 Nina: You were counting in your head just like you showed?

17:34 Luke: [Nods, indicating agreement.]

17:36 Nina: And how did you know to stop at 15?

17:41 Luke: Because I counted by fives and I really like that number, and that's why I knew it was 15.

17:49 Nina: But how did you know to stop at 15? Just because you liked fives?

17:52 Luke: Wait, I was counting by fives, 5, [touches the first tower] 10, [touches the second tower] 15. [Touches the third tower and looks up at Nina indicating the task is complete.]

18:00 Nina: And you stopped at 15 because … ? [Nina has established explanation as the norm for the group prior to this episode. Her question here prompts Luke to explain his thinking.]

18:04 Luke: Because that was how many I figured out.

18:09 Nina: [Includes a suggested answer in her question to further support Luke.] Because there were no more towers or because you figured it out?

18:12 Luke: Because there was no more.

While it is not possible to be certain that Luke's actions in Excerpt 3 initially consisted of counting by fives, his actions and movements (16:55) indicated that he did not count each cube by 1s. When initially solving the task, Luke neither looked at nor gestured toward the available cubes. He did not make any movements with his hands or body that indicated how he had figured out his answer of 15. This indicates he was operating on a mental image of the towers coupled with his number sequences. He also spent less than six seconds figuring out the answer, whereas counting by 1s would likely have taken longer.

When asked to explain his thinking, Luke said (17:09) he counted and then made three gestures over the cubes - possibly indicating an internal, mental template of each

CU of five. When asked what he counted by, Luke first said 15 (indicating his most recent focus of attention on the total) and then immediately corrected himself to five. When asked to show how he arrived at 15, he counted by 1s. Luke, like Jake, might not have been aware of his goal-directed activity. However, when questioned by Nina he used the towers and seemed to purposefully count by fives to keep track and know when to stop his count at the last tower (17:52). Based on the time it took him to provide his initial response, his fluency with the multiple of 5s sequence, and his ability to monitor where to stop his counts of five, Excerpt 3 suggests that, even though Luke did not appear aware of his own goal-directed activity, he could operate multiplicatively with a unit rate of five.

Two days after the events detailed in the first two excerpts, Nina served as a sender while she and Jake engaged in playing the PGBM game. Based on Jake's facility with double counting in tasks with unit rates of five or two, she chose to promote his progress by sending him to bring 3 towers with 4 cubes in each tower (3T4). This choice of numbers introduced slightly harder numbers (for him), which she presumed he could still work out by using each hand separately—one to account for accrual of CUs and the other for accrual of the 1s in each CU. Excerpt 3 starts after Nina asked Jake how many cubes he brought in all.

Excerpt 4: Folding back to additive reasoning when finding the total number of cubes (student: Jake; task 3T4; date: October 9, 2014).

6:12 Jake: I brought 11 cubes altogether.

6:16 Nina: [Lays the towers down on the table closer to Jake] Can you double check? (Researcher's note: Nina might have better asked Jake how he found 11 rather than asking to double check which can connote that the given answer was incorrect; nevertheless, asking him to check could have helped in orienting his Type-1 reflection.)

6:21 Jake: [Takes apart each tower into its individual cubes while counting them out loud but without keeping them as distinct groups.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12!

6:34 Nina: So how many (cubes) did you bring altogether? (This question supports Jake's Type-1 reflection but does not give any hints about which of the two answers was correct.)

6:38 Jake: 12.

6: 41 Nina: I want to know how you figured that out. [She picks up four cubes and reassembles them into a tower.]

6:42 Jake: [Reassembles another tower, albeit made of **5 cubes instead of 4 as asked for by the task**.]

6:56 Nina: [Reassembles another tower from the remaining 3 cubes, and asks Jake to give her a cube so she can make a tower of 4. Then, she tries to re-orient Jake's attention to how he keeps track of towers and cubes.] Jake, when you counted the first time, I saw you use your fingers [she folds her left hand's fingers to emulate his motions]. Can you tell me what you were doing?

7:08 Jake: [Folds down 4 fingers on his left hand, then folds down a finger on his right hand, apparently to stand for 'having counted the first tower's cubes'; repeats the process for the second and third towers, while left-hand fingers seem to stand for 1s and right-hand fingers for the compilation of composite units/towers.]

7:14 Nina: And why did you do this? [She replicates Jake's motions.]

7:25 Jake: Each finger was a tower.

7:28 Nina: Each finger was a tower? So can you show me again?

7:30 Jake: [Holds up his right hand with the palm face up.] Each finger was a tower. [He folds down three fingers on his right hand, one at a time.] That's the first tower, that's the second tower, and that's the third tower. [He then adjusts his hand motion, and folds down four fingers on his left hand, one at a time] If we add them all up we go 1 [folds down a finger], 2 [folds down a second finger], 3 [folds down a third finger], 4 [folds down a fourth finger], 5 [looks at his hand in a seeming 'oops' experience. After 2 seconds, he opens his hand] No, [He then raises three fingers on his right hand] No; this is 3 towers; and you [have to] count by 4s [He uses his left hand to put down one of the fingers on his right hand.] 4, [He raises his left hand counts four fingers] 5, 6, 7, 8 [He then uses his left hand to fold another finger down on his right hand and opens his left hand] 9, 10, 11, 12. [He folds down the third finger on his right hand and his body language indicates that he has finished.]

7:58 Nina: And you knew to stop at 12 because these were like the towers [She touches the fourth finger on *his* right hand] and this was like the 12th.

8:02 Jake: [Completes her sentence incorrectly]… The 12th one.

8:04 Nina: [Inquires about the unit type.] The fourth tower and the 12th what?

8:07 Jake: Tower

8:08 Nina: [Prompts for a response of cube.] The 12th?

8:13 Nina: [Places her hand on the real towers] How many towers do you have here?

8:16 Jake: [Corrects himself in response to her prompt] That was the last, third tower.

8:17 Nina: It was the last, third tower; but it was the 12th what?

8:21 Jake: Cube.

8:02 Jake: [Completes her sentence]… The 12th cube.

Excerpt 4 provided further evidence that numbers chosen for multiplicative tasks can both afford and constrain mDC as SLDs evolve in the ways in which they operate to solve multiplicative tasks. When challenged to solve a task with four as the unit rate, Jake's mental system was no longer freed from focusing on the accrual of 1s (as it had been when the unit rate was 2 or 5). Thus, he folded back to the counting of 1s (single cubes) that composed each unit. When prompted by Nina to reassemble the single cubes back into towers and then prompted to find out the total number of cubes Jake said, "this is three towers and you have to count by four." (7:30) but he required multiple prompts from Nina to successfully count the composite units of four. The differences in Jakes counting acts when presented with four as the unit rate including his initial reaction to take apart the towers and proceed to count in a single number sequence (1 through 12) without any allusion to their grouping (6:21) and his inability to independently keep track of his "counting- by" acts indicated that, with a unit rate of four, Jake was operating, at most, in what Steffe (1988) termed the Initial Number Sequence and was no longer reasoning multiplicatively. When reasoning multiplicatively (excerpts 1 and 2) Jake was able to double-count the items, simultaneously keeping track of the sequence of counting by 2s and 5s, and more importantly keeping track of where to stop. This coordination of at least two units in activity (Norton et al., 2015) was distinctly different from his activity with a unit rate of 4. With 2 and 5s, Jakes actions seemed purposeful in the sense of anticipating the need to coordinate two simultaneously accruing number sequences: CUs (towers, counting by 1s) with unit rate (cubes distributed over each of the towers, counting by 2s), with 4 he was unable to anticipate the coordination of units.

Jake's solutions to this more challenging (for him) task differed from his coordinated actions to solve the previous tasks (with unit rates of five and two). Specifically, as both his initial solution and past 'oops' solution indicated, he seemed to fold back to a previously constructed scheme, in which counting by 1s would be his way of accomplishing the goal of finding a total of singletons organized in a given number of CUs. This suggested that Jake's goal-directed coordination of counting two types of units (CUs, 1s) was still evolving and thus dependent on the numbers used in the task (in the sense of his facility with the multiples of the unit rate). As noted above, this way of operating supports the inference that his number concepts were rooted in the INS (Steffe, 1988).

While Jake initially folded back to an additive scheme when asked to double check his answer, his ability to model an mDC using both of his hands suggested transition (when prompted) toward an enactive form of a scheme for coordinating the number of CUs with the unit rate of 4. Jake (7:30) identified his left hand as the number of towers but counted past the unit rate of 4, stopped at 5, and started over. His unprompted 'oops' experience suggested Type-I Reflection, however, Jake remained unable to operate on the harder unit rate of 4 in the same way he could operate on the easier rates of 2 and 5.

In his second, adjusted attempt at the activity of coordinated counting, Jake was able to accurately use his fingers to keep track of the unit rate of 4 (7:30). He began his count not from 1 (cube), but from the first multiple of four, indicating the coordination of a first CU with the numerical value of the unit rate. Due to lack of facility with the next two multiples of four, he turned to using the left hand for counting accrual of 1s while keeping track of the CUs with fingers on the right hand and was successful in tracking three units: the 3 towers, the 4 cubes per tower, and the total number of cubes. This excerpt suggested that Jake's mDC could possibly be applied to solving tasks not constrained to the familiar unit rates of 2 and 5. Excerpt 5 below continues as Jake is engaged in a task of finding the total number of cubes in 3 towers with 6 cubes in each tower (3T6).

To further demonstrate the interplay between numbers chosen for a task and a child's ways of operating, Excerpt 5 presents Jake's solution to the task that immediately followed his solution to the previous task (4T3), finding the total of singleton items (cubes) in 3 towers of 6 cubes each. Here, the teacher/researcher decided to keep the number of towers at 3, but increased the unit rate (6 cubes per tower) so it exceeded the number of fingers on one hand. This choice was intended to test the hypothesis that the number of composite units of which Jake had to keep track may hinder his use of the mDC scheme. That is, she set out the task to examine how Jake would cope with the challenge of keeping track of a compilation of composite units while using fingers on both hands to keep track of a sequence of multiples with which he was not facile. It should be noted that, at this early stage of Jake's construction of a way to coordinate CUs and 1s, taking his previous solution as material for further operating on just two more towers of 3 cubes each was considered beyond his capacity (e.g., "In 4 towers I already have 12 cubes, so 6 towers imply 2 more towers, hence 6 cubes, so 18 in all"). It should also be noted that author of this study re-created the figures in Excerpt 5 based on her field notes, because the student's original drawing could not be extracted from the video recording.

Excerpt 5: Folding back to additive reasoning when finding the total number of cubes (student: Jake; task 3T6; date: October 9, 2014).

22:49 Nina: How many cubes did you bring altogether?

22:52 Jake: [Silently touches each cube in the first tower, perhaps re-counting them all. He then raises and stares at his two hands for 1 second, put his hands down, reaches for a pen, looks back at the cubes, points to the cubes for 1 second, and writes on a small white board the number 6 first and then the letter T above it.]

T
6

23:14 [He looks back at the towers on the table and writes on his white board another set.]

T	T
6	6

23:19 [He looks back a third time at the towers, and completes his writing to correspond with the numbers given in the task.]

T	T	T
6	6	6

23:34 [Seemingly not knowing how to proceed by using this organization on paper, he turns back to the real towers, taps on each of 8 cubes individually (all cubes in the first tower and two from the second), stops, and erases the white board completely. At this point, he points with his finger to each individual cube in the first tower and begins counting by 1s, then shifting to counting by twos to expedite the process] I brought 18 cubes altogether.

23:53 Nina: [Not indicating the correctness of the answer.] How did you figure it out?

23:55 Jake: [Picks up all of the towers and then puts them back on the table] I figured it out a different way; I just counted it this time. I used the cubes [picks up one tower]. I pretended I broke it up like … [breaks off individual cubes from each tower]; like 1, 2, 3, 4, 5, 6; that is a tower I counted.

24:43 Nina [Asks the question of the whole group.]: Did it get a bit harder today when the numbers got harder?

24:48 Jake: Not for me.

24:49 Nina: Was it harder with 4s?

24:50 Jake: Six

24:52 Nina: The six was hard for you. Why was the six hard?

24:54 Jake: It was hard for me because I wanted to use the way I did last time [holds up both hands and shows the hand motions that he made when using his fingers to re-present a coordinate, double-count with 4s]; but I don't have as much fingers.

25:04 Nina: Oh yeah, so you couldn't, you couldn't, you don't have as much fingers so does that mean you couldn't? What do you mean by that [shows holding up of both hands]?

25:11 Jake: I had to do it a different way.

25:13 Nina: What do you mean you don't have as much fingers?

25:15 Jake: I only have 10 fingers and … [rebuilds the three towers of six using the cubes on the table] Since … So if there are towers of six [holds up both hands] and I went like that [begins to count out six on his right hand]; so I wouldn't have enough fingers.

Excerpt 5 provides further evidence that numbers Jake operated on made a difference in his goal-directed activity. When presented with a unit rate of 6, for which he had neither a mental number sequence nor enough fingers to create a figural re-presentation in the way he had for the easier unit of 4, he was unable to complete a coordinated count (22:52-23:53) as he could use for solving tasks with unit rates of 2 and 5.

