ASSESSMENT FOR STUDENTS' CONCEPTUAL READINESS AND MULTIPLICATIVE REASONING

by

ERICKA J. LEGNARD

B.S., Colorado State University, 1992

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Ericka J. Legnard

has been approved for the

Mathematics Education Department

by

Ron Tzur, Advisor

Heather Johnson

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Legnard, Ericka J. (M.S.Ed., Mathematics Education)

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ABSTRACT

This research thesis addressed the problem of how teachers can assess students' readiness to begin learning and progress through multiplicative reasoning. Addressing this problem is important, particularly because this type of reasoning serves as a conceptual foundation for higher-level mathematical topics, such as fractions, ratios, proportions, and algebra. When students and teachers struggle with these higher-level topics, assessing conceptual prerequisites plays a key role in identifying and eliminating the root causes of difficulty. This constructivist premise, of the need to assess students' current knowledge as a basis for implementing instruction adaptive to the students, is considered essential for effective teaching and underlies this thesis study. Accordingly, a first aim of this study was to determine the reasoning students need, both additive and multiplicative, to learn the aforementioned higher-level concepts. A second aim was to create an assessment instrument for teachers and researchers that would provide essential information about students' preparedness to engage in further, meaningful study of mathematics. The researcher used studies on students' thinking and learning to create assessment items that would either bring to light their available reasoning or reveal its absence. Additionally, a theory that describes the stages of student understanding in developing multiplicative reasoning was used to better place concepts in a framework that depicts current knowledge, and what is needed before instruction can fruitfully move ahead. The assessment items were presented in a format and in situations that should be accessible to elementary and middle grades students and teachers. An instrument consisting of these items was then developed to serve assessors of students' multiplicative reasoning. This instrument includes differentiating items to fit with various grade levels and to determine the extent to which students can solve these tasks independently or with assistive prompting.

The form and content of this abstract are approved. I recommend its publication.
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CHAPTER I

INTRODUCTION

This research thesis addressed the problem of how students' conceptual readiness for and progression through reasoning multiplicatively can be assessed. The research problem grew out of an interest in the issue of student conceptual preparedness for learning fractions. The teaching and learning of concepts such as fraction and ratio seem quite ineffective in many American school settings (Fan and Zhou, 2006). To learn fractions, many researchers believe that students need to have previously constructed ways of reasoning multiplicatively (Confrey and Smith, 1995). It may be that some students are not prepared with the necessary foundation, namely multiplicative reasoning, for instruction in more advanced concepts, such as fractions, at the time that the instruction is delivered. Tzur's (2004) work indicated that one of the crucial steps a teacher can take to best serve his/her students is to assess the underlying reasoning necessary for students to build a concept before attempting to teach that concept. Assessment of students' prior knowledge is considered important before instruction in any case, therefore, if we consider multiplicative reasoning as an important basis for fraction, assessing students' conceptual preparedness for multiplicative reasoning seems critical for enabling their progress through the study of fractions.

Weak foundational skills can hinder a student for years to come (Confrey and Smith, 1995). In particular, a lack of sufficient prerequisite knowledge may cause students to learn the rules and procedures for concepts, such as fraction operations, without being able to take in the fundamentals of fraction concepts. This may allow the student to be somewhat successful in the short term, but as time goes on there can be frustration in trying to understand the concepts that come later. The need for a rich, thorough learning of fractions is clear when we consider the wealth of future mathematical topics that students will encounter that require an understanding of multiplicative reasoning and fractions (Thompson and Saldanha, 2003). Ratio and proportion follow soon after fraction instruction, but the concepts of similarity, slope, exponential and logarithmic functions, just to name a few, also draw on multiplicative schemes. Confrey and Smith (1995) traced such schemes extending through calculus and beyond, and noted that they should be based on multiplicative concepts instead of additive ones. Teachers often lament that introducing an algebra problem including fractions seems to dissuade many students, even if using algebra with whole numbers seems solid for those students . This indicates that there may be a problem in the way that students learn the foundational knowledge for fractions, and that there is an urgent need to correct this problem.

Students who are able to efficiently and independently solve problems involving multiplicative situations may be more able to create conceptions that support the learning of future concepts, and those who do not can find these future topics increasingly difficult to understand (Confrey and Smith, 1995). It seems an important issue that students are prepared with a solid conception of multiplicative reasoning, leading to a solid conception of fraction, and provided a clear path to future studies such as algebra.

In the context of a regular classroom experience, it is difficult for educators to correctly determine the reasoning used as students solve problems. There are factors, discussed in this section, which make this especially true when the assessment is taking place mainly with pencil and paper tests, or when it is done informally through brief conversations with individuals or groups of students. This is not to say that these means of formative assessment are not valid. Rather, that when teachers need to know that students are prepared to undertake what seems to be such important new learning, there is a need to get an authentic, detailed picture of students' readiness.

There are many factors that make the assessment of reasoning difficult in the classroom. These include the time-consuming need for careful selection of the problems to be used, unintentional prompting within the problem or by the teacher or other students, and the ways that students can demonstrate what seems to be evidence that they understand, when in fact they may not. The ability to remove as many of these factors as possible and to assess a student in a way that is likely to give a clear picture of each student's capabilities is important.

As a middle school teacher, I have come to recognize that teachers need to know the importance of this multiplicative foundation, how to detect it, and ways to support students at their current level. A user-friendly assessment of multiplicative reasoning, which gives specific information about what students do and do not understand, would be a tool to help teachers pinpoint where their students may need more experience before fraction instruction begins. It is proposed in this thesis that having these assessment results is important for teachers and students. This is not a simple matter, however, because students who appear to have multiplicative reasoning may not have it. A reliable assessment that helps clarify the depth of students' understanding could be valuable to ensure their conceptual preparedness.

The instrument developed as part of this thesis study, the Assessment for Multiplicative Reasoning (AMR), was designed to allow assessors (e.g., teachers) to determine not only the student's level of multiplicative reasoning, but also whether a student is able to reason independently or with support. The results of the AMR allow a teacher to determine not only students' preparedness for fractions but also whether what is needed is to lay the groundwork for a multiplicative scheme, building on the previous scheme, or help to progress to the next stage within a new scheme. This is something that as a teacher I will certainly incorporate into my practice. The knowledge and tools that I have gained in this study, both the awareness of multiplicative reasoning as an essential part of math education, as well as a way to assess this knowledge, will serve my students well in the future. I feel better prepared to address the issues that too often arise when I begin the teaching of fractions. I intend to use this assessment to determine the readiness of students according to the schemes that they can access, as well as their level of understanding within the scheme, to better target instruction for my students.

A discussion of the meaning of scheme in this context will follow in the Review of Literature section in Chapter II. The next section depicts the conceptual framework that guided this research, because some of the constructs from this framework are needed to better delineate the problem addressed in this study.