Jake's first, spontaneous attempt to re-present the towers was to write a "T" (for tower) and the number "6" under each "T" to indicate how many 1s constituted that CU (22:42). This indicates his ability to properly re-present the quantities involved in the task, that is, 3 CUs, each made of six 1s. While insightful and resourceful, this symbolized re-presentation did not proceed to a double-counting activity. Seemingly having no other recourse, Jake erased the whiteboard, indicating abandonment of this initiative (23:34). This indicates that, at least with 6T3, Jake was yet to call forth the double-counting activity for figural items of those towers and cubes. Instead, he thus folded back to counting each of the tangible cubes (albeit shifting from counting by 1s to counting by 2s to expedite the process). Jake's explicit utterance, about not having enough fingers (24:54), indicated an acute awareness of the need coupled with inability to operate in a coordinated way when the number in one unfamiliar CU precluded using each hand for a different component of the coordinated count. This was evident in his show of two hands, used to this end for 3 towers of only 4 cubes each, and the statements that followed ("I had to do it a different way"; "I only have 10 fingers."). Coupled with Jake's inability to operate on re-presented (diagrammatic) figures that could, potentially, replace the need for fingers, Jake's explicit utterance indicated a significant state of conceptualization for him. For him, re-enacting the actual count of both types of units depended on availability of action-enabling figural items, not simply "idle" items that he could create, but not yet operate on.

Next, Excerpt 6 provides further data on the constraints that harder numbers can place on a child's construction of multiplicative thinking early in the participatory stage of development. In this excerpt, Luke was asked to determine the total number of cubes in 6 towers with 3 cubes in each tower (6T3). Although he had established that he had brought 6 towers and that there were 3 cubes in each tower, he was unable to attempt a solution. Furthermore, when the teacher uncovered the cubes, Luke counted them by 1s. This excerpt thus serves to further illustrate the impact of numbers chosen for a task on a child's operations.

Excerpt 6: folding back to an additive scheme when finding the total number of cubes (student: Luke; task 6T3; date: October 16, 2014).

0:35 Nina: How many cubes do you have in all?

0:40 Luke: [Looks forward and moves both of his index fingers up and down under the table. He brings his fingers closer to the covered cubes and moves them up and down but does not make any overt motions with his hands or body that could indicate he is counting. When a student sitting next to him raises her thumb to indicate she has found an answer, Luke puts his head down on the table to look at the cubes under her paper.]

1:00 Nina: Luke, do you have an answer?

1:11 Luke: [Puts four fingers out on each of his hands (his hands are under the table). He looks at his fingers and moves them appearing to count four fingers on each hand.] Twelve (12, whereas the correct solution should be 18).

1:21 Nina: How many cubes are there altogether?

1:27 Luke: I …brought…. 3…12 cubes together.

2:00 Nina: [Returns to Luke after asking the other students in the group how many cubes they think there are; they gave different answers than Luke's.] I want you to lift up the cover and check your answer. Did you notice everyone got a different answer?

2:02 Luke: [Removes the cover and counts the cubes by 1s.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18. Eighteen (18); oh, I got it wrong.

[Indicates that he recognizes that his initial response of 12 was incorrect—an instance of externally-prompted reflection Type-1].

5:24 Nina: [After checking the answers with the other students, prompts Luke for an explanation of his first answer of 12 when he seemingly used groups of four to figure out the total number of cubes.] Luke, how did you work to figure out that you thought the answer was 12 cubes altogether?

5:39 Luke: I thought that the cubes were six to 12 because there was a problem with 6+6 and it equaled to 12 and I thought it was the answer.

Luke's initial hesitation to solve this task (0:40) showed that the combination of harder numbers (6 towers of 3) diminished his ability to create a figural representation. With these numbers, not only was he unable to operate on the CUs but also (unlike Jake) unable to re-present the CUs. Luke thus folded back to the additive scheme of counting all single items (1s) and was not able to create a figural representation of the 6 towers and 3 cubes per tower that he had previously established. Compared to Luke's ability to create such a re-presentation in previous tasks with "easier" numbers, this suggests that for a student like Luke, more difficult numbers may not only impact the student's shift from a multiplicative to an additive scheme, but they can impact the child's available number sequence.

When allowed to see the towers and cubes, Luke did not re-present all or part of the 6T3; rather, he put out four fingers on each of his hands and put forth an answer of 12. Luke's mental operations are difficult to infer to this point. Without the cubes, he was initially unable to engage in even additive schemes of operation, which was in contrast to the ease with which he could operate multiplicatively with the unit rate of 5. The less familiar numbers presented in the 6T3 task constrained his ability to reason multiplicatively. In solving the 6T3 task, Luke's way of operating when solving multiplicatively seemed consistent with Steffe's (Steffe, 1988, 1992) notion of the Initial Number Sequence (INS) in that Luke anticipated a larger than 1 being a composite unit (Luke could state that under the cover there were 6 towers with 3 cubes in each tower), but he had not yet constructed three as an iterable item in and of itself and therefore could not coordinate (operate on) the six towers with three cubes per tower.

With the covers removed, and a prompt from the teacher that answers from other students were different from his answer of 12, Luke counted the cubes by ones (folded back to an additive scheme). This count of each and every unit of 1 enabled him to reach the correct answer (18), while indicating no attention to the embedding of each six 1s within a unit (tower). On the other hand, having reached the different answer, Luke immediate indicated his recognition that the initial answer (12) was inaccurate. This externally-prompted reflection Type-1 suggested that Luke could reflect on his goals and resulting mental activity but only when prompted to do so. The change in the number of composite units and the unit rate involved in this task seemed to reveal Luke's construction of numerical operations consistent with Steffe's (Steffe, 1988, 1992) notion of INS, which does not provide for operating on "harder" numbers.

What supports this analysis is that, when prompted by Nina to explain how he got his original answer of 6, he easily doubled 6 (5:39), which was not necessarily helpful in the context of this 6T3 problem as seen by the child. That is, "doubling" as an "easy" operation was available to him for the number 6 (indicating enactive operation on the given CU). His use of doubling, however, gave a strong indication that Luke's initial response grew out of a constrained activation of the mDC scheme due, specifically, to the choice of numbers. Whereas Luke's solutions to tasks with easier numbers could suggest he was able to operate multiplicatively on figural items (representations of cubes on his fingers), in this multiplicative situation his newly evolving scheme for coordinating the activity on 1s and on CUs could not be applied. Thus, he folded back to counting perceptual objects (singletons) to determine the total number of cubes.

Based on Luke's folding back to operating on 1s, Nina recognized that tasks with more difficult numbers *and* with the cubes covered did not allow Luke to operate multiplicatively as he had in the 4T5 problem (Excerpt 2). Accordingly, for the following task she presented him with five towers with four cubes in each (5T4) and asked him to leave the towers uncovered, hypothesizing that making the countable items available to Luke might make a difference in his actions and operations. In Excerpt 7, with the covers removed, Luke again counted by 1s to figure out the total, which provides further support to the claim that his newly evolving multiplicative scheme was limited to CUs (unit rate) of "easy" numbers.

Excerpt 7: Folding back to an additive scheme when finding the total number of cubes (student: Luke; task 5T4; date: October 16, 2014).

15:52 Nina: How many towers did you bring?

15:54 Luke: Five (5).

15:55 Nina: Five towers. Everybody should have five towers. [Nina instructs some students to cover and asks Luke to leave his towers uncovered.] How many cubes [are] in each tower?

16:15 Nina: [Asks Jake, another student in the group] How many cubes [are] in each tower, Jake?

16:17 Jake: There are four cubes in each tower.

16:32 Nina: [Asks Luke] How many cubes altogether?

16:36 Luke: Ahh… [He holds 2 towers of 4 in his left hand and 3 towers of 4 in his right hand. He moves the towers about in his hands, but does not make any finger, head or body movements that could suggest counting.] There should be 4 cubes. [Possibly referring to the number of cubes per tower.]

17:04 Nina: [Reorients Luke to the task of finding the total number of cubes.] There are four cubes in each tower. How many cubes altogether?

17:11 Luke: [Looks at each of the cubes and repeatedly nods slightly, which indicated he is counting all cubes by 1s.] Eighteen (18, which is incorrect as there are 20 cubes in all).

17:14 Nina: [Prompts Luke to the class norm of answering in complete sentences.] Can you say, 'there are eighteen (18) cubes altogether'?

17:15 Luke: There are eighteen (18) cubes altogether.

17:23 Nina: Can you tell me how you figured it out?

17:26 Luke: [Shows awareness of counting as a solution strategy] By counting

17:28 Nina: Can you show me what you did?

17:30 Luke: [Places the towers flat on the table and begins to count by 1s. He loses track in the middle of his count and recognizes that his count is off when he reaches 17 and still has several cubes left to count. He starts counting the cubes again.] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17… awww, [Nina prompts him to keep going so he continues counting by 1s until all cubes are accounted for] 18, 19, 20; wrong answer again!

[Again, Luke shows an awareness that his answers are not consistent.]

In this excerpt, Luke's difficulty with the question of how many cubes there were in all 5 towers (16:36) indicated that it was not just the covering of towers that made tasks with harder numbers difficult. Even uncovered, the 5T4 task with the harder number 4 (for the unit rate) changed the way Luke could operate. When Nina prompted him for a second time (17:04) to figure out the total number of cubes, he interpreted the task correctly. Luke's re-presentation of the cubes with head nods suggested that with these numbers he was able to generate figural units (his head nods), however he again folded back to operating on units of 1 to determine the total number of cubes. Similar to his response to the previous task (6T3), his "awww" (17:30) when reaching 17 in his count of 1s indicated an 'oops' moment. Although he did not explicitly say which answer would be the correct one (18 or 20), his continual to count after that 'oops' experience indicated he considered his original answer of 18 as incorrect.

Excerpt 7 suggests that Luke was yet to conceive of CU (numbers larger than 1) as iterable units. This is the reason suggested for why his ability to operate (pseudo) multiplicatively when given 5T4 was different than when given the tasks of 6T3 and 5T4. In the 4T5 task, Luke could form a rudimentary anticipation of the need to coordinate the accrual of the compilation of 4 CUs with the unit rate of 5 possibly because his familiarity with the multiples of 5 allowed him to focus on at least one count. In contrast, the multiples of 3 in the 6T3 task created a situation in which Luke was pushed to operate mentally on both a unit rate and a compilation of CUs. It appeared that the difficulty of the numbers in this task did not allow Luke to use his evolving coordination of 1s and

CUs, so he folded back to counting all of the available figural objects (the cubes) by 1s.

Theme 1 summary. In the seven excerpts presented in this section, all pulled from early teaching episodes, both Jake and Luke demonstrated an enactive anticipation of mDC when tasks were presented with the "easy" numbers of two and five for the unit rate and the number of composite units did not exceed five. When given the easy numbers, both students initiated and completed a coordinated count, which was supported by the ease with which they could count by those known sequences and the awareness of the counting by operation. (Both students used the term "counting by" when explaining their solutions and thinking when the unit rates were 2 or 5: Jake in Excerpt 1: 13:53; Luke in Excerpt 3: 17:15 & 17:52.) The introduction of more difficult numbers, particularly for the unit rate, (three and four initially), provided enough of a difference in task difficulty that both students were unable to independently initiate and complete a coordinated count and instead folded back to counting all, an additive approach to finding the answer that did not promote multiplicative thinking. In this sense, Jake and Luke's cases support the idea that children may use more "mature strategies" when given easier problems (Gersten, Jordan, & Flojo, 2005), while adding specifics to that idea in the process of shifting from additive to multiplicative reasoning.

Jake and Luke's folding back to additive strategies of operating on 1s suggests two important points. First, it highlights the need for attention to maturity of scheme development, not just maturity of a solution strategy. Said differently, when "harder" numbers chosen for a task seem to be a factor in the child's solution strategy, a teacher may conclude the child's scheme is yet to be constructed. Second, in early stages of mDC construction, easier numbers, defined in this case as numbers for which SLDs sequence of multiples was available (e.g., unit rates of 2 and 5), seemed to open the way for initial progress toward multiplicative thinking. At that point in their development, harder numbers seemed to halt Jake and Luke's construction of the mDC scheme. Key for teaching is the realization that, in spite of a child's ability to correctly solve a given task with harder numbers (e.g., by folding back to operating on tangible units of 1), children at the early stages of constructing mDC may need to first focus on cementing their new way of operating by solving tasks with easy numbers.

Theme 2: Transition to Figural Representations with Numbers that Exceeded the Number of Fingers on One Hand

To illustrate the impact of number choice on the progression of student multiplicative thinking, this section includes analysis of three data excerpts from Jake as he constructed a participatory stage of mDC. Excerpts 8-10 are from a single teaching episode with Jake, approximately two weeks after he had folded back to operating on units of 1 when solving the task with 3T6. In these excerpts, Jake demonstrates his transition from folding back when faced with harder numbers, which exceeded the number of fingers on one of his hands[6], to a participatory stage of a multiplicative scheme when solving a task with the harder numbers (7T6).

Additionally, these episodes show how Jake developed new figural representations for problems in which the compilation of CUs, the unit rate, or both exceed the number of fingers on one hand. Excerpts 8-10 provide evidence that Jake did not fold back to operating on 1s when faced with the more difficult numbers. Rather,

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[6] In this study, such harder numbers are denoted "*5+n*," wherein the *5* accentuates the importance of the number of fingers on one hand of the student and the *n* represents any natural number added to five. In this study the largest number used was 12, or *n+7*.

while Jake required prompts and experienced many ‘oops’ moments, these excerpts provide evidence that harder numbers could then support his development of a figural re-presentation for numbers that exceeded the number of fingers on one hand. Consequently, using those numbers seemed to promote his transition toward an anticipatory mDC, evidence of which would be the transfer of the mDC scheme to contexts other than the PGBM game with cubes and towers.

In Excerpt 8, Jake makes several starts and restarts, each with ‘oops’ moments and each indicating emerging awareness that he was not reaching his own goal (which had been easy for him when given problems with a unit rate of 2 and 5). In these excerpts, Jake explained his construction of a method to keep track of two counts even when one of the counts exceeded the number of fingers on his hand. This is considered an important landmark for Jake, because in Excerpt 7 he stated that there was not a way to keep track of this type of task. In this instance, Nina’s choice of numbers greater than five for both the compilation of composite units and the unit rate did not cause Jake to fold back to counting by 1s. Rather, it created a situation for which Jake had to engage in a goal-directed activity while finding novel ways to keep track of CUs that exceeded the number of fingers on one hand. His use of multiplicative thinking despite the harder numbers indicated advances in his participatory stage of mDC.

Excerpt 8: Multiplicative reasoning when finding the total number of cubes (student: Jake; task 7T6; date: October 23, 2014).

7:45 Jake: So 7x6 [Writes 7x6 on his paper then raises both hands above the table. On his left hand, he folds down five fingers to indicate five counts starting with his thumb and using all of the fingers on that hand. He then counts the last two counts on his right hand using his right thumb and index finger to keep track of the 6th and 7th counts. He repeats this process a second time and then pauses, puts his hands down, and erases the 7x6 that he had written on the paper. Next, he raises both hands again, then puts them down, and re-erases what's written on his paper. Forty seconds elapse. He then raises both hands again and counts five counts on his left hand as he had done initially and makes two more counts on his right hand. He draws one vertical line on his paper and sets down the pencil and raises both hands. Jake begins to count again. He folds the index finger of his left hand down and then folds the other four fingers on his left hand and then continued to count the thumb index and middle fingers of his right hand. He stops again, puts his hands down, and then erases the line that he had drawn on his paper. He stops and looks toward Nina. He raises his paper to show her.] I don't have an answer.

9:11 Nina: That's OKAY, take your time.

9:17 Jake: [Returns to his paper and writes 7x6 again, then counts five on his right hand and two on his left hand twice seemingly counting by ones to 14. He places both hands on his head and taps his head as he looks down at the paper. He begins counting again, this time counting all five fingers on his left hand once and then recounting the thumb of his left hand (finally establishing a way to keep track of 6 cubes per tower). He lowers the thumb of his right hand and then counts another six on his left hand by counting each of his fingers once and then recounting his thumb. He lowers the index finger of his right hand (indicating a second tower) stops his count at 12 and places both hands on his head again.]

Jake's repeated attempts to keep track of 7 towers of 6 suggested an initial construction of a participatory stage of the mDC scheme; each of Jake's stops and starts (7:45) suggested he was monitoring his own goal-directed activity of counting both the 1s and the CUS. In his first attempt (7:45), Jake counted two sets of 7, suggesting an initial confusion about which number in his number sentence represented the number of towers and which number represented the number of cubes per tower. Without a teacher prompt, he made a correction to his work (7:46) that seemed to promote his own ability to solve 7T6 rather than his original attempt of counting towers of 7. In his second attempt (9:17), Jake firmly established a way to keep track of the unit rate (6 cubes per tower) for at least the first two of the seven towers. It is noteworthy that, while Jake had not yet been able to answer this task, he did not count all by 1s as he did in Excerpts 4 and 5 with these even more challenging numbers.

The behaviors observed in Excerpt 8 support the inference that now, even with harder numbers, Jake had constructed an enactive anticipation of mDC. In Excerpts 4 &5, harder numbers inhibited his ability to operate on (coordinate) counts of both the compilation of CUs and the unit rates. In contrast, in Excerpt 8 he showed an enactive anticipation of the need to somehow keep track of the unit rate. Key to this claim is not the extent to which Jake was successful in keeping track, but rather in his persistence to find such a method, which provided a compelling evidence for the inference he anticipated the need to do so. Moreover, he could then transform his goal-directed activity to include counting of six items on just a single hand, by counting all five fingers on one hand and then recounting his thumb. This latter strategy seemed to indicate an enactive anticipation of the need to keep track of the compilation of CUs. Similarly, to keep with his novel goal of coordinating the counts, he also successfully kept track of the accrual of 1s – at least until 12. In this sense, data of Jake's work in Excerpt 8 mark the first instance of *his initiation* of a multiplicative operation on more difficult numbers – numbers that previously (excerpt 7) were precluding his intentional, coordinated operation on both types of units (CUs, 1s in each CU).

Based on Jake's work as described in Excerpt 8, a shift in Nina's instruction and how it further fostered Jake's transition is described in Excerpt 9. Here, she elected to orient Jake's reflection on the accrual of 1s, while continuing to refine the ways in which he tracked the compilation of CUs (towers) and the unit rate that both exceed the number of fingers on one hand.

Excerpt 9: Tracking towers that exceed the number of fingers on one hand (student: Jake; task 7T6; date: October 23, 2014).

10:19 Nina: Can I help you?

10:21 Jake: [Nods, indicating that he would like help.]

10:22 Nina: [Orients Jake's attention to his available methods for the simultaneous tracking of towers and cubes.] Which hand are you using for towers?

10:23 Jake: [Holds up his left hand.]

10:25 Nina: And the cubes?

10:25 Jake: [Raises his right hand.]

10:27 Nina: When you have one tower [raises the thumb of her right hand] how many cubes do you have?

10:34 Jake: [Raises the thumb of his left hand and looks at his right hand.] Six

10:35 Nina: Six cubes. And when you have two towers [raises the index finger and thumb of her right hand and motions for Jake to do the same] how many [cubes] do you have? [Notices that Jake has only raised his thumb and prompts him to raise two fingers to represent two towers.] Two towers, two towers.

10:40 Jake: [Raises his index finger so that now both the thumb and index fingers of his left hand are up.]

10:42 Nina: How many cubes?

10:50 Jake: [Appears to guess] Four? Eight?

10:56 Nina: [Raises her right thumb prompting Jakes attention toward how many cubes he had in the first tower.] You had six here, right?

10:58 Jake: Four?

11:00 Nina: [Continues to orient his attention to the established 6 cubes in one tower.] You had six here, right? Yeah?

11:03 Jake: [Ignores the question about how many were in one tower but accurately answers how many cubes are in two towers. Looks at his thumb and index fingers.] 12.

11:04 Nina: [Raises the middle finger of her right hand so that her thumb, index finger and middle fingers are now up. Wiggles her middle finger.] What about here?

11:08 Jake: [Places his right hand on his head and counts under his breath.] 13, 14, 15, 16, 17, 18; Eighteen!

11:12 Nina: [Orients Jake's attention to the accrual of towers.] So tell me how many times have you counted now?

11:13 Jake: [Raises the thumb, index, and middle finger of his left hand, indicating 3 counts of a tower, that is, having counted 3 towers.]

11:14 Nina: Now what about four towers? [Does *not* continue raising her fingers as a demonstration, but rather lets Jake continue the process of keeping track of towers on his own.]

11:18 Jake: Holds his right hand up to his head and accurately continues the count.] 19, 20, 21, 22, 23, 24 [using each of his fingers on his right hand once and his thumb twice.] Twenty-four!

11:21 Nina: That's four [towers], right? What about five towers?

11:27 Jake: [Holds his right hand up to his head and counts] 25, 26, 27, 28, 28, 29 [using each of his fingers on his right hand once and his thumb twice.] Thirty (30)?

Excerpt 9 provided evidence of the tentative nature of student thinking when in a participatory stage of an mDC scheme. Just minutes before Excerpt 8 (9:17), Jake had independently established a figural re-presentation of two towers on his right hand and had independently stopped the count for the accrual of 1s at 12. Yet, Excerpt 9 shows he was not able to answer Nina's question about how many cubes were in 2 towers (10:50); instead, he answered with the numbers 4 and 8. Thirteen seconds later, there was some evidence that Jake could find the effect of an activity of counting, through figural re-presentation of the covered objects, 2 towers with 6 cubes each. When Nina re-oriented

Jake to the number of cubes in one tower, he answered 12 without appearing to count figural items as he did in subsequent counts. This seemed to indicate Jake was beginning to operate multiplicatively on at least the first 2 towers of 6 cubes. It should be noted that, initially, Jake did not seem to assimilate Nina's prompt of how many cubes were in the second tower; yet, he corrected the unit rate (from 4 to the proper 6) and found the total number of 1s. As the session focused on teaching and not on fine-grain assessment of Jake's mDC, Nina's use of tasks does not allow to determine whether Jake was able to hold the coordinated count independently. With Nina's prompts, however, he was able to first follow her demonstration and then independently continue a coordinated count. Specifically, he continued using *his fingers intentionally* to keep track of the 4^{th} and 5^{th} composite units ('towers') in the compilation while also counting the total accrual of 1s (11:08, 11:18, and 11:27).

Though unfamiliar with the multiples of 6 past 'doubling' to make 12, with Nina's explicit prompt Jake began using a figural re-presentation of the CUs (hidden towers) to keep track of the compilation of those units of 6, while simultaneously tracking the accrual of 1s. It seems that Jake had thus established a way to re-present numbers greater than the number of fingers on one hand, which allowed for at least an enactive mDC operation on numbers greater than 5. This is in contrast to Excerpt 5, where his ability to operate on CUs was constrained by figural representation on one hand (he could only re-present a unit rate of 5 or less because his other hand was busy tracking the compilation of composite units).

Excerpt 10 below follows Excerpt 9 within the same teaching episode. Here, Nina oriented Jake's attention onto tracking the accrual of towers beyond the initial five as he tracked each of the unit rates and the total accrual of 1s. She did that because, although Jake had thus far established a way to track a unit rate that exceeded the number of fingers on one hand, he was yet to simultaneously track the compilation of CUs that exceeded 5 without heavy teacher prompting.

Excerpt 10: Orienting attention to the accrual of towers beyond 5 (student: Jake; task 7T6; date: October 23, 2014).

11:36 Nina: [Interacts with Jake to provide a model for continuing beyond the number of fingers in one hand. Jake already uses this model to keep track of the 6 cubes per tower and Nina builds on his established thinking to promote a way to keep track of the total number of towers.] Ok, You've got five towers, right? [She raises her right hand with all five fingers extended to indicate Jake has already counted 5 of the 7 towers with 6 cubes each.] I'm going to use this thumb as a six and I'm going put it down to say six, so in six towers how many are there? That way I can use this hand [wiggles fingers of her left hand] to say cubes [She folds her right thumb down.]

11:47 Jake: [Considers this method but does not move either hand.] Hmmmm.

11:49 Nina: We were at 30 for five [towers], and now we're counting the sixth tower. [She folds down her thumb again.]

11:54 Jake: 36

11:59 Nina: Now we need to count the ….[Emphasizes folding down her index finger to meet her folded down thumb twice.]

12:06 Jake: [Answers incorrectly] Second tower.

12: 09 Nina: This was six towers [Folds in her thumb] …this is going to be? [Folds her index finger up and down toward her thumb supporting Jake to continue the count at 7.]

12:11 Jake: The fifth tower?

12:12 Nina: [Raises her left hand with all five fingers extended.] This was five [towers], which was 30 [cubes], and you said the sixth one was 36 [cubes]; and now we're going to count the… [Moves her index finger down toward her folded in thumb again.] Which tower are we counting next? [Proceeds to presenting an incorrect prompt to determine if Jake can detect 'her error'.] The tenth tower? [She motions her index finger up and down.] Is it the tenth tower that is next? [Continues to motion up and down with her index finger.]

12:40 Jake: [Shakes his head no, indicating disagreement.]

12:40 Nina: What number [of] tower are we counting next?

12:45 Jake: The seventh.

12:46 Nina: The seventh tower. The sixth was 36 cubes. How many are in seven?

12:53 Jake: [Uses all of the fingers on his left hand to five and then reuses his thumb for the sixth count] Forty-two (42).

12:57 Nina: Write it down so we can see.