CHAPTER II

REVIEW OF LITERATURE

In this chapter, I situate the research problem depicted in the previous chapter within the larger body of research literature. I begin by summarizing the constructivist theory of learning that underlies all other ideas in this work. Next, I discuss the prevailing theories of best practice in fraction instruction. A discussion of the facets of multiplicative reasoning and why it is important for fraction concepts follows, to link the best practices to what we know about multiplicative understanding. Then, I will detail the learning theory that I suggest using to support students as these practices are employed, as well as an explanation of the participatory (prompt-dependent) and anticipatory (prompt-less) stages of understanding that students would arrive at using this theory. Finally, I will explore methods of assessment that can be used to determine a student's knowledge, as well as the previous work on assessments specific to multiplicative reasoning.

Constructivist Theory of Learning

This thesis used a constructivist theory of learning as a guiding framework, because the focus of the assessment is on cognitive aspects of students' learning, namely, reasoning multiplicatively. A constructivist theory asserts that students must construct their own knowledge, and that it cannot be transferred to them through some other means (Dewey, 1938; Piaget, 1985; von Glasersfeld, 1995). With this idea in place, it seems that to guide students in the learning of new concepts, such as fraction, ratio, and the like, teachers need to be aware of activities that help students make their own meaning and understandings, and allow students to do the work necessary to create this new knowledge. In constructivism, the student's thinking is a key goal for any lesson, and effective lessons and teachers will help the student to do the thinking required to learn.

Piaget (1985), the originator of constructivism, argued that anything that is to be understood should be rooted in practical experience. Von Glasersfeld (1995) interpreted this belief put forth by Piaget, Radical Constructivism, as "knowledge being built up by the cognizing subject" (p. 51). If teachers are not able to relate multiplication, or fractions, or ratios, to something that students already understand, and help them see how to operate on fractional quantities using something tangible or figural that is well understood, then it is unlikely that students will be able to create meaningful understandings of fractions, let alone apply any of their understandings to other situations. A further development of these constructivist principles follows in the next section about the development of schemes, a key building block of logico-mathematical thinking in the constructivist theory.

Construction of Schemes

One of the foundations in the constructivist theory of learning is the concept of scheme (Piaget, 1985). A scheme is thought of as a 'miniature framework' in the mind, which allows thoughts, experiences, and knowledge to be organized and connected in a way that makes sense to the learner. Von Glasersfeld (1995) described the action on schemes as a three-part system, comprised of a perceived situation, an activity, and a result. The perceived situation is the learner's reality in the moment, which may be something that they notice on his or her own, or their view of a situation created by a teacher to inspire learning. The activity is something that is done to an object in that situation, which may involve physical manipulation, such as pouring of liquid in a geometric model, or a mental activity completed on an available scheme, such as considering the effect of adding two negative integers when the addition of two positive integers is understood. The result is what happens after the activity is completed, or what the learner notices about these effects, and how they relate to the object or scheme. This three-part system describes the way that assimilation and accommodation can take place in Piaget's constructivist theory.

Schemes begin as simple things, such as the scheme for what a table might be, and then move on to be much more complex, such as a scheme for things that are related proportionally. They can involve the concrete or the abstract, but the fundamental idea regarding schemes is that they are necessary for new learning to take place. New information, such as the relative size of fractional quantities, must be taken in to the consciousness of the student in one of two ways. The processes of assimilation and accommodation, described below, take place to allow students to learn via transforming their existing schemes into novel ones (Piaget, 1985).

The student assimilates new information into an existing scheme, meaning that it relates so naturally for the student that they incorporate it into what they already know. For instance, if students already know the concept of unit fractions such as 1/5, 1/8 and 1/6, and the unit fraction is well understood, when they encounter a new unit fraction, such as 1/7, this should be a simple assimilation. The new idea of 1/7 can be incorporated right into the present scheme because it fits nicely with the "rules" that the student has constructed to understand unit fractions.

If, however, the student does not have a scheme for unit fractions, they will need to accommodate, or make sufficient changes in an existing scheme, such as how a whole number is composed by iterating the unit of 1, so that they can develop an understanding of the unit fraction 1/7 (Piaget, 1985). Accommodation is a much more complex and laborious task, what we usually call learning, and requires the student to alter the “rules” which currently govern their scheme to allow the understanding of the new information, or to create a new scheme that will house the new information. This accommodation, or linking of the known to the unknown, is what is referred to in the future sections describing the ways that multiplicative reasoning can be used as the scheme which will be accommodated for a student to understand fractions.

It was further conjectured by Piaget (1985) that students relate new information to what they already know, even if what they know is incomplete. If not properly guided, this connection may be tenuous and without any mathematical meaning, causing the new learning to be unstable or misdirected. Teachers see this phenomenon when students express misconceptions or convoluted reasoning in arriving at a solution that seems to the teacher to be inexplicable. Presumably, the student made some connections in past work, which functioned for them at the time in some way, but perhaps only coincidentally or tangentially. This may happen if the underlying scheme is not properly developed, such as may be the case with multiplicative reasoning and future concepts.

In the context of this thesis, the development of a scheme as an accommodation of previous schemes is central, because it builds the case for educators to be mindful of the schemes that student have available, through some sort of assessment, and how we expect them to elaborate those schemes to include new knowledge. This understanding helps to guide the later sections, to provide a foundation for the need that drove the creation of this assessment, and explore the theories with which it was constructed. Giving students a foundation in multiplicative reasoning, and creating experiences for them that will show fractional quantities as multiplicative relations may be a solid path towards real understanding of concepts such as fractions, ratios, rates and proportions. I suggest in the following sections that one of the problems that teachers face in teaching fractions is that the multiplicative schemes needed to proceed are not fully in place, and cause students to have great difficulty in learning. Evidence for that difficulty is presented next.

Fractions as Problematic for Students and Teachers

As students are attempting to assimilate and accommodate their schemes throughout their mathematics education, some seem to come more easily than others. The conception of fraction is often difficult, as the studies in this section will show.

The familiar International Mathematics Studies, including SIMS in 1976 and TIMSS beginning in 1995, as well as the International Assessments of Educational Progress (IAEP1 and 2), showed US students lacking in mathematics in general, and fractions, ratios and multiplicative reasoning, a prominent part of all of these assessments, were no exception (Fan and Zhou, 2006). The National Mathematics Advisory Panel stated, "The most important foundational skill not presently developed appears to be proficiency with fractions …" (2008, p.18). It seems that this is an area where American teachers and students need assistance and continued focus to create improvement.

Fractions are a topic often dreaded by students and their parents, and cause teachers from middle school through high school difficulty and frustration when students who should have learned the concepts long ago struggle with using them to solve problems. Students have been seen in studies to avoid operations such as multiplying by fractions, even when this would be the most efficient strategy for solving a problem, and they have had experience with these methods before (Fischbein et al., 1985). This is presumably because they felt uncomfortable with the meaning of such an operation and often preferred to cling to more familiar, additive methods. This may be due, in part, to the evidence that multiplicative quantities are not intuitive in the way that additive ones are, and that students will generally not develop these understandings without careful and explicit instruction (Dehaene, 1997; Fischbein et al. 1985; Steffe, 2002). It has been seen in multiple studies that in order to come to a clear understanding, students should be assisted in their learning process, usually in school by their teachers and through social interactions with others, in order to develop the schemes necessary (Dewey, 1938).

While there is no single universal theory for what should be done to bolster the fraction proficiency of US students, there are many theories that intend to guide educators in this endeavor. Many researchers have examined the teaching and learning of fractions as they attempted to unravel its complexities. Some of the main theories are presented next in order to understand which among them seems to be the most promising, and what prerequisite knowledge is required in each case.

Practices for Developing Multiplicative Understanding of Fractions

If fraction education is problematic, it makes sense to explore the prevailing research about how the teaching and learning of fractions might be approached and understood. The theories presented here are highlighted to show the rationale for the type of reasoning that I tried to assess using the AMR. There are many options for introducing fractional concepts to students, including, but not limited to, part-of-a-whole, fractions as division operations, whole number bias, and the reorganization hypothesis

(Siegler et al. 2011; Stafylidou and Vosniadou, 2004; Steffe, 2002; Thompson and Saldanha, 2003;). As a teacher, I found exploring these theories indispensable for understanding the different ways of thinking about how to teach fractions, and to determine which theory seemed most likely to produce effective learning. The sections that follow explore what types of schemes students need to develop during the study of fractions, and then to consider the underlying schemes that would need to support that study. From this understanding, the assessment developed to focus on these underlying schemes.

<u>Part-of-a-Whole Model</u>

A prevalent method used to help students think about fractions in American schools is the part-of-a-whole model. In this model, the student sees a whole divided into a given number of equal-size parts, and consider the fraction as one part, or several parts, of that greater whole. This part-of-a-whole conception breaks down, however, when students are confronted by a quantity that does not fit this model, such as 8/5, because 8 cannot be a part of five (Thompson and Saldanha, 2003). When students are asked to operate on these often mysterious quantities, they may be forced to rely on taught or assumed rules, which may have little grounding in mathematical theory. This process is understandably confusing, and may contribute to student thinking that math is a set of arbitrary rules to be followed, regardless of intuition or prior knowledge (Erlwanger, 1973). The true understanding of mathematical concepts is a major purpose of valuable mathematics education (NCTM, 2000), and when students are not able to make sufficient meaning, not only does their understanding break down, but their enthusiasm for the discipline as a whole may break down as well. It is considered crucial in this thesis that students understand fractions as multiplicative quantities instead of as part-of-a-whole. For example, they need to think of a unit fraction, say 1/6, as a unit such that the whole is 6 times as much.

Fractions as a Division Operation

A related understanding of fractions as multiplicative constructs, in place of or in addition to the part of whole model, is to see them as the effects of a division operation. In such a model, 5/8 means five divided by eight, which is a valid way to consider this notation in certain circumstances, and which will also yield the decimal form of this quantity. However, such a notion can be problematic when one encounters, for example, 5/ (8/3), because the operation to be completed is not clear (Thompson and Saldanha, 2003). This way of introducing fractions may also be dangerous to student understanding because it does not provide a way to think about the value of the quantity presented, nor a way to compare and order fractions without completing the operation and comparing the decimal. This lack of understanding of the magnitude of fractional quantities can create difficulty for estimation and using intuitive means to determine whether the solution to a problem involving a fraction operation is reasonable or not. On the other hand, 5/8 could be thought of as a unit that is 5 times as much of another (1/8), where the whole is 8 times greater than 1/8, as explained for unit fractions. Such a conception (e.g., 5 units of 1/8) supports understanding of fractions in a way that does not seem possible with the division operation model alone.

Linking Additive and Multiplicative Situations

Another theory, proposed by the Rational Number Project (Behr, et al., 1983; Cramer et al., 1997), is that the comparison of additive and multiplicative situations be made explicit, so that students see examples of each, side by side. The theory supposes that the students are then better able to use this knowledge to understand that fractions are multiplicative quantities and can be treated as such. This avoids the common lack of understanding by students, for example, that in making an equivalent fraction you may just as easily add or subtract the same quantity to or from the numerator as multiply or divide by the same quantity. This type of misconception may show that the student does not see the fraction as a quantity involving a multiplicative relation.

One theory related to this stance is whole number bias, or whole number interference (Stafylidou and Vosniadou, 2004). In this approach it is argued that when students apply what they already understand about whole numbers to understand fractions, confusion occurs, and that we should not try and connect the two for students. For example, students may mistakenly believe that division by a fraction smaller than 1 results in a quantity smaller than the dividend, similarly to the result of division by a whole number. While drawing upon prior knowledge is the only way for students to create new knowledge in the constructivist view, it does present problems if students make associations that are not helpful in understanding the concept. Stafylidou and Vosniadou (2004) and Streefland (1991) stated that the conceptions students have about natural numbers create problems for students when they encounter fractions, because they generally draw incorrect conclusions, or make connections between ideas that are not mathematically sound. For example, students sometimes reason that as the numbers in a fraction get larger, the value of the fraction become larger as well (e.g., 9/10 vs. 90/100, respectively), regardless of the invariant, multiplicative relation between numerator and denominator. The study conducted by Stafylidou and Vosniadou (2004) asked students to compare fractions and give a rationale for their conclusions. The responses were categorized according to their conceptions. The conception called Relation between Numerator/Denominator was the only one in which students showed a comprehensive understanding and strong conception for all types of fraction relationships, including improper fractions. Interestingly, even though this theory contrasted with the one that follows, the importance of multiplicative reasoning prevailed in both cases.

The Reorganization Hypothesis.

An alternative theory to the whole number bias, and that supports the development of fraction from whole number multiplication, is called the reorganization hypothesis (Steffe, 2002). It states that the number system that children learn in early grades, should be expanded, or reorganized, to include all rational numbers instead of separating whole number and fractional quantities for students (Siegler et al. 2011; Steffe, 2002). This theory finds value in the struggle to integrate fractions and whole numbers as members of a continuous number system, and to create a comprehensive mental number line through which students not only gain a strong understanding of the values of rational numbers, but also a richer understanding of whole numbers. This "marriage" of whole number and fraction is also supported by recent neurological research which indicates that the areas of the brain in which fraction magnitude decisions are made occurs in the same area of the brain as whole number processing, and that absolute magnitude and relative magnitude are processed with "the same analog code" in the brain (Jacob and Nieder, 2009, p. 4656).

Steffe's reorganization model (2002) asks students to equipartition to find the unit fraction, and then iterate it to create the desired fraction. As Tzur (2004) emphasized, the key in such an approach is building on the child's activity part of a scheme (here, iteration of unit, which underlies the creation of whole numbers) instead of on the child's abstracted construct of number per se. Another aspect of Steffe's theory is to use splitting, which asks students to coordinate both partitioning and iterating as inverse operations. This is context-independent and applicable to many other situations, including improper fractions, and underscores the multiplicative relationship by focusing on the relation between the unit rate and the number of units.

Ni and Zhou (2005) also advocate a merging of teaching whole number and fraction concepts concurrently, to emphasize and link the known whole number concepts with the newer fraction concepts. This can allow students to accommodate their whole number schemes effectively to understand rational numbers.