Excerpt 10 showed that, with a very heavy, specific prompting from Nina, Jake was able to continue the count of 5+n towers. Nina modeled a method for tracking intended to orient Jake's reflection specifically on how he could use his fingers to keep track of numbers greater than 5 and still use one hand for towers and the other hand for cubes. She did this because Jake had previously demonstrated that he could re-use his thumb to keep track of the sixth count when keeping track of the *unit rate* (cubes in each tower). She realized, however, that for counting how many CUs have already been accounted for, he continued to struggle past the fifth tower. While eventually capable of attributing the proper number to the sixth and seventh fingers (for towers), Jake's initial responses of "two" and "5" for the finger Nina raised for the 7^{th} tower indicated that, for him at the time, the composite unit of 6 was not iterable. This was yet another example of a child who could follow and use the strategy of keeping track, which he carried out intentionally up to 5 CUS, but not yet using a scheme indicating a CU, not just 1, is an iterable unit for him.

Nina attempted to model re-using the thumb and index finger of the hand on which towers were counted for the sixth and 7th counts (11:36) because, in her mind, it was similar to Jake's way of keeping track of the units of 1 (6 cubes) that constituted each CU (tower). What she did not consider was the difference in the nature of units Jake could operate on; 1s were already iterable for him whereas CUs were (at least 6s) were not. Thus, she counted all five fingers of one hand and then continued the count re-using the thumb for the sixth cube, which she expected Jake to easily assimilate and use, too. As noted above, Jake did not show an understanding of her demonstration, indicating the lack of iterable CU for the mDC scheme past 5. *For him*, the additional fingers *Nina* had raised for CUs did not constitute a continuation of the count of the first five CUs.

To further analyze students' (limited) transition to a participatory stage of tracking of accrual of both CUs and 1s we return to Luke's case. In Excerpt 11, which occurred in the same teaching episode as excerpts 8, 9, and 10 with Jake, Luke solved a task of how many cubes are in all-hidden 3T4. This teaching episode took place one week after the previous teaching episode, in which Luke had folded back to counting all (6T3 task, see Excerpt 7). Excerpt 11 provides evidence of a new way of operating for Luke. Here, instead of reverting to counting all 1s, he could successfully use his fingers to re-present the hidden cubes and was able to coordinate counts and determine the total number of cubes (the set of 3T4 was covered before the students could determine the total number of cubes).

Excerpt 11: Creating a figural re-presentation for covered towers (student: Luke; task 3T4; date: October 23, 2014).

4:13 Luke: [Uses his right index finger to point to the tip of thumb, index, middle and ring fingers of his left hand three times, (he does not use the right hand to clearly indicate that he has counted three towers) stops, and writes 12 on his paper.]

4:28 Luke: Easy e easy! [Taps his pencil in excitement.]

Excerpt 11 marks a transition in Luke's construction of an mDC scheme. He was able to successfully create a figural presentation of the cubes for a number, 3, that for him (a) was not a known counting sequence for him and (b) had previously been a more difficult number (Excerpt 6). In this case, Luke did not fold back to operating on 1s, but rather independently created a unit of 4 through sequentially pointing to parts of his finger. Furthermore, he could keep track of three counts of those units of 4 on only one and the same finger, suggesting that both four (1s) and three (CUs) were no longer numbers that constrained his ability to coordinate the accrual of those units.

Based on Luke's "easy" solution to the 3T4 task, Nina decided to further promote, and assess, his construction of a participatory stage of the mDC scheme. As shown in Excerpt 12, she thus engaged Luke in solving the 7T6 task. His work suggested further construction of the participatory stage, namely, not folding back to operating on 1s even with numbers greater than the number of fingers on his hand. Instead, he constructed a unique way to attempt to re-present and keep track of 5+n towers and cubes, similarly using different spots on his fingers as possible markers for each of the towers. In turn, this strategy allowed Luke to use the fingers of both of his hands to keep track of the accrual of cubes. It should be noted that, although Luke's answer was inaccurate, this attempt to coordinate counts and his awareness of the need to keep track of the towers marked a transition in his goal-directed activity when presented with multiplicative situations in which numbers were not memorized as a sequence of known multiples.

Excerpt 12: Creating a figural re-presentation for towers and cubes with 5+n numbers (student: Luke; task 7T6; date: October 23, 2014).

7:50 Luke: [Rubs his palms together for two seconds and begins using his right index finger to count the tips of the five fingers of his left hand and then raises the thumb of his right finger and counts quietly.] 1, 2, 3, 4, 5, 6. [He then starts over with the thumb of his left hand but points slightly lower on each of the fingers for another six counts.] 7, 8, 9, 10, 11,12. [He repeats the procedure each time pointing to a slightly lower place on his left fingers and then counts past the seven towers to eight towers and finishes with a total of 48 cubes.]

8: 30 Nina: Why did you stop at 48?

8:33 Luke: Because it's seven towers.

8:35 Nina: So what told you to stop?

8:38 Luke: Because [Shakes his head and shakes both of his hands possibly suggesting that he knows the answer may not be correct] I couldn't get it because it's seven towers and I don't know if I kept track of seven towers. It's hard [moves both of his hands forward and backward in front of him again and then moves his right hand to his forehead and then back down to the table in a gesture suggesting he was not clear on how to keep track.]

8:46 Nina: You had trouble keeping track of seven, yeah, that's ok.

8:50 Luke: [Gestures with his right hand toward the covered cubes] Yeah, it's just… I just don't like covering them up.

8:52 Nina: I totally understand.

Excerpt 12 provided evidence that Luke has began anticipating the need to keep track of CUs accrual while establishing figural representations for 5+n numbers—an important step toward constructing the mDC scheme. In excerpts 11 and 12, Luke no longer folded back to operating on 1s (counting-all), not the least because he seemed to have established an anticipation of the need to keep track of the CUs so he could stop at 7 (towers). This need was indicated specifically in his utterances related to the extent he was not sure if he actually did count 7 towers. These utterances suggest Luke was aware of the need not just generally to keep track but specifically to also do this for the number of towers given in the problem. Indeed, his method for accomplishing such a goal was successful for smaller numbers (Excerpt 11), but numbers greater than the number of fingers on one hand was still constrained by the difficulty of executing the activity toward the anticipated effect of stopping the count of CUs. In this regard, Luke's awareness

(Reflection Type- I) that his stoppage was not certain to be seven towers indicated an important transition, from folding back (counting-all) to maintaining the newly constructed activity even though it could produce an effect of which he was uncertain.

Returning the focus to Jake, here two episodes after the one presented in Excerpts 8-10, Excerpt 13 provides data on how he solved a task involving 8 towers with 12 cubes each (8T12). During this teaching episode, the students were asked by Nina to pose tasks for each other and chose larger numbers than Nina had presented thus far. Jake's solution suggests two changes in his construction of a participatory stage of the mDC scheme: adjusting the method for keeping track past 5 and handling a unit rate of 12 (which exceeded two hands). To accomplish his goal once reaching the fifth tower, Jake created a tracking system whereby he switched hands. For towers past the sixth, he used his *right* hand to keep track of towers and his *left* hand to continue to keep track of cubes.

Excerpt 13: Developing ways to re-present towers and cubes with 5+n numbers (student: Jake; task 8T12; date: November 18, 2014).

25:00 Jake: [Raises his left hand to keep track of towers and his right to keep track of cubes.] 12, [says something inaudible then starts his count again, folding his right thumb in] 12, [Counts the five fingers on his right hand twice and then his index and middle fingers a third time for the 12 counts representing the unit rate.] 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 [Folds down the middle finger of his left hand and uses his right hand to keep track of another count of 12.] 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36 [He folds down the fourth finger of his left hand and uses his right hand to keep track of another count of 12.] 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48 [Folds down the pinkie finger of his left hand and keeps track of a fifth count of 12 on his right hand] 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60.

25:22 Nina: [Purposely pauses Jake mid- count to attempt to establish a conscious awareness of the number of towers he has counted so far.] How many towers have you counted?

25:24 Jake: Five.

25:25 Nina: Ok, How many are you trying to count?

25:29 Jake: Eight [Continues counting on his right hand.] 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72… 72.

25:39 Nina: [Continues to orient Jake's attention on total number of towers thus far.] Ok, yes; so how many towers is 72?

25:43 Jake: Six.

25:45 Nina: So how are you going to show that?

25:47 Jake: I run (sic) out of fingers.

25:49 Nina: [Supports the same model of keeping track that Jake currently uses to keep track of the 12 cubes per tower.] Is it possible you could show it on this finger on this hand? She pointed to Jake's *left* thumb- this is the hand that he had been using to keep track of towers up to this point.

25:52 Jake: Oh yeah; did I say 72? [Jake raises the thumb of his *right* hand; the hand he had, to this point, consistently used for keeping track of cubes per tower and begins counting on from 72 on his *left* hand] 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84. Jake raises the third finger of his *right* hand and begins counting on his left hand for another 12 counts.] 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96 [He raises a fourth finger and continues counting 97, 98.[At this point Jake has developed his own, novel way, to keep track of towers beyond the fingers in one hand. He has switched hands for the sixth count and beyond.]

26:10 Nina: [Interrupts] How many towers is that?

26:12 Jake: Eight.

Excerpt 13 indicates that Jake's goal of finding the total of cubes in 8 towers of 12 cubes each led to an anticipation of the need to keep track of both CUs and 1s in the given task. This was a challenging feat, because to execute the activity of counting while keeping track of both units required a sophisticated adjustment. First, independently, he introduced a way of counting what for him seemed iterable units of 1—by raising fingers on a single hand once (five 1s), a second time (ten 1s), and two more fingers (twelve 1s in all). This is an important change, as it reflected his decomposition of 12 into 3 sub-units within the activity. Most importantly, it suggested this method came about as an anticipated way for resolving the perturbation of keeping track of 1s on just one hand so the other could serve for keeping track of composite units.

With this novel method of keeping track of a larger number of 1s, Jake's anticipation of the need to also keep track, simultaneously, of accrual of CUs was enabled. As expected, this method could go smoothly for him during the count of the first five towers, as indicated in his response to ability to monitor how many towers he had Nina (25:24). This prompted stoppage prior to completion of the coordinated count, as well as Nina's continual prompting of the number of towers he had counted later, seemed to open the way for his novel activity of keeping track of CUs past those that matched fingers on just one hand. In this sense, it was the first time Jake was observed to decompose, in activity, a compilation of CUs (8 towers) into two sub-compilations (5 towers and then 1, 2, and 3 more towers that he added to the five towers on the other hand). Most importantly, during the entire process Jake never seemed to be confused about the units on which he was operating, as seen in his correct responses to Nina when asked about what fingers showed in terms of towers and cubes he has been counting.

Theme 2 summary. The excerpts presented in this section shed light on the difficulty that 5+n numbers can create for SLDs as they establish methods for double counting. Initially, for tasks with hard numbers both Jake and Luke seemed to rely on the use of one hand for the compilation of CUs (towers) and one hand for the iterable unit rate (cubes per tower). That is, in the absence of a memorized sequence of multiples, the operation they brought forth was one in which each hand stood for (figural) a different type of unit in the process of coordination—one hand for 1s, the other for CUs. It is precisely this reliance on dual representation of the two types of unit that seemed to initially preclude mDC for numbers of CUs larger than 5. The necessity of a dual representation when the numbers in both the CU and the unit rate exceeded five made operating multiplicatively (say, on 7T6) much more difficult than operating on the known sequences of 2 and 5 and the harder (but still less than 5) numbers of 3 and 4. Accordingly, the students folded back to operating on 1s (counting-all), indicating that the CUs were not yet iterable for them. The gradual change in Jake's and Luke's abilities to create and use figural re-presentations of the units required for larger numbers may point to the construction of representations of 5+n numbers as a key step in SLDs' transition to an anticipatory stage of the mDC scheme.

Theme 3: Dual Anticipation of Stops and Starts in mDC

This section consists of six excerpts that demonstrate further advances in the SLDs' construction of goal-directed activities of coordinated counting to solve multiplicative tasks with harder numbers (8T12, 5T6, 7T6, 5T8, and 6T8). These excerpts pertain to the work of 3 students (Jake, Devin, and Dana) who solved tasks in the PGBM game context. The data from Jake, Devin, and Dana were selected for this theme because, at this point, each of them could independently call up an anticipation of activity-effect relationship suitable for figuring out the total of 1s in a compilation of CUs via a coordinated count of the dual accrual of 1s and CUs. Jake and Devin were yet to independently anticipate knowing where to stop the count of CUs and thus were still at the participatory stage, whereas Dana could readily monitor the counts for more difficult numbers and was thus was identified as being in an anticipatory stage of the mDC scheme.

Jake and Devin's ability to anticipate the start of their counts but not yet anticipate where to stop their counts is identified here as an intermediate stage in constructing mDC by SLDs. This distinction builds on the distinction between knowing where to start vs. knowing where to stop in counting-on (Tzur and Lambert, 2011). In their study, the challenge for students was that solving a given situation using additive reasoning (e.g., 8+5) required the anticipation of both where to start (the next number after eight) and the anticipation of where to stop that count (after they had tracked the five 1s to be added). In the 5+n multiplicative situations in which numbers exceeded the number of fingers on one hand, the challenge for the students was in anticipating where to stop while different counts are executed. The harder numbers in these tasks foreground the difficulty in (a) keeping track of 1s while monitoring where to stop each time, (b) keeping track of the compilation of the CUs and the unit rate while monitoring where to stop both counts, and (c) coordinating the stops and starts for each. The student responses to the harder numbers used in three tasks allowed analysis of possible sub-schemes related to the dual anticipation of stops and starts.