The basis for all of these strategies is multiplicative in nature, and calls for students to have a strong background in the ways quantities can be related multiplicatively in order to make sense of the tasks presented. It is considered for the purpose of this thesis, and the associated instrument, that reorganization is preferable as a method for guiding students in their understanding of fraction. This thinking, guided the development of the assessment, discerning between schemes available to students that can be reorganized further toward intended mathematical concepts.

It is important to note the conclusions of researchers regarding the most effective ways for students to learn fractions, because many students, as well as adults, assume that multiplication is simply repeated addition. Such a conception does not seem to support fraction understanding as effectively as a more robust conception of multiplicative relationships. The next section focuses on ways of reasoning multiplicatively that were seen to be paramount in the theory of reorganization, and how such reasoning may support students in their future learning.

Multiplicative Reasoning

In the previous section, I have provided a rationale for multiplicative reasoning as the conceptual basis for strong fraction understanding. In this section, I provide more details about what that reasoning entails. The understanding of multiplicative reasoning for whole numbers that is used in this paper was summarized by Steffe (1992) in the following statement; "For a situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one of the composite units is distributed over the elements of the other." To make sense of this foundational assertion, one must understand the idea of number as an abstract, symbolized, composite unit.

A composite unit is an abstract construct, or scheme, that a person uses to symbolize potential (but not actual) results of counting, such as a numeral or number word, which means for him or her, a collection of units of ones. When a student says "4", and is using it to refer to a unit of four ones considered as a group, the student understands composite unit. In earlier stages of development, a student may recite the word "four" in rote counting, or know that it goes with the numeral 4, but she must develop the idea that the number four is a group of four ones which can be assembled and re-assembled (Olive, 2001). For example, understanding that 5x7 is not simply 5 repeated 7 times, but five units, each composed of seven units of 1, or seven units of five units of 1, and why they yield the same number of 1s in total, is necessary for students to understand multiplication.

A repeated addition model misses the coordinated units idea of multiplicative reasoning. Several of the core ideas that help students understand multiplication in a flexible, complete manner, such as the importance of units as a point of reference, are not supported by repeated addition. In a repeated addition model for multiplication, we can consider, for example, three pies made of six apples each, which total eighteen apples (or, in repetition, 6 apples + 6 apples + 6 apples =18 apples). In a multiplicative model, we coordinate the one composite unit of six apples per pie (unit rate) with three pies as the number of composite units, which when distributed as explained above produce eighteen apples as the total of 1s, coordinating two different units to yield a third unit). In the multiplicative situation, one must keep track of whether the number refers to apples, apples per pie or pies themselves, and distribute the six apples across each of three pies to find the result. When thinking of this situation as repeated addition, this coordination can be completely missed by students.

The apparent weakness of repeated addition is confirmed by Ni and Zhou (2005), who stated that not only does repeated addition prove a weak and limited view of multiplication, but that this reasoning is specifically harmful to students in learning fractions. One issue is that seeing 5x7 as five groups of seven is conceptual, which is essential for building schemes and understandings, while seeing 5x7 as seven added five times is simply a command to act. When seen as a command, an expression such as 5 times x, or 5x, becomes problematic for students, as they are not able to carry out the specified computation (Thompson and Saldanha, 2003).

Unfortunately, multiplication as the coordination of units explained above is not something that teachers can take for granted as understood by their students. In a study by Clark and Kamii (1996), only 49% of fifth graders had strong multiplicative reasoning, at a time when formal instruction in multiplication is likely finished in the classroom. Furthermore, Siemon, Breed, and Virgona (2005) found that far too many students at the secondary level were using simplistic additive models when multiplication would have been more efficient. It was considered likely by the authors of the study that part of the reason for their lacking proficiency was that they understood multiplication as repeated addition. Steffe (1994) said that when students use an additive model to multiply, it is an indication that they have yet to dissociate the idea of multiplication from more primitive counting schemes.

The difficulties pointed to by the aforementioned studies show that there could be benefit derived from an instrument to determine where students seem to be on the continuum from additive to multiplicative reasoning before multiplication is considered solid enough to begin further instruction in fractions. In this sense, multiplicative reasoning is a conceptual prerequisite for middle grades mathematics (Thompson & Saldnha, 2003; Tzur et al., 2013). Its absence may create significant problems for students who are not yet equipped to construct these understandings. Most importantly, an assessment instrument is needed because teachers may not realize that this gap in reasoning multiplicatively is present. It is often assumed that students are ready for further instruction if they are proficient in executing calculations of multiplication and division, and can give some reasoning that supports their work. The notion that these students may not indeed have a full multiplicative understanding, complete with the ability to track and anticipate the coordination of units involved in problem situations, was paramount in the development of the AMR.

Another reason to develop an assessment instrument of multiplicative reasoning is that many teachers may not be aware of the schemes and stages involved in multiplicative understanding. This is especially true at the secondary level, in which these concepts are not usually taught. Jacob and Willis (2003) commented on the need for teachers to be aware of and able to distinguish between additive and multiplicative reasoning in order to better serve their students. Seeing students "skip count," which may reflect essentially a facility with repeated addition, can give the impression that students understand the concept of counting groups of objects. However, if this is simply a learned method, and not rooted in an understanding, it may be only an indication of a willing student. The next section will focus on the specific schemes that students develop when constructing meaning for multiplicative situations.

Schemes for Reasoning Multiplicatively

The multiplicative reasoning described above does not develop all at once, but on a continuum of schemes. The framework for student understandings discussed in this work, came from the six schemes of multiplicative reasoning proposed by Tzur et al., (2012). These schemes begin with multiplicative Double Counting (mDC), during which students are able to keep track of two quantities when one is distributed over another. This first of six schemes, mDC, is the highest level assessed in the AMR instrument, because it focuses on the shift in thinking from additive reasoning to multiplicative reasoning for the student. For example, being able to find the total number of flowers in four vases, if there are three flowers in each, would require a student to be able to count how many composite units of three flowers are present, while at the same time counting how many units of 1 single flower are accumulating respectively. They would then need to knowingly stop when the count of the composite units of threes reaches four, and understand that the number obtained is total flowers, not vases nor flowers per vase.

This distribution is an advancement in thinking from the additive, in which a student would either need to count each flower, or add 3 + 3 + 3 + 3 to find the result of 12, but misses the idea of the units being coordinated. The meanings of the answer and the two numbers 3 and 4 are critical, as well as which is being distributed across the other. Failure to understand this can become problematic in later studies if this concept is not solidified early on. The six schemes, of which mDC is first, have laid a foundation for educators to not only determine where students may be in relation to multiplicative reasoning, but to ensure that understandings necessary for meaningful work on fractions are in place.

Activity Effect Relationship

One elaboration of the constructivist learning theory was articulated by Tzur and Simon (2004) in their depiction of the mechanism of learning, namely, reflection on activity-effect relationship (AER). This depiction described the way that a student would engage in an activity, which may be visible to an observer or occur mentally, to achieve a goal or desired outcome that the learner has in mind. This activity is performed on an object, which again may be physical or simply a prior conception (i.e., number as an abstract, composite unit). The effect is what follows from the activity performed on the object. It stands to reason that if there is not a viable object, the activity cannot proceed fruitfully, and that the teacher must be aware of the mental objects available to the student. When the object is acted upon an effect will be created, but the effect and learning may not be optimal, and possibly even harmful, if it creates misunderstandings on the part of the student due to insufficient prior conceptions.