As mentioned in Theme 2, for SLDs the mental activity required for the dual anticipation of an mDC while working with harder numbers raises the issue of working and/or short-term memory (STM) for SLDs. This analysis hinges on the construction of multiplicative schemes and the mental operation required in multiplicative reasoning (rather than the memorization of associated factors and products). In each of the cases presented in this section, the children's mental energies were needed for executing the dual anticipation of mDC with numbers that were (a) not known multiples and (b) greater than the number (5) of fingers on one hand. In all cases of mDC, children have to remember the stops and starts of the compilation of CUs while running the count (monitoring each of the stops and starts) of the unit rate repeatedly *and* tracking the total number of 1s. As numbers exceeded 5, the mental activity of the child has to change so it includes a way of decomposing and monitoring the count past 5. Thus, anticipation of each of these coordinated stops and starts (in 5+n situations) can become more challenging. One possible reason for the added challenge may be that the child has to "disembed" (Steffe, 1992) the five fingers of one hand standing for either the CUs or the unit rate from the total number of either cubes or towers while representing those numbers with their fingers. While the increased challenge to execution of activity toward anticipated effect could create a possible suppression of meaning, due to the short term and/or working memory demands, the difficulty seemed to diminish as the students progressed in their construction of the mDC scheme (from participatory to anticipatory stage). This suggests that issues in the construction of multiplicative schemes may not solely be explained by issues of working or short-term memory, but may be related to the conceptual advancement in schemes (such as mDC).

Excerpt 14 provides data that follow immediately from Jake's work at the end of Excerpt 13, as he counted on from the 6th tower of 12. This excerpt provides evidence that working with these hard numbers constrained his ability to keep track of the stopping of the compilation of CUs, while simultaneously tracking the starts and stops of the unit rate and tracking where to stop in the accumulation of 1s.

Excerpt 14: Counting past the last (8th) tower (student: Jake; task: 8T12; date: November 18, 2014).

25:52 Jake: [Continues the coordinated count from 6T12, suggesting an advance in his ability to remember the counts to this point.] Oh yeah did I say 72? [Jake raises the second finger of his right hand indicating initiation of the count for the 7th tower and continues counting.] 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84 [He raises the third finger of his right hand indicating the initiation of the 8th tower, as he already counted 5 towers on the other hand, and continues counting on his left hand for another 12 units of 1] 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96. [Without a pause, or any other indication that he knows he had finished accruing the total number of cubes for 8 towers, Jake raises a fourth finger for counting cubes in a ninth tower and, incorrectly, continues counting 1s in that tower] 97, 98...

26:12 Nina: [Interrupts to orient Jake's attention on the number of towers counted thus far] How many towers is that?

26:14 Jake: Eight [There are no gestures or body language to suggest that he knows that he has counted past the eighth tower.]

26:15 Nina: How many towers were you trying to get to?

26:16 Jake: Eight- Oh I forgot my answer. [Indicates that he is trying to remember the last number, 96, which he uttered upon completion of the 8th tower.]

26:18 Nina: Do you remember what you said?

26:19 Jake: No, [Begins to work again counting on from the first tower of 12.] 12…

As analyzed above for Excerpt 13, Jake's ability to continue a coordinated count from the 6th multiple of 12 (25:42), and his continued ability to anticipate the starts and stops of each 12 (while running a 5-10-12 count on one hand's fingers), suggested his initial ability to operate multiplicatively (to a point) on these more difficult numbers. What is different in this mental operation is that the limit of his ability to monitor the stops of the compilation of CUs was at the 8th tower, at which point he counted past the total number of towers with an accurate figural representation of moving on to the 9th tower (raising a 4th finger on his right hand). This limitation may indicate that Jake's cognitive energies were engaged in the goal-directed activity of double counting 12 eight times – each of those counts requiring a stop and start *and* a figural representation that required using each hand multiple times – to the extent that, perhaps, the global goal of the activity (finding the total) may have been lost. However, when asked how many towers he was trying to get to (prompted by Nina), Jake was able to answer 8, suggesting that he could hold the number of towers required for this task in his short-term memory. Yet, multiplicative reasoning requires more than the recall of factors and products – it requires the challenging mental activity of anticipating the stops and starts of *two* coordinated counts, while at the same time keeping some focus on the larger goal of the collection of total units.

The next excerpt (15) continues from this point as Dana, another student identified as SLD in the intervention group joined in the conversation. This excerpt is included for two reasons. First, it supports the claim that when mDC is further constructed (as in Dana's case), the issues that may have appeared to be related to working or short-term memory diminish while this scheme supports connections across and within tasks. Second, it demonstrates the impact of interaction between two students (Dana and Jake) at different stages in the construction of the mDC scheme, as Jake responded to her suggestion of how to track these larger numbers.

Excerpt 15: Dana (a student from the group) orients Jake's reflection to 6T12 (students: Dana and Jake; task: 8T12; date: November 18, 2014).

26:24 [Dana joins in the conversation. She prompts Jake to use five towers, with a total of 60 cubes, as his starting place rather than going back to the first tower of 12 indicating her understanding that 8T12 can be found using 5T12 as a starting place- plus an additional 3T12- essential for working with 5+n numbers in these situations.]

26:24 Dana: You know [do you recall] when you said 60 and that was five towers?

26:35 Nina: [Orients Jake's attention to Dana' suggestion] So did you hear that? She is helping you out, five towers was how many cubes?

26:41 Dana: 60

26:45 Jake: Okay, oh yeah 60; so 60 is 5 towers, [Makes a fist with his left hand and begins to continue the coordinated count with his right hand again] 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72 -72. [He switches hands (as he had done before), and folds the thumb of his *right* hand down to symbolize the completion of a count of all cubes in the 6th tower. Then, he continues the coordinated count with his left hand] 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84. [Upon uttering '84' Jake also folds down the index finger of his right hand to symbolize the completion of counting all cubes in a 7th tower] 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95; 95? [He folds down a third finger on his right hand and looks at Nina.]

27:16 Nina: Is that answer different?

27:18 Jake: [Unable to answer the question, begins counting again at 60.] So 60, 60 is 5 towers, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77 [Counts past the unit rate of 12, stops and looks at Nina indicating awareness of an error—an internally prompted 'oops' experience.]

27:45 Nina: Why don't you try drawing it out?

27:48 Jake: Wait no, no. [He motions his hands across the table in indication that he wants to start over.]

27:51 Nina: Ok, so 5 towers is 60 [cubes].

27:53 Jake: [Begins counting at 60 again, makes a fist with his left hand and counts 15 counts on his right hand] 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75 -75. [He again counts past the unit rate of 12 and in a seeming oops moment stops and looks at Nina for confirmation of the unit rate.]

28:00 Nina: What are you counting by?

28:04 Jake: 12's? [He remembered the unit rate, albeit with a questioning tone in his answer.]

28:05 Nina: [Confirms] 12's.

28:07 Jake: [Begins counting at 60 and again makes a fist with his left hand, counting 12 counts on his right hand] 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72 [Looks to Nina for confirmation of the total. He Switches hands and puts up the thumb of his right and begins a count of 12 on his left hand.] 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84. 84!? [He looks to Nina for confirmation then raises the index finger of his right hand and continues counting.] 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96 [He raises the middle finger of his right hand and looks to Nina for confirmation. She gestures for him to continue but does not indicate whether he has finished.] Wait so that's eight towers. [Indicating recognition that he has completed his count of 8 towers.]

28:33 Nina: [Confirms that Jake has stopped because he knows he has completed counting the 8 towers.] So are you stopping or keeping going?

28:35 Jake: Stop.

Excerpt 15 indicates that Jake continued his operation on larger numbers with increasing success. When asked about the unit rate after counting past it (28:04), he could say that it was 12, but his recall did not yet guarantee independent operation. After a prompt from Nina confirming the unit rate, Jake independently resumed his activity by counting from 60—not from 12 (28:07). This indicates he could at least use Dana's suggestion in the context of *his* activity, hence his growing anticipation of the possibility to decompose the 8 CUs (8T12) into two sub-compilations (5T12 and 3T12) during this episode. That is, Dana's suggestion to help Jake with a count he seemed continually be starting all over seemed to provide an opportunity for his Reflection Type I in that he recognized the possibility to build on partial results of his own activity. While Jake's working or STM may have initially been burdened to the extent that he couldn't operate on these larger numbers while keeping track of his own regulated activity toward the goal of how figuring out the total number of 1s (cubes), Dana's suggestion seemed to be assimilated and operated on to allow progress in his monitoring actions. Specifically, he learned to use what was available in his short-term memory (e.g., 60 coordinated with 5T12) as a re-start point for counting the next sub-compilation (on the fingers of his other hand). His progress could thus be seen also in the relative freedom, without confusing units, gained through applying each type of unit counted (1s or CUs) to any of his two hands. That is, he seemed to have created a strategy that separated between counts of 1s within the composite unit of 12 (5, then 5, then 2) and counts of composite units (5, then 3 more). This strategy could be both rooted in and support cementing the anticipation of the dual start-stop of the two types of units.

Jake's accomplishment after Dana's contribution, as discussed in the previous paragraph, was gradual. In his first attempt to continue the double count (26:45) from the

5^{th} tower per her suggestion, Jake accurately tracked the coordinated count for the 6^{th} and 7^{th} towers, but then stopped the unit rate count at 11 instead of 12. His appeal to Nina indicated that he knew the answer was incorrect—an instance of an internal prompt for Reflection Type I. In the second attempt that followed this reflection while going back to the double count from the 5^{th} tower (27:18), Jake could again coordinate the double anticipations up to the 7^{th} tower, and then, instead of stopping short as in the previous attempt, he counted past the 12^{th} unit in the unit rate on the 8^{th} tower. His 'oops' moment during this attempt provided additional evidence of his awareness of errors in situations involving these more difficult numbers. In his third attempt to coordinate the count from the 5^{th} tower (27:18), Jake counted past the 12^{th} unit for the 6^{th} tower and again had an 'oops' moment that indicated recognition of his error.

On one hand, these continued instances of error self-recognition support the claim that these larger numbers created a difficulty in mDC but not such a difficulty that Jake folded back to additive thinking. On the other hand, they also seemed to have opened the way for a Reflection Type II. After the third attempt, Nina suggested that Jake "draw it out" using some of the notations of towers and cubes that she had previously presented to the students. Jake dismissed this suggestion in a possible indication that he did not know how it might help him in this situation. Instead, plausibly due to the three instances he has then experienced as lacuna in his monitoring activity, he set out to carry out the fourth attempt to coordinate the count while starting at 60 for 5T12 (26:45). Jake was successful in his dual anticipation and his utterance, "Wait, that's eight towers," combined with his confirmation that this was the place to stop (28:35) provided insight into his ability to reflect (across three incorrect instances) on the results of tracking the total number of towers. His utterance suggests that, at this point, he was satisfied with the (new) way of operating in which a start at a sub-compilation (e.g., 5T12) would allow him to keep track of two accruals of units without confusion.

Within Excerpt 15, Jake seemed to have operated multiplicatively even with the harder numbers. The availability of prompts (from Dana, from Nina) and his multiple ‘oops’ moments, indicated that with harder numbers Jake was still in a participatory stage of constructing the mDC scheme. This claim rests on the difference between his ways of operating before and after assimilating and using Dana’s prompt to think of 5 towers with 60 cubes as a single item (and thus – a possible re-starting point). Before Dana’s prompt, Jake repeatedly resumed his counting from the first tower (26:19) rather than from the 5^{th} tower for which he had already established the total of 1s (60). After Dana’s prompt, he assimilated her suggestion (26:45 and 27:18) and his ensued, prompt-dependent activity, which started at 60-for-5-towers, indicated some coordination of the composite rate and unit rates both from the initial and the 5^{th} multiples of 12. For Jake, with “easy” numbers the role of each unit in regulating activity was available (e.g., independently counting 12 for each of the first 5 towers). Yet, with “hard” numbers, his effort to focus on accrual took over these roles: although he knew the count was 12, he initially had difficulties knowing where to stop his count of twelve 1s.

Excerpt 16 features another subject, Devin, who also needed a prompt to monitor where to stop. Like Jake, Devin was facile in the anticipation required for the accumulation of up to 5 towers, but his difficulty was in the anticipation of where to stop in the accumulation of the unit rates.

Excerpt 16: Keeping track of where to stop when counting the unit rate (student: Devin; task 5T6; date: October 15, 2014).

11:21 Nina: How many towers are there, Devin?

11:28 Devin: There are s….there are … [Do you ask] how many towers? Five towers.

11:30 Nina: How many cubes are in each tower?

11:32 Devin: There are six cubes in each tower.

11:34 Nina: [Covers the towers with a piece of paper only after Devin has confirmed the number of towers and the number of cubes per tower.] How many cubes are there in all?

11:45 Devin: [Under his breath.] Six, twelve [He presses the thumb and index finger of his right hand on the table indicating two towers and counts six counts with his left hand starting with his thumb.] 13, 14, 15, 16, 17, 18 [He presses the third finger of his right hand on the table and continues to count *five* counts using the fingers on his left hand.] 19, 20, 21, 22, 23 [then presses the fourth finger of his right hand and continued to count *five* more.] 24, 25, 26, 27, 28 [Then presses the fourth finger of his right hand and continues to count six more] 29, 30, 31, 32, 33, 34. [He raises his hand indicating that he has finished.]

Excerpt 16 presents how Nina began the work with Devin in the typical way PGBM is played, particularly when towers are hidden, namely, asking him to repeat (and firmly establish) the number of towers and the number of cubes per tower. This common practice when playing the game can help mitigate short-term memory issues.

Devin's response to the question about how many cubes are there in all indicated an anticipation of the need to coordinate the counts of 1s and CUs. Specifically, he independently initiated a start of the counting from the second multiple of 6, indicating his coordination of the compilation of CUs and the unit rate. That is, with the first two easy (for him) numbers in the sequence of multiples of 6, he seemed to distinguish the number of 1s in each CU from the number of composite units for which he had been accounting so far. As he tracked the accumulation of cubes past these two easy numbers (6, 12), however, he shifted to counting only five (instead of six) items per composite unit while keeping track of six towers (instead of just five). That is, whereas with "easy" numbers the role of each unit in regulating Devin's activity was clear and proper, with 'hard' numbers his effort to focus on accrual of both types of unit took over these roles. In this sense, Devin provides another example of a student for whom a claim that the more difficult (5+n) numbers impact construction of mDC can be demonstrated.

Excerpt 17 provided data from an instructional prompt from Nina designed to capitalize on Devin's work (and errors). In this excerpt, Nina oriented Devin's reflection to the way in which he accurately used his left hand to keep track of six cubes (Reflection Type I).

Excerpt 17: Teacher/researcher supporting tracking the unit rate (student: Devin; task 5T6; date: October 15, 2014).

12:42 Devin: There are 34 cubes altogether.

12:57 Nina: Now I want you to double check.

13:00 Devin: [Uncovers the cubes and starts counting at 12 cubes for two towers (confirming his mDC even when the cubes were available to him). Moves the first two towers and continues counting the rest of the towers by 1s, arriving at 30. Smiles and looks at Nina.] I got it wrong.

13:10 Nina: [Orienting Devin on his own tracking methods.] So, Devin, what I saw you doing, which I thought was so good, is this. [Holds her left hand over the table and begins to fold each finger down.] You had this hand and were going like this [Holds her left hand over the table and begins to fold each finger down.] Can you tell me what you were doing?

13:24 Devin: [Places both hands on table and moves his right hand] That was towers [moves his left hand] and this was my cubes.

13:26 Nina: [Models Devin's error.] And what I saw you doing was something like this. [Folds over each of 5 fingers on her right hand]

13:34 Devin: Oh it was 5 [Indicates possible recognition of his error of counting 5 cubes instead of 6 cubes.]

13:37 Nina: So you were doing… almost like…[motions Devin to show his count.] Can you show me?

13:42 Devin: Um, oh so 5 towers [presses his right hand on the table] and cubes [presses his left hand on the table] so 6+6 is 12.

13:56 Nina: So how many towers is that?

13:57 Devin: [Demonstrates his ability to track towers.] Two.

13:58 Nina: Ok.

13:59 Devin: So my third tower is [presses each finger of his left hand on the table and counts his thumb twice for an accurate 6 count.] 13, 14, 15, 16, 17, 18.

14:06 Nina: Ok … [Does not indicate whether the response is correct or incorrect, but encourages him to continue.]

14:08 Devin: My fourth tower is 19, 20, 21, 22, 23, 24.

14:13 Nina: Ok …

14:15 Devin: My fifth tower is 25, 26, 27, 28, 29, 30.

14:20 Nina: [Compares Devin's answer to another student's response they both heard previously.] So you got 30, too.

Excerpt 17 shows that, with Nina's prompting, Devin appeared to recognize the inconsistency in his counting of the unit rate in this task (13:34). Furthermore, he was able to re-orient his attention to systematically monitoring the subsequent stops of each of the unit rate counts at 6. When Nina oriented Devin's reflection on his own tracking methods (an essential part of Nina's adaptive pedagogy), she allowed him to monitor his own goal-directed activity (Reflection type I). In this case, Devin was monitoring a stop at 6 for each tower as he monitored the accrual of the total number of cubes. A crucial point in Devin's counting activity is that, for each tower, he first stated its number in the sequence of accruing CUs and only then interjected the 1s that constituted that CU. This is a subtle but important difference from first counting the 1s and then raising a finger for the CU, in that the count of 1s after stating the CU's ordinal number may indicate distribution of those items into each of the CUs (as shown in the study by Clark and Kamii, 1996).

Excerpt 18 provides data from Devin's work on a task during an episode that took place three teaching episodes after the one presented in Excerpt 17. In Excerpt 18, Devin still needed prompting to regulate his stoppage of the count of the unit rate when solving mDC tasks with harder numbers. This provides additional evidence of the difficulty of developing the dual anticipation (start, stop) required for an mDC when faced with larger numbers. Despite Devin's acceptance of Nina's suggestion and his subsequent ability to monitor the stops in the unit rate for 5T6 in Excerpt 17, Devin remained inconsistent in his ability to anticipate where to stop counting the unit rate. This suggests that in that previous episode Devin's construction of the coordinated count was at the participatory stage only.

In Excerpt 18, Devin first determined the total number of cubes in 7 towers of 6 cubes in each tower (7T6). In this case, the difficulty of the numbers had increased because both the compilation of CU and the unit rates exceeded five. Devin readily anticipated where to start both counts, but again was unable to regulate where to stop the unit rate count past the second tower.

Excerpt 18: Anticipating the stop when tracking the unit rate (student: Devin; task 7T6; date: October 23, 2014).

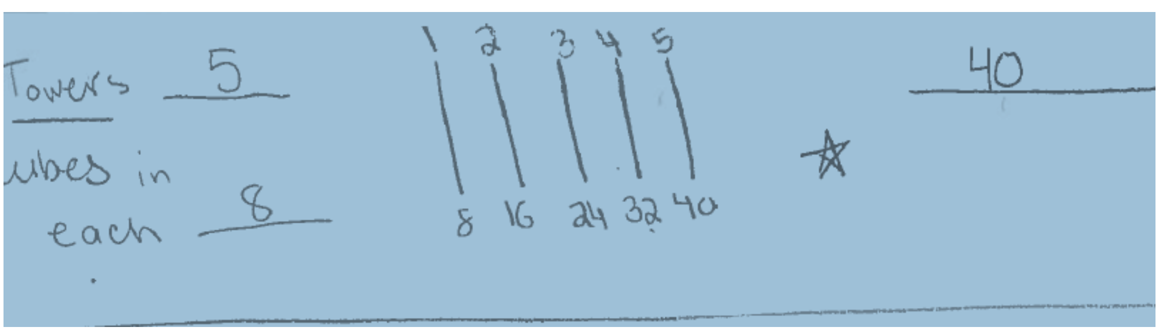
15:24 Devin: [Raises the thumb and index finger of his right hand] So 6+6 is 12; [then] 13, 14, 15, 16, 17, 18 [Raises the middle finger of his right hand and continues counting.] 19, 20, 21, 22, 23, 24 [Raises the ring finger of his right hand and continues counting albeit with *5* counts this time.] 25, 26, 27, 28, 29, [Raises the pinkie finger of his right hand and continues counting 5 counts again.] 30, 31, 32, 33, 34 [He folds all fingers on his right hand and raises his thumb again. At this point, the thumbs of his right (towers) hand and left hand (cubes) are raised.] He counts six counts.] 34, 35, 36 37, 38, 39…. I got 39 but I got 40 before.

Excerpt 18 indicates that, in this task, Devin anticipated and could initiate the coordinated count while beginning from the second (known) multiple of 6, just as he had in Excerpt 17. This indicated his anticipated coordination of the compilation of CUs and the unit rate. That is, with the first two "easy" (for him) numbers in the sequence of multiples of 6, he seemed to clearly distinguish the number of 1s in each CU and the number of CUs for which he had been accounting so far. As he tracked the accumulation of cubes for the fifth and sixth multiples, however, he shifted to counting only five (instead of six) items per CU. At the seventh multiple, he recounted 34 (the last number he said when inaccurately counting the 6^{th} multiple), and then accurately counted 6 counts. Similarly to Jake, Devin's anticipation when solving a problem with "easy" numbers included both where to stop and where to start each. Yet, with "hard" numbers, his mental goal of accrual took over and he made errors in both stops and starts when monitoring the accrual of the unit rate.

To culminate this section and theme, Excerpts 19 and 20 present data from the work of Dana, the student who assisted Jake in Excerpt 15 and oriented his reflection on the 5T12 contained in the 8T12. In these excerpts, Dana solved tasks involving 5T8 and 6T8. The ease with which she anticipated and monitored the stops and starts even with the harder numbers in these tasks suggested that she was in an anticipatory stage of mDC. Thus, Dana's case supports the assertion that the cognitive advance involved in constructing the mDC scheme may 'free' more mental energy to track the multiple stops and starts required to solve these problems with more difficult numbers. Finally, Excerpt 21 presents data of Dana's solution of a task (7T8) that included two 'oops' moments. That final Excerpt further supports the claim that overcoming the difficulty in the dual anticipation of mDC, when solving tasks with harder numbers, is not a once-and-for-all event.

Excerpt 19: Anticipatory mDC (student: Dana; task 5T8; date: November 6, 2014)

0:58 Dana: [Presses the thumb and index finger of her left hand together (based on previous problems it was inferred that she is doubling 8 to get 16), then silently continues the coordinated count of 8 more by 1s using her right hand to keep track of the 8 counts by counting each finger once and then re-using her thumb, index finger and middle finger. She uses the fingers on her right hand to track the accrual of an additional tower. She repeats this process until she reaches eight towers and 40 cubes and then writes 40 on her paper. After she finds the answer using her fingers to represent the towers and cubes, she draws a representation of the towers and cubes on the paper, which was a requirement of the day's task.



Excerpt 19 shows how Dana carried out her mDC using doubling (8, 16) for the first two multiples followed by a continued coordinated count (0:58). Her dual anticipation of the coordinated counts even with these more difficult numbers indicated that she was in an anticipatory stage of mDC. Said differently, she was inferred to have constructed the anticipatory stage of the mDC scheme based on the spontaneous, facile process by which she carried out the dual anticipation (start, stop) of the counts required in this operation.

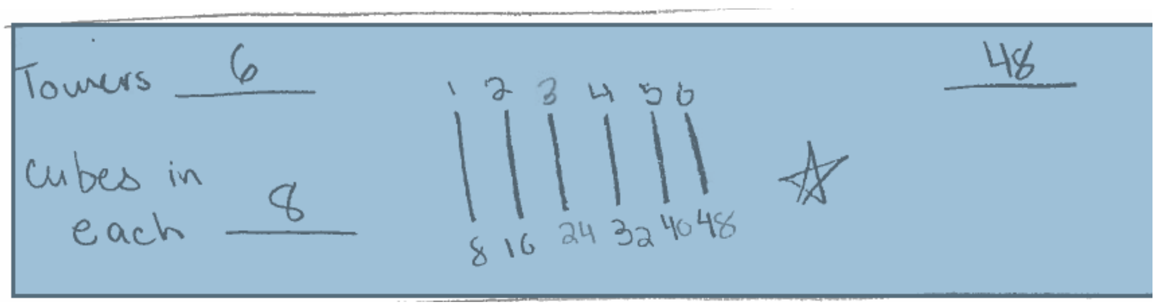
In the next problem, Dana recalled the number sequence for 3 towers of 8 cubes in each tower (3T8) without doubling or counting each of the cubes and then continued the count by keeping track of the 8 cubes on her right hand and the total number of towers on her left hand. It is interesting to note that, a month after she oriented Jake to the 5T12 in the 8T12 problem (suggesting her ability to operate on 8 towers as 5 towers and 3 towers), she did not begin with 5 towers of 8 from the previous problem in this situation and instead continued her coordinated count from 3T8.

Excerpt 20: Anticipatory mDC (student: Dana; task 6T8; date: November 6, 2014).

2:12 Nina: Good job. Can I draw a line? [Draws a horizontal line below the five towers of eight problem.] Now this was 5 towers with 8 cubes in each. Ok. Now lets do this. [Writes 6T8 (a convention for representing towers and cubes that Nina had taught the group once they had reached a participatory stage of mDC) on the paper.]

2:50 Dana**: [**Holds up both hands and puts together the thumb and index finger of her left hand and mouths 16. She then presses her middle finger with the thumb and index finger and mouths 24. She begins counting under her breath 25, 26, 27, 28, 29, 30, 32, using her right hand to keep track of the 8 cubes by counting each of her fingers and then re-using her thumb, index finger, and her middle finger. She continues this process until she reaches 6 towers and 48 cubes. She keeps track of the total number of towers using her left hand. When she runs out of fingers she re-uses her thumb for the 6th tower. When she finishes, she looks up at Nina and grins (seemingly knowing that she had the correct answer) and begins to draw a representation of the towers and cubes.