To illustrate the working of this theory, consider a student who has a multiplicative scheme such as mDC, as may be measured by the AMR, attempting to solve a fraction problem, and how that student may be instructed using the model of AER. A student with solid multiplicative reasoning could independently solve a problem such as this one: Maria wants to have juice boxes at her birthday party for all of her guests. The juice boxes come in packs of 4, and the store has 6 packages on the shelf. How many juice boxes will Maria have if she buys all 6 packages of four?

A student who has multiplicative reasoning could solve this problem by iterating the composite unit of four six times to efficiently find that there are 24 juice boxes to be purchased. When it has been established that the student is able to operate in this way, the multiplicative scheme being used can be extended to allow the student to think about fractions, such as considering what fraction is just one pack of all 24 items). This is exactly the type of reasoning that the AMR can detect to determine whether a student is ready to advance to fraction. A student may then be presented with a problem such as this one: Maria is cleaning up from her party, and finds that some of the juice boxes she served her friend are not finished. She finds five juice boxes that are each ¼ full. If she puts them all together, how many boxes could she fill completely?

Using a continuation of the previous problem can help students see the connection between the two, and begin to activate the scheme, but they will likely need further direction to fully utilize the current scheme for this problem. The student may be prompted to draw the boxes, and to describe what they see, redrawing the juice boxes when combined. The student could then be prompted to use the same idea that was used in the multiplication problem to describe the total. By reflecting on the commonalities of six units of four juice boxes, and five units of ¼ juice boxes, a student can come to see that the total would be 5*1/4 , or 5/4 of a juice box. This can mean for the student that it takes four of these units to make a whole, and that five are being considered, so I will be able to fill up one juice box with four units and will then have one left over, the same as 1 ¼ juice boxes. If the student is comfortable with the multiplication to find the number of juice boxes in the previous problem as 6 * 4 or 24 juice boxes, this same conception can be used to understand fraction operations.

This transition is not likely to occur quickly or easily, but can occur with continued focus on the conceptions that students have, and continued emphasis by the teacher on the common patterns with iteration of whole numbers and fractions. By asking the student to reflect upon their work for each type of problem and reporting on their findings, a solid foundation for the understanding of fractions can occur. The key in this process is for the student to reflect upon, and make sense of what they see as it relates to what they already know. It is important that they do this reflection while being directed toward reactivating the pre-existing schemes that will best support this new learning, using multiplication to support an understanding of fraction. This understanding may begin with students needing support and prompting from the teacher, but with assistance, students will become independent even when the situation or context is different. The reflection on AER model helps us to understand these two stages, called anticipatory and participatory (see below), as well as the need for fine-grained assessment, which is described in a later section as a way to discern between these stages.

Anticipatory and Participatory Stages

That students may be at different stages when constructing a new scheme was considered in the development of the assessment in this thesis, and these two stages are explained in the section that follows. Participatory and anticipatory stages were considered important because in order for students to be able to function effectively in their multiplicative schemes, they would need to be at the later stage described here. Also included in this discussion are the ways to determine the stage in which a student seems to be functioning, a critical part of the AMR.

Tzur & Simon (2004) postulated that students may have constructed a new scheme at a stage, called participatory, which requires prompting in order for them to access, and use, the newly forming scheme. Later, the students may progress to a more advanced stage, called anticipatory, in which they can independently use the new scheme. This distinction implies that students may be able to solve a problem given manipulatives, following a lesson on that concept, or with prompting, but they would not be able to solve independently, or when the memory of the activity has faded. Tzur & Lambert (2011) related this cognitive stage to Vygotsky's notion of Zone of Proximal Development (ZPD), suggesting that the ZPD is a reflection of students' cognitive need for prompts.

A student who is at the participatory stage may be able to solve a problem about the number of flowers in four vases with three flowers each, during or immediately following a lesson in which these types of problems were posed, and worked through with guidance from the teacher (e.g., "How about using our fingers to keep track of how many vases and how many flowers we have in all?"). After some time, however, the student may seem not to be able to solve problems of this type at all. This, sometimes called the "next-day phenomenon", is a common stumbling block for teachers and students, causing frustration for both parties. Possibly, the issue here is that the student might have been only at the participatory stage, meaning that they must be prompted, at least mentally, to be able to reason about the problem. The student seems to understand, and may be able to solve independently after prompting, but the next day in class the student may not remember how they should proceed on a similar problem. For example if the teacher asks questions such as "How many flowers were there in one vase?" "Now how many would be in two vases?" to lead the student's thinking, it may quickly result in a student's ability to resume their previous way of reasoning and solving the problem. The student continues to need prompting, however, and will likely not be able to solve these tasks as time goes on and the activity becomes a distant memory.

At a higher level of the participatory stage, a student may prompt oneself upon realizing that he made an error, sometimes called the 'oops' experience. For example, students may begin the vase problem by saying there are seven flowers (e.g., adding 4+3), and then correct themselves as they realize that there are three flowers in each of the four vases and answer correctly. This is still not considered anticipatory, because the student needs to self-prompt. However, this is a higher level of the participatory stage, because the prompt has now come from within the students' mental system (Tzur & Lambert, 2011). One difficulty in discerning between the levels is that unless the teacher is very involved with the student during the solution process, the self-prompting may not be evident. The student will correct and then move on, leading the teacher to assume independence in using the new scheme, and thus anticipatory status.

Unlike at the participatory stage, a student at the anticipatory stage does not need to consider the activity, or to be prompted. Their understanding is more solid because it occurs in the student's mind as a well-formed scheme, independent of working through the activity in the familiar context. In the example above, the relationship between flowers and vases is clear to the student, and they don't hesitate to proceed, anticipating the entire process from the start. They anticipate the need to coordinate counting of one composite unit (4 vases) with another (3 flowers/vase) to calculate the total (e.g., 1 vase is 3 flowers, 2-is-6, 3-is-9, 4-is-12). This anticipatory stage can become a conceptual foundation for meaningful memorization of the fact 4 x 3 =12.

It may be that in the classroom some students do not reach the anticipatory stage of a concept, such as multiplicative double counting, before the instruction moves forward. For students not at the anticipatory stage a problem arises, because though they have been exposed to a concept, and made progress toward understanding at the participatory stage, they are not prepared to independently use that understanding as a building block for new learning before the topic of study changes. This situation is also frustrating for teachers, who feel as if the student should have learned the material already, and showed signs of understanding, but cannot access this concept or scheme when they attempt to move forward. In the teaching of concepts such as fraction and ratio, a participatory understanding of multiplicative relationships is not sufficient to create understanding about the meaning of fractions, so the detailed assessment of this reasoning is crucial. The next section discusses the type of detailed assessment that can shed light on these distinctions, and was used in the development of the AMR.