3:05 She draws the lines for the towers first, then writes the numbers on top of each line to represent the number of towers.



She then writes the 8, 16, 24, 32, and 40 immediately, without re-using her fingers. She uses the fingers of her right hand to count out 8 then writes 48 under the line labeled 6.]

Dana's facility with the first three multiples of 8, the ease with which she had a dual anticipation of the stops and starts even with these more difficult numbers, her confidence with the answer (2:50), and the ease with which she re-presented the work she had done (3:05) provided further support to the inference she was in an anticipatory stage of mDC. As a result, the numbers presented in these tasks were no longer hard for her as they were for Jake and Devin. In summary, Dana was able to anticipate the effects of her double counting activity required by an mDC even in situations involving harder (5+n) numbers.

While Dana showed evidence of an anticipatory scheme for mDC, it can be inferred that she had yet to construct an anticipation of the invariant relationship between her activity in solving 5T8 with her activity in solving 6T8. At least initially, and in spite of her suggestion to Jake in a previous episode, these two tasks seemed to be different situations, that is, she did not yet interiorized 5T8 as a sub-compilation of 6T8, which would allow her to recognize the link between these two situations and thus solve the second task with hard numbers (8 as unit rate) similarly to how she suggested to Jake to solve the problem of 8T12. One reason this interiorization might have been missing is the difference in relationship between the sub-compilation and the global compilation. In seeing 5T12 as a sub-compilation of 8T12, Dana could operate on the sub-compilation as embedded within the global one. In contrast, the 5T8 was a compilation in and of itself, disembedded from the compatible sub-compilation of 5T8 that would have to be distinguished within 6T8.

Theme 3 summary. Multiplicative Double Counting requires a coordination of more than one count and *each* of the coordinated counts requires an anticipation of where to start and where to stop. This dual anticipation and the difficulty in monitoring each of the stops and starts illustrated a possible intermediate stage distinction in the construction of mDC, particularly by SLDs. The analysis of Jake and Devin's work in this sub-section amounts to suggesting that mDC requires the learner's (and teacher's) attention to *both* an anticipation of where to shift each count of a CU (unit rate) and of where to stop a count for the compilation of CUs. It is this dual anticipation that makes operating on harder numbers in which CUs and/or 1s exceed the number of fingers on one hand a challenging feat to overcome, because the count of figural items for each type of unit initially requires the child to use 1-to-1 correspondence between each finger and what it stands for.

In addition, Both Jake and Devin's ability to recall the number of towers and the number of cubes per tower, while not yet being able to anticipate the stops of their counts when operating on those units, provided some evidence that short term memory may not be the sole factor at work in SLDs' construction of mDC for harder numbers. When prompted, and later without prompting, Jake could remember and accurately continue a coordinated count from the 5^{th} multiple of 12. With prompting, Jake's operation distinguished the number of 1s in each CU and the number of CUs for which he had been accounting so far (5T12). However, in several places Jake's action in the subsequent coordinated counts highlights the ongoing difficulty of anticipating stops and starts within the operation of a double count in the participatory stage of mDC. This finding seems consistent with Tzur and Lambert's (2011) identification of two intermediate sub-schemes in children's progress from counting all to counting on in additive situations. Specifically, it suggests a possible extension to include anticipation of the start and stop of two simultaneous counts in multiplicative situations, which is discussed further in Chapter V.

Summary

In this chapter, data were analyzed to better understand the impact of numbers used in tasks on SLDs' construction of the mDC scheme. Specific attention was paid to the ways in which these students' conception of number as a CU afforded or constrained the transition to multiplicative reasoning and the ways in which specific numbers supported or interfered with SLDs' progression from additive reasoning to mDC. Three themes emerged from the analysis:

Theme 1: Early in the construction of an mDC scheme, students seemed able to coordinate a multiplicative Double Count when given the numbers 2 and 5 for the unit rate and when given numbers of 5 or less for the CU. These students were not yet able to coordinate units in mDC tasks involving what, *for them*, seemed to be harder numbers (three and four). When faced with the more challenging numbers, students tended to fold back to a lower level strategy.

Theme 2: As students progressed in the construction of a multiplicative scheme, numbers that exceeded 5 (the number of fingers on one hand) for the unit rate and the CU required the construction of a revised figural representation of the units required for a composite count. This suggested that the construction of figural representations for numbers in an mDC scheme does not occur once-and-for-all, and that attention should be paid to the ways in which children represent more difficult (5+n) numbers throughout the construction of the mDC scheme.

Theme 3: As students progressed in the participatory stage of an mDC scheme, they no longer tended to revert to operating on 1s (counting-all) when given harder numbers. However, the harder numbers chosen for the CU and unit rate impacted whether they could anticipate where to stop with each of their coordinated counts: the compilation of CUs, the unit rate and the accrual of 1s. This suggested that the construction of an mDC scheme requires attention to both anticipation of where to start and stop a count for the CU and anticipation of where to start and stop the accrual of each of the unit rates. This double anticipation for a dual count seems a unique multiplicative feature. Attention paid to the numbers used in tasks can support SLDs' reflection, particularly on keeping track to know where to stop the counts for composite unit and unit rates in an mDC scheme.

CHAPTER V

DISCUSSION

This dissertation addressed two research questions: (1) In what ways do SLDs' conception of number as a composite unit afford or constrain transition to multiplicative reasoning? And (2) which specific numbers, used in instructional and/or assessment tasks support or interfere with SLDs' progression from additive reasoning to multiplicative Double Counting (mDC)? The data were collected in an intervention class with fourth grade students in an urban elementary school in the western United States.

In the previous chapter the results of the study were analyzed and three primary themes emerged:

Theme 1: Early in the construction of an mDC scheme, students seemed able to coordinate a multiplicative Double Count when given the numbers 2 and 5 for the unit rate and when given numbers of 5 or less for the composite unit (CU). These students were not yet able to coordinate units in mDC tasks involving what, *for them*, seemed to be harder numbers (unit rate of three or four). When faced with the more challenging numbers, students tended to fold-back to a lower level strategy, such as counting all.

Theme 2: As students progressed in the construction of a multiplicative scheme, numbers that exceeded 5 (the number of fingers on one hand) for the unit rate and the CU required the construction of a revised figural representation of the units required for a composite count. This suggested that the construction of figural representations for numbers in mDC scheme is not accomplished once-and-for-all, and that attention should be paid to the ways in which children represent more difficult (5+n) numbers throughout the construction of an mDC scheme.

Theme 3: As students progressed in the participatory stage of an mDC scheme, they no longer tended to revert to additive schemes when given harder numbers. However, the harder numbers chosen for the compilation of composite units and for the unit rate impacted whether they could anticipate where to stop with each of their coordinated counts; the compilation of composite units, the unit rate and the accrual of 1s. This suggested that the construction of an mDC scheme requires attention to both kinds of anticipation: where to start and stop a count of the composite units, and of where to start and stop the accrual of the unit rates. This dual anticipation is unique multiplicative thinking and attention paid to the numbers used in tasks can support SLDs' reflection particularly on keeping track to know where to stop the counts for composite unit and unit rates in an mDC scheme.

In this final chapter of the study, these three themes are discussed along three central claims:

1. The construction of an mDC scheme is not accomplished in a once-and-for-all manner, and attention should be paid to the numbers used in tasks to support the construction of multiplicative reasoning.
2. The ways in which children construct figural re-presentations impacts their ability to reason multiplicatively and may suggest figural representations for harder numbers as an important sub-scheme in the construction of anticipatory mDC.
3. The construction of an mDC scheme requires attention to both anticipation of where to start and stop a count for the composite unit, and anticipation of where to start and stop the accrual of each of the unit rates. This dual

anticipation is unique multiplicative thinking and attention paid to the numbers used in tasks can support SLDs' reflection particularly on keeping track to know where to stop the count of composite units and of unit rate in an mDC scheme.

Accordingly, for each claim this chapter is organized into (a) a summary of the findings related to that claim, (b) connections to related research and theory, (c) implications for practice, and (d) implications for future research. The chapter concludes with an overall summary and a review of the responses to the research questions.

Numbers Can Both Afford and Constrain the Construction of Multiplicative Thinking

The findings of this study suggest that the participating students operated both additively and multiplicatively when given what, to an adult, could be seen as multiplicative tasks. A central finding of this study is that, especially early on in the construction of mDC, numbers chosen for the tasks seemed to influence whether the student used additive or multiplicative reasoning. In the excerpts presented in the previous chapter, evidence was presented that both Jake and Luke had at least a participatory mDC scheme when given easier numbers (initially, 2 and 5). When given similar (from an adult's point of view) tasks with more difficult numbers—numbers for which the students had no facility with the multiples, they were either unable to attempt the solution independently or "folded-back" (Pirie & Kieren, 1994) to the additive solution of counting all.

The inconsistent use of strategies and reliance on immature strategies has been noted as common for SLDs (Geary & Hoard, 2003; Geary et al., 2004; Zhang et al., 2011). Findings of this study suggest that the choice of numbers used in tasks may be one factor that can influence the reliance on immature solution strategies and the movement to more developed thinking and strategies.

Additionally, this study's findings suggest that SLDs' construction of mDC is consistent with a model of development ("folding back") suggested by Pirie and Kieren (1994). The students in this study "folded-back" to less developed conceptions when solving problems with more difficult numbers. Tzur and Simon (2004) noted that "folding-back" might serve as a good behavioral indicator for a stage in constructing a new scheme at which the child's evolving anticipation (here, multiplicative coordination/distribution of composite units) is yet to become independent and spontaneous. Specifically, Jake and Luke were able to use their newly constructed understanding of the coordinated count of composite units and begin operating in multiplicative situations when using "easy numbers." Yet, they demonstrated a case of a child who would fold-back to operating on units of one (1s) with numbers for which they had no facility with the multiples.

Implications for practice. This study's findings provide a means to evaluate instructional practices for multiplication for SLDs. First, these findings draw attention to a distinction between the construction of multiplicative thinking and the use of additive strategies to solve multiplication problems. Although Curriculum Based Measurements are often used for progress monitoring for SLDs, these findings suggest that the use of timed, answer or digits-correct based assessments may be inadequate to measure shifts in children's *thinking* from additive to multiplicative reasoning. Initial knowledge that students tend to fold-back to less developed (additive) schemes when given more difficult numbers, particularly when still at the participatory stage, might influence the design of assessments that track student's use of multiplicative schemes using problems with both familiar and unfamiliar rates. To some extent, correct or incorrect answers can reflect not mainly a student's mastery of the unit-coordination involved in multiplicative reasoning but only the extent to which she or he have mastered the multiples of a given number.

The second implication relates to instructional design that supports the construction of schemes for reasoning multiplicatively, specifically the mDC scheme. If a teacher's goal is to focus student reflection on the mental action of multiplicative unit coordination, it may be useful to begin instruction with number sequences that are known to the student. Using problems with more familiar ("easy") units in multiplicative situations can support SLDs' focus and reflection on the intended operation (in this instance keeping track of both the unit rate and the compilation of composite units) and may promote students' competence in keeping track of how many composite items they have counted. Even if, early on, the students are not fully aware of keeping track, the easier numbers provide a context in which they can engage in Type II reflection. Giving SLDs tasks with less familiar ("harder") units too early in an instructional sequence may actually result in the perpetuation of their additive thinking, which could hamper their learning not only in multiplication and division but also later in rational numbers.

Implications for future research. The findings from the four cases in this study suggested that easier numbers promoted the shift from additive to multiplicative reasoning for SLDs. Future studies with a larger sample of students are necessary to determine whether this model of student thinking is generalizable to a larger population of SLDs and perhaps to a larger sample of NAPs.

Secondly, the findings from this study were limited to students moving from additive reasoning to the initial multiplicative scheme mDC. Future research is necessary to determine the importance of easy (known number) sequences in each of the other schemes in multiplicative thinking – Same Unit Coordination, Unit Differentiation and Selection, Mixed Unit Coordination, Partitive Division, and Quotitive Division (Tzur et al., 2013). For each scheme, it seems important to determine if the choice of numbers for tasks continues to influence the development of thinking as the child progresses, or if it is most important in the initial stages of the construction of mDC.

Finally, future research may be focused on cognitive studies to better understand the impact of "harder" numbers on the mental resources required by a child to construct a novel scheme (i.e. mDC). In other words, in the early stages of construction of a participatory scheme, are a child's mental resources only available for one of the required operations (can they either operate on the more difficult multiples *or* operate on the unit coordination) in mDC?

Claim 1 summary. These findings suggest that the construction of mDC is not a "once-and-for-all" event, and that "folding-back" to lower level strategies is to be expected when "hard" numbers are used for "similar" tasks SLDs can solve with "easy" numbers. When a shift in a child's way of operating is to be promoted, such as the cognitive leap from additive to multiplicative reasoning, initial emphasis in task design and use need to be placed on orienting the child's mental powers onto the novel coordination of operations on units. To this end, choosing "easy" numbers seems productive, as it promotes the construction of the intended goal-directed activities (a new scheme). Once constructed, at least at the participatory stage, the novel scheme can serve (as seen Excerpts 8-10) as an invariant way of operating the child would apply to solving tasks with numbers that required engaging more complex mental capacities.