Fine-Grained Assessment

The assessment technique used in the AMR is one developed by Tzur (2007) to better assess the precise level of student understanding, by distinguishing between the anticipatory and participatory stages explained in the previous section. In fine-grained assessment, it is important that the items begin without prompting, and progress to less difficulty and more support, in the form of prompts and questioning, to determine whether the student is able to reason independently. When students are presented with problems, it is without any prompting or introduction by the assessor. If the student is unable to solve the problem, prompts are then given in a way that gives the student clues about how to act, while being cautious to give as little away as possible. The intent is to find the highest level at which the student can function. If the student is given an easier problem first, or is led toward that higher understanding by instructions, the reasoning seen may be as a result of the unintentional prompting the student receives, instead of the student being able to anticipate the entire process on her or his own.

For example, if one desires to know if a student can find a missing second addend, the problem should be presented without any prelude. The student may be asked, “I had three candies, and my friend gave me some more. Now I have seven candies. How many did my friend give me?” If the student is able to provide reasoning and solve this independently, they can be considered at the anticipatory stage. If they are not able to solve this problem, the assessor could prompt, beginning with calling attention to the numbers in the problem and what they mean. The assessor might ask: “How many candies did I have to start with?” and then ask the student if they can solve. A follow up could be: “How many candies do I have at the end?” and then: “Can you use the number

I have now and the number I started with to find how many my friend gave me?" If the student can use these prompts to solve, they are at the participatory stage for this conception. They have some knowledge of how this problem can be solved, but need to be prompted to arrive at a solution. If the student is not able to solve even after prompting, it is likely that they have not begun to build the concept for themselves at all.

Additionally, it is important that student assessment not follow an activity in which they were engaged in solving similar problems. This is to avoid the participatory student appearing to be anticipatory. Immediately following a lesson, the student can simply recall the activity, and use it to solve the problem. It may be impossible to tell whether the student can reason this way independently or only following the lesson.

The importance of this distinction between participatory and anticipatory students is that even though the participatory students may appear to understand the concept in the context of the current class period, or with subtle prompting, these participatory students will likely not be able to draw upon this reasoning to develop later concepts, and could be left behind. This shows the importance of an assessment that can distinguish between these stages, so that planning for these students can be optimal and directed specifically at their area of need. The next section shows previous work in assessing multiplicative reasoning, which was used to guide the development of the AMR, and to show the differences between these attempts and mine.

Previous Research on Assessment of Multiplicative Reasoning

There are some existing instruments that have been developed for the assessment of multiplicative reasoning necessary for future concepts. Here I will present these instruments, with what I consider to be their strengths, and then explain how the AMR attempts to go one step further to achieve a fine-grained assessment of multiplicative reasoning.

One instrument, developed by Siemon and Breed (2010), was a pencil and paper test to address many of the same concerns expressed in this paper. The assessment considers many of the same conceptions, including distributing one composite unit across another. That instrument has excellent potential to determine whether students are reasoning multiplicatively, and includes rich tasks for them to explore, moving beyond what is assessed in the AMR to include proportional reasoning. The intent of the Siemon and Breed's (2010) assessment is similar to the AMR in that it tries to capture student thinking and elicit strategies used to determine the sophistication with which the student is approaching the problem, and the resources that they employ to solve them. Scoring of this assessment was rubric based, and completed after students had finished the problems. Their instrument, however, did not include a distinction between participatory and anticipatory stages in a child's use of the mDC scheme, and did not provide the same feedback as a clinical interview situation could.

Another assessment, developed by Bright, Joyner and Wallis (2003), has similar features to the Siemon and Breen instrument. This assessment included a multiple choice paper and pencil test, but mentions interviews to follow up and question students about their answers to delve deeper into their thinking. It was noted in their work that it is possible that questions which are easier to answer, such as multiple choice questions, may induce more sophisticated reasoning, while open ended questions may cause students to revert to more primary methods which they feel most comfortable using. Their point may be captured when using the participatory/anticipatory stage distinction, for which specific, finer considerations and ordering of the tasks is needed (Tzur, 2007). This consideration guided the development of the AMR, to allow assessors to differentiate between anticipatory and participatory stages of mDC. The AMR attempts to be more specific than the previous instruments in assessing the smaller increments in reasoning. In the Methods section, the development of the AMR is detailed, using this stage distinction as an important part of the assessment.

CHAPTER III

METHODS

This chapter chronicles the development the AMR, including the selection of tasks, the administration method, the administration guide, and scoring. The Assessment for Multiplicative Reasoning (AMR) was developed to give teachers and researchers a way to assess the level of a student's multiplicative reasoning, or preparedness to engage in that reasoning. The intent is that instruction can then be focused at the appropriate level to give students the experiences that they need to develop the next level of reasoning required to support multiplication, or to indicate that the student is ready with a strong basis to experience other math concepts for which multiplicative reasoning is a prerequisite, such as fractions, ratio and proportion, and slope.

The AMR was developed with assistance from experts, and then tested briefly on a small sample of fourth grade students. The full final version of the assessment to date with all of the components mentioned here is provided in Chapter IV, but examples are given in this chapter to illustrate the development process. This chapter explains the process by which the assessment grew from tasks to target specific schemes of multiplicative reasoning, possible distractors that were taken into consideration, and the design of prompts to bring to light the possible participatory and anticipatory stages at each conception. Later sections describe the development of an administration guide for assessors and considerations for scoring the results. Refinement based on feedback from experts and from the initial testing are discussed in each section as appropriate, as well as their implications for the improvement of the AMR.

Development of Tasks

Tasks were created based upon theoretical, literature-based analysis and on my interpretation of the performance that students should exhibit if they have completely developed the scheme in question (Zazkis and Hazzan, 1998). Through conversations with Dr. Ron Tzur, I decided to include prior schemes in the assessment: Counting On, Missing Addend, and Composite Unit Iteration (shown in Chapter IV), to screen students and determine whether the additive schemes, rooted in number as composite unit and necessary for multiplicative concepts, were accessible to the student. One multiplicative scheme was included as well, namely Multiplicative Double Counting (mDC), to assess the progression of students from additive to multiplicative reasoning.

The tasks were differentiated into grade bands, in an attempt to allow for flexibility in administration by providing multiple entry points for students and assessors to use, being responsive to previous knowledge about student understanding. For example, a student at the fifth grade level may be tested at the 3-4 grade band, if the teacher or assessor knows that the student will struggle with the numbers involved at their grade level. It is also advised, and stated in the Administration Guide in Chapter IV, that in general the assessor should begin with the grade level task, but move down to the previous level if the student is unable to answer. The intent here is to determine whether the struggle is due to the numbers involved, or to the conceptual scheme or stage being assessed. The following section provides a description of the tasks used to assess these schemes, and information about the choices made in their development.

Composite Unit Iteration Task

Composite unit was explained earlier, and is important for both the additive and multiplicative schemes. The Composite Unit Iteration task asks students to use these units and iterate them to find a new quantity, without counting all of the items. This requires students to coordinate two quantities, the unit to be iterated and the number of iterations that take place. This task is administered first because it involves composite unit, which is important to both additive and multiplicative thinking, but also because it represents a midpoint in the continuum of the assessment as a whole. This approach was used although there was not enough data from student responses to create a proper Guttman Scale (Trochim & Donnelly, 2008) for these tasks. Such a scale requires one to use data to create a linear ordering of tasks. However, an attempt was made to arrange the tasks in a developmental order that would most likely occur for the student conceptually. Thus, if a student can properly solve and reason about the first task the assessor can move to higher levels, and if not, the assessor would go back to prior conceptions. For instance, it is unlikely that a student who can perform Multiplicative Double Counting would be unable to solve a Counting On problem.