Developing Figural Representations for Harder Numbers in mDC

The findings suggested that a milestone in the construction of an anticipatory stage of the multiplicative double counting scheme was the child's creation of a figural representation for the numbers required in a coordinated mDC for numbers that exceeded five (the number of fingers that each of these students had on each hand). As discussed in claim 1, early on in their construction of mDC, the students folded back to additive strategies when presented with tasks with more difficult numbers. As their participatory mDC evolved, they no longer folded back to additive strategies when given "harder" numbers. However, they had difficulty constructing figural representations as they were faced with numbers for which they were unfamiliar with multiples, and had to establish a way to track CUs, unit rate, and the total number of cubes. As the students in this study constructed a coordinated mDC, their primary strategy for keeping track was to use one hand for CUs and the other hand for unit rate. The construction of a figural representation for 5+n numbers presented a particular challenge because the use of one hand for cubes and one hand for towers had to be revised. Although Jake and Luke no longer folded back to additive strategies when given more difficult numbers, the mental operation for the coordination of units in an mDC scheme (Tzur et al., 2013; Jerry Woodward et al., 2009) was inhibited. Thus, they could not anticipate a coordinated mDC with 5+n numbers until they had constructed a new way to figurally represent 5+n towers and/or 5+n cubes. Of central importance in this claim is, again, that the construction of mDC is not a once-and-for-all event, and 5+n units require a recreation of figural models for double counting as the previous models no longer work for the students.

Implications for practice. These findings suggest the importance of monitoring the ways in which SLDs can represent the numbers in an mDC task. In the cases of Jake and Luke, while they no longer “folded back” to an additive scheme, the shift from numbers 5 and less to numbers greater than 5 inhibited their ability to coordinate units, and required the teacher to employ adaptive pedagogy methods that supported the children’s construction of useful figural representations. For example, in excerpt 10, Nina modeled a method for tracking intended to orient Jake’s reflection on how he might “re-use” fingers on one hand to continue a count of towers past five. Of critical importance to this teaching move is that Nina chose to model this for Jake *only* because she had seen him re-use the fingers on his other hand to keep track of the cubes per tower and therefore knew this model could be available for Jake. Nina’s close observation of Jake’s current conceptions allowed her to provide effective instruction- instruction that built on Jake’s current conceptions.

Implications for future research. As with claim 1, these findings are based on data collected from a small sample of SLDs. Future research is needed on a larger sample of both SLDs and their NAPs to better understand the ways in which figural representations of numbers greater than five can impact the construction of multiplicative schemes.

Additionally, this research focused on the impact of number on the shift from additive to multiplicative reasoning and explore the cognitive shifts and cognitive resources required for mDC tasks with numbers greater than 5. Jake and Devin were asked to work with these 5+n numbers only after they had constructed a participatory mDC scheme and had therefore established, at least to a degree, the coordination of composite units required in mDC. With this research design it was not possible to determine the impact on shifts from additive to multiplicative reasoning if "hard" numbers are introduced before the students have established, at least, a participatory mDC scheme.

Claim 2 summary. As in claim 1, these findings suggest that the construction of mDC is not a "once-and-for-all" event. Specifically, including 5+n numbers for the number of composite units in a compilation and/or the unit rate required the children's construction of novel ways to re-present these units (e.g., towers and cubes). In these cases the children did not fold-back to additive reasoning when given the more difficult 5+n numbers, suggesting an invariant way of operating multiplicatively. However, the 5+n tasks shifted the child's mental powers (at least for a time) to solving the problem of not enough hands and fingers for 5+n towers and cubes.

Anticipating Where to Stop in mDC

The findings of this study confirmed previous research that students' conception of number as a composite unit was required for, and promoted, advancement to multiplicative thinking (Steffe, 1992; Steffe & Cobb, 1994; Steffe, Cobb, & von Glasersfeld, 1988). However, student responses in this study suggest that, even when number as a composite unit is well established in additive situations, and students have developed a figural representation, they still may not anticipate *where to stop* (counting composite units) when faced with more difficult or unknown number sequences in multiplicative situations. As explained above, at issue for them is the mental focus on simultaneously counting both a compilation of composite units and a composite unit rate.

In mDC, monitoring the stops in the compilation of composite units and the unit rates with more difficult numbers presents a challenge. For example, when the compilation of composite units and unit rates were less familiar (eight towers of 12 cubes, six towers with three cubes, five towers of six cubes and seven towers of six cubes), Jake and Devin were both able to anticipate the start of the mDC, but had difficulty with anticipating where to stop (Excerpts 12-16). The tasks in each of these excerpts required a coordination of counts and *each* of the coordinated counts required an anticipation of where to stop. This illustrates a possible, intermediate stage distinction in SLDs' construction of mDC. That is, this study's findings suggest that SLDs' construction of mDC requires a dual anticipation: an anticipation of where to start and stop a count for the compilation of composite units, and an anticipation of where to start and stop the accrual of each of the unit rates. This finding seems consistent with Tzur and Lambert's (2011) identification of two intermediate sub-schemes in children's progress when operating on units of one (1s), namely, from counting all to counting on in additive situations. This study's findings thus suggest a possible extension to their study, in which anticipation of the start and stop of two simultaneous counts may be applicable to multiplicative situations, too.

Implications for practice. These findings foreground possible constraints to SLDs multiplicative thinking as they progress in a participatory stage of development. When designing lessons to promote mDC, the use of larger numbers may provide a context that can support the orientation of children's mental activity toward keeping track of the start and stop of two simultaneous counts. It is for teachers to consider which numbers support an anticipation of where to stop and which constrain an anticipation of where to stop for each child. In other words, teaching to promote students' progress from additive reasoning to mDC may be sequenced by commencing the activities with "easy" numbers and then using "harder" numbers to foster further abstraction of the novel operations they have been constructing with the "easy" numbers through application to the "harder" ones.

Implications for future research. The primary focus of this dissertation was on SLDs construction of mDC as a result of the teacher's number choice. Future research may shift focus to the teaching perspective and explore what considerations are involved in a teacher's choice of numbers for tasks when designing and/or adjusting (real-time) instruction to support SLDs' progression from additive to multiplicative reasoning.

This research was conducted using a small sample of students identified with learning difficulties in mathematics. Additional research is needed to determine whether the teacher's choice of number is also relevant with a larger sample of SLD and with NAPs construction of mDC.

Claim 3 summary. As students progressed in the participatory stage of the construction of mDC, the introduction of more challenging numbers supported by the teacher's focused questions resulted in advances in the participatory stage of mDC. Particularly, this progression seemed to bring forth the conscious reflection and anticipation of keeping track of a collection of composite units and unit rates as they counted.

Overall Contributions to Theory

This research has provided support to three core constructs in mathematics education, and suggested possible extensions for multiplicative reasoning. First, these findings are consistent with Pirie and Kieren's (1994) notion of folding-back. Specifically, the findings suggest that number choice can be a factor that affords or constrains children's move from additive to multiplicative thinking.

Second, this dissertation's findings support and expand on a construct introduced by Tzur and Lambert (2011), who suggested an intermediate stage of counting on. In that intermediate stage of additive reasoning, students may or may not be able to anticipate where to start counting on, and may or may not be able to anticipate where to stop the second count. This study's findings suggested that, in multiplicative situations, anticipation of where to stop is of particular challenge. In such situations, number choice can both afford and constrain a child's ability to anticipate where to stop.

Third, this study's findings seemed to lend further support to Tzur's (2011) assertion about types of reflection that underlie learning. Specifically, SLDs may need teacher intervention in order for type-II reflections to occur in multiplicative reasoning. This study's findings indicated that such interventions could include an explicit choice of numbers in tasks along with teacher questioning.

Overall, this study demonstrated the influence of number choice on children's construction of multiplicative thinking in the participatory stage of the construction of mDC. In the early evolution of a participatory stage of the mDC scheme, familiar ("easy") numbers could support the transition from additive to multiplicative thinking. As the child progresses through the participatory stage, more difficult numbers could help focus a student's attention on the anticipation of where to stop in mDC. As students moved closer to the anticipatory stage of the mDC scheme, number choice used in problem sequences with a consistent unit rate supported reflection and generalization across related multiplication problems.

Limitations and Delimitations

The nature of a Constructivist Teaching Experiment involves some limitations. The work of the teaching experiment is directed toward understanding the conceptual progress that children may make in their mathematical understandings over time (Steffe et al., 2000). Central to the Constructivist Teaching Experiment is using a pedagogy that is responsive to students' current understandings and thus drives the selection of goals and activities for upcoming teaching-learning processes. For that reason, the numbers presented to students were not exhaustive. In this setting, not all possible numbers and combinations of numbers in composite units and unit rates were tested to determine how they might support or interfere with the progression from additive to multiplicative reasoning, or support or interfere with the progression within multiplicative reasoning.

Additionally, it cannot be determined from this study whether or not the children might have responded differently to the numbers presented in problems they were asked to solve had they been working in a different environment. The environment in which these students worked had cubes, whiteboards, and markers provided with the students throughout each episode. Initially, the cubes were available for the students to look at, touch, and manipulate. As students progressed, those tangible items were covered and, most often, the students used their hands (fingers) to re-present the cubes. The whiteboards were always available but were used only in a few instances. Had the children been working in different environments, with different tools, the results might have differed.

Delimitations. This teaching experiment involved four students who participated in an intervention setting over the course of three months. It was designed to look at changes from additive to multiplicative reasoning only as far as mDC. The impact of number on multiplicative schemes beyond mDC was not addressed (but was suggested as a fruitful area for future research).

Research Questions Revisited

The results of this study suggested that number sequences that were well known to the students supported SLDs' progression from additive reasoning to (mDC). In early participatory stages, using familiar numbers (in this research, 2 and 5) promoted multiplicative solution paths, e.g. counting units of five while simultaneously keeping track of how many such units they have counted. This allowed for student reflection on their activity of double counting two different sequences of numbers and its effects in terms of the total of 1s accrued. Introduction of more difficult numbers—any numbers that are not known number sequences—tended to limit the multiplicative thinking. Such limitation was indicated through the child's folding-back to known (additive) solution strategies.

In later participatory stages, the introduction of "harder" numbers promoted the progression within mDC. "Harder" in this study meant numbers for which the child did not have a memorized sequence of multiples and/or numbers that were larger than 5 and required re-inventing a figural way of keeping track. Those challenging numbers created situations where the children required teacher prompting to orient reflection (type-II) on where to stop each of the counts and create a perturbation. Careful sequencing of problems with the introduction of problems that had the same number for the unit rate and different numbers for the composite unit supported student reflection on similarities between the problems and supported student movement within mDC. Reflection across related multiplication problems supported movement to more advanced multiplicative thinking.

Concluding Remarks on the Importance of this Research Study

The process and results of this research study have personal implications beyond the scope of the specific research questions (focus on number) and the claims made thereof. I entered this doctoral program with a teaching certificate, an MA in curriculum and instruction, a special education endorsement, several years of experience as a classroom teacher, many additional workshops and classes in mathematics education, and several years as a district level mathematics curriculum coordinator. With all of my coursework and experiences, I was shocked to realize I little to no knowledge of the differences between multiplicative reasoning and additive reasoning. I had not been introduced to these differences in any curriculum that I had used, or in coursework from a variety of organizations and institutions. It now seems critical to me that the understanding of the development of multiplicative reasoning be a core construct in mathematics teacher education programs, in curriculum development, and perhaps most importantly in special education (SPED) programs. Such SPED programs serve students who, as I shared in chapter two, pay a huge 'price' when left behind with only additive schemes to support their mathematics (or, worse- left behind without the concept of number.)

One illustration of the absence of multiplicative reasoning in special education programs could be found in the goals set for SPED students' Individual Education Plans (IEP). Some sample goals, found in recommendations published by some large urban districts in the United States, are consistent with the goals I have seen in practice. For example, goals such as, given basic multiplication flash cards, the student will recall multiplication facts through the nines with 80% accuracy in 4/5 trials, seem pervasive in the field. This goal focuses on completion and mastery of multiplication problems; attention to the reasoning a student might be using to procure her or his answers are noticeably absent. As was apparent from the literature review and the results of this study, it is possible for students to solve multiplication problems using additive reasoning. Furthermore, there is a critical difference between one's fact recall and her or his multiplicative reasoning, and SLDs' learning seemed hampered if they do not progress to additive reasoning. This aforementioned IEP goal, which appears to be representative of the specific and measurable goals required with an IEP, is inadequate when teachers are charged with promoting student progress in multiplicative reasoning.

As was suggested in this study, having a robust concept of number as a composite unit is a core consideration in promoting SLDs' progress in multiplicative thinking. However, this consideration also seems absent from IEP in special education. Moreover, this consideration can only be used effectively when learning environments for SLDs provide the opportunities for the construction of multiplicative reasoning. The creation of these learning environments seems to point in the direction of direly needed collaboration among mathematics educators, SPED researchers and practitioners, and curriculum (e.g., IEP) designers—so that goal development and instructional planning for SLDs encompass the various facets of multiplicative reasoning (and mastery).

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