The original Composite Unit Iteration Task I had created, seen in Figure 3.1, was problematic for students. It was intended to give students a visual model and see if they could use the groupings of four beads to iterate and find the total, but students were generally not able to use this information to iterate correctly. It was suggested by Dr. Tzur that I eliminate the task, and it no longer appears in the final version. Instead, a task involving iteration of the smallest composite unit (i.e., 2) was created, with a context using pairs of socks that seemed familiar to any student. This task is further described in Chapter IV.

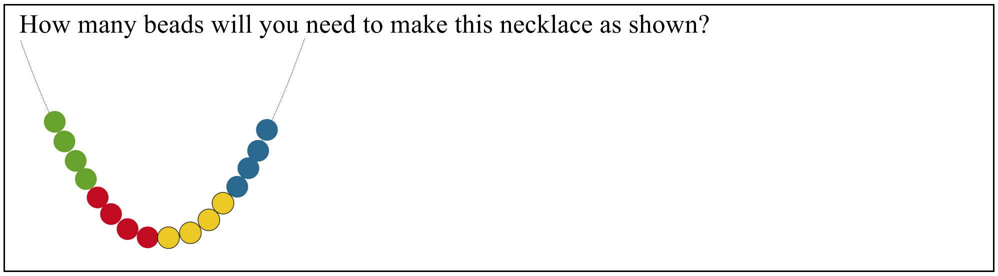

Figure 3.1. Previous Version of Composite Unit Iteration Task

Additive Tasks

The additive portion of the assessment, including the schemes of Counting On and Missing Addend, is not intended to be an exhaustive assessment of additive reasoning, but to give a rough idea of where a student might be functioning in regard to additive schemes in general, and in terms of her use of composite units in particular. These schemes are explained next.

Counting On refers to the process by which a student can begin with a quantity, such as 7, and then add to that quantity, perhaps 4 more, and find the total. The student does this not by starting to count from 1, but by beginning with the known number 7 and counting from there. This counting on strategy is sometimes called double counting, to emphasize that the student is purposely keeping track of 1s in the second addend and knowing to stop after they have counted four more (Tzur & Lambert, 2011). The student might say something like this: “I have 7 already, and four more; so eight is 1, nine is 2, ten is 3, and eleven is 4. I have eleven all together.” The Counting On tasks in this assessment were constructed so that the second, added quantity is smaller. This was done to make representing these problems with figural, pictorial or manipulative strategies more accessible. Assessment of students’ abilities to reverse the quantities for counting on when the larger quantity is given first was not a part of this assessment due to the limited nature of this portion.

The next two schemes are assessed using Missing Addend problems, where a student is given one of the quantities to be added and the total in a problem situation, and is asked to find the other added quantity. In the first type of missing addend problems, the student is given the first addend and the total, and is asked to find the second addend. In the second type of missing addend problems, the student is given the second addend and the total, and is asked to find the first addend. These problems are considered different for students, as the Missing First Addend requires more sophisticated thinking. The Missing First Addend Tasks are considered more difficult, because of the way that the composite units are nested within the whole (Fuson et al., 1988). If the second addend is missing, the student can know where to start her counting, as they do in counting on, and proceed by counting up from the first addend unit the total is reached. If the first addend is missing, students do not have a way to start an activity of counting upward using their previous way of operating. They thus need to either reverse the order of the addends (in effect, creating a missing second addend situation), or count down from the total until they reach the second addend. Both of these operations are more advanced and less likely to be available to many students due to insufficient abstraction of the total as a composite unit containing of both addends as sub-composite units. Additional considerations for the additive tasks are discussed next, in an attempt to clear away distractions that might cause the assessment to give invalid results.

It is understood that some tasks may be much too simple for many students in the upper grade bands, but to be thorough, they are included. It is also understood that the concept of multiplicative reasoning assessed later in the instrument may not be present in students in the earlier grades. Yet, the task is designed in such a way that if younger students are able to reason multiplicatively, the numbers and situations should not be a barrier.

The levels in each task were created to target values and situations that were developmentally appropriate for students who are achieving at the grade-levels indicated. The example seen in Figure 3.1, the Missing First Addend task, is differentiated to four levels to accommodate grades K through 8. I chose the bouncy ball situation in the same way that all situations were chosen, with an attempt to provide familiarity and context to the problems. The numbers for each level (grade-band) were selected (a) in consultation with Dr. Tzur, (b) using expert feedback from 15 teachers, and (c) guided by the work of Sherin and Fuson (2005). The latter, empirical study focused on ways in which chosen numbers and problem types affect an assessment. An effort was made not to repeat and addition or multiplication facts, so that regardless of which grade level tasks were given, or in which order, there would not be any repeats. Also I did not include consecutive facts, such as 3 x 5 and 4 x 5, so that there would not be reliance on the previous problems to answer the later problems.

For example, in the Missing First Addend Task (Figure 3.2), the lower grade levels are given the addend 3 and a sum of 8, so that the total does not pass ten and involve place value. These are numbers that younger students are likely to be familiar with, yet avoid the facts that they might know, such as 5 + 5 = 10. For the oldest grade band (7-8), the AMR uses the numbers 25 and 38 to discourage use of known facts and encourage reasoning. It is unlikely that the student has the math fact 25 + 13 = 38 in their memory, as they may have 4 + 5 = 9, and therefore will make it more likely that they will use additive reasoning to find the solution (e.g., 25+10-35, and 3 more is 38—a strategy indicating the student's operation on 13 by decomposing it strategically into 10 and 3). Each task was designed with these considerations in mind, in an attempt to hone in on the reasoning available without interference from memorized facts or inaccessible numbers and situations.

Missing first addend task

<u>Level 1, K-2</u>

10. Anna has some bouncy balls. Her mom gives her three (3) more bouncy balls. She then has eight (8) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 2, grades 3-4</u>

11. Anna has some bouncy balls. Her mom gives her thirteen (13) more bouncy balls. She then has nineteen (19) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 3, grades 5-6</u>

12. Anna has some bouncy balls. Her mom gives her sixteen (16) more bouncy balls. She then has twenty-four (24) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 4, grades 7-8</u>

13. Anna has some bouncy balls. Her mom gives her twenty-five (25) more bouncy balls. She then has thirty-eight (38) bouncy balls. How many bouncy balls did Anna have in the beginning?

Figure 3.1.Missing First Addend Task

There are some prevailing strategies commonly taught in schools to handle these types of addition problems. The strategies seen in early trials of the problems were using known doubles, such as 4 and 4 make 8, and building on them, so that 4+5 could be seen as the double of 4 plus one. Another strategy is to 'break' or decompose the numbers to make ten, so that 8+5 could be seen as 8+(2+3) = (8+2) + 3 = 10 + 3 = 13. Clearly, many of these strategies that do not involve counting on allow students to use them in any situation, and could not be avoided completely. A resourceful student can use these strategies on any problem. In fact, this adaptability shows a high level of proficiency with additive situations, particularly the operation on and with composite units, and thus indicates that the student is ready to proceed.

These problems require students to be able to either use a modification of the counting-on procedure, where they know to stop at the given total, and must find how many more they need to count from the given addend. Alternatively, they can use counting down, or beginning at the total and counting backwards until they reach the known addend. The Missing Addend problems were created with the missing addend as a quantity that is less than ten. As mentioned in the section on Counting On, this was done because students are being asked to show their reasoning using figural or manipulative means, so it was necessary that the missing portion be kept to a number that would be reasonable for students to represent in this way. Only two of these tasks were included, both considered Joining Problems by Carpenter et al. (1981). The reason is twofold—the AMR was not intended to be a comprehensive assessment of additive reasoning, and the study by Carpenter indicated that this deviation in problem structure did not affect the solution methods of students. It is, intended that both tasks be administered as a reliability measure. It is possible that a student could miss the first problems, the easier missing second addend problem, for various reasons, but correctly solve the more difficult second problem with a missing first addend.

Multiplicative Tasks

Multiplicative Double Counting, as described in the section on Multiplicative Schemes in Chapter II, represents the first stage in the transition from additive to multiplicative thinking. This is the first and only multiplicative stage assessed by the AMR in this prototype.

Due to the fact that Multiplicative Reasoning was the primary goal of the development of this assessment, there are more tasks to lend reliability to this section. The three tasks were all created with the unit rate, how many items per group, being less than the number of composite units given in the problem. For example, in the first MDC task for level 1, there are three cars per box and four boxes. The tasks were constructed in this way to make it more conducive to the creation of a figural, pictorial or manipulative solution. At the time of the defense of this thesis, it was suggested by Dr. Heather Johnson, that the multiplicative tasks be adjusted to be consistent as to the relative size of the unit compared to the unit rate. This additional expert feedback for task adjustment was made and is reflected in the final instrument.

Task Refinement

Initial refinement of these tasks was done through feedback from 15 teachers of K-12 students. One suggestion they gave was to include picture cards for younger students. A picture was included on the Composite Unit Iteration task, a frequent suggestion, to help clarify the meaning of a pair of socks mentioned in the problem. I decided not to include picture cards for the remainder of the problems, because it may provide too much prompting via visible units. This would fail to show whether a student has the reasoning or simply relies on the picture, counting 1s while visually and silently scanning the picture. I believed that part of the scheme necessary for students would be the ability to create the pictorial or figural representation, and that providing a picture might detract from their ability to show this crucial step in operating on units. It would be appropriate for a kindergartener to be unable to solve even the additive reasoning problems, so I did not feel that providing this scaffolding was necessary.

The remainder of the teachers' suggestions involved the numbers chosen for the tasks. There were some repeats in the number families chosen in the initial draft, such as 4+5=9 being used three times on different problems. This was corrected in the later version to include more addition facts, carefully avoiding the doubles and other patterns described earlier. It was also suggested that a problem on the initial version involving earrings be changed to a more gender-neutral item such as mittens. This change was made to have the problem ask about socks, after much consideration as to an item that students might have multiple pairs of at home.

After receiving the expert feedback from Dr. Tzur and the K-12 teachers, the instrument was tested with fourth graders. As a result of that initial testing, the grade bands were narrowed to accommodate students in a more differentiated way. The first draft of this assessment included three levels, K-2, 3-5 and 6-8. During the initial testing of the AMR it was found that some fourth grade students struggled with the values given, and that the grade bands may have been too large. This prompted the further differentiation of levels in elementary grades to include only two grade levels instead of one. The current structure includes levels for K-1, 2-3, 4-5 and 6-8.

Construction of Prompts

In this section, I will explain the way that prompts were developed for each of the aforementioned tasks. These prompts are meant to assist a student who may not be at the anticipatory stage in solving the problem with support, and thus gaining data about the participatory or pre-participatory stage in which the student operates for the scheme being tested. This part of the assessment, in which prompts are provided gradually, is designed to give the assessor an idea of where the student is on the participatory continuum, and what specific support they may need. This is important because a student who has no prior knowledge of the reasoning at this stage will need to be approached differently than one who is nearly anticipatory, and a nearly anticipatory student will likewise need different learning experiences than one who is just beginning to be participatory. This type of stage distinction was the goal of scripting the prompts to be given while the tasks are being completed, instead of simply presenting the problem and prompts to the student as typically occurs in a pencil and paper test.

As explained in the Fine-Grained Assessment section of Chapter II, the first step is to administer the problem without prelude or prompts. This is intended to allow students who have achieved an anticipatory stage for the scheme to demonstrate it. Using the example of the Missing First Addend task above, the student would be given the problem as is and asked to solve while explaining their thinking. If independently doing so, it allows the assessor to infer construction of the scheme at the anticipatory stage.

Whether or not the problem is solved successfully, the student may then be prompted to use figural, pictorial or manipulative means to prove their solution in an attempt to understand the reasoning used by the student. This may be omitted sufficient evidence was given that the student understands the concept, but it provides an opportunity to delve deeper into what the student is thinking about the task. The use of these means can also be a way for the student to self-prompt, or catch their own mistakes. For a student who has used known facts, drawing or using manipulatives allow the assessor to see if these facts are supported by reasoning, or simply reflect rote memorization.

If the problem cannot be successfully solved independently, the assessor will then begin to prompt the student in an attempt to determine whether the student has constructed a participatory stage for the scheme as indicated by solving the problem with support. The prompts are scripted purposely to provide the student information that should help to orient their thinking to the activity involved in the scheme and thus to a correct solution if they have begun forming that underlying conception. To the extent possible, assessors are expected to avoid deviating significantly from these prompts, in part because a teacher or researcher who works regularly with students may unintentionally give additional information that would lead the student to score more highly than is accurate.

An example of the prompting created for the Missing First Addend Task (Figure 3.2), which begins after the student has attempted to solve independently, is seen in Figure 3.3. Some prompts were included to elicit specific reasoning from the student, if the correct answer is given but sufficient reasoning is not.

For both correct responses without reasoning and incorrect responses:

- o Ask students to explain their thinking on this task. It is important to discourage counting all as a solution. Prompt the student to use counting on by covering the first composite unit, or asking them to make a drawing that does not include all of the ones in the unit. If the student is using marks or manipulatives, it may be necessary to cover the first composite unit and then see if the student can then proceed to count on.
- o If at any time during the administration of this task you believe that the number values are impeding the student, move down to a lower grade band and test again.
- o If at any time the student answers correctly using counting on or counting back, go to the next problem.

If response is correct, but the students use an unknown strategy, cannot articulate one, use counting all or known math facts, prompt student using these questions:

1. How many did she have before her mom gave her more?
2. How many does she have in the end? Can you use the number in that she had before as a starting place to see how many more she got from her mom? How?
3. Can you use the number she had in all and count backwards to find out how many she got from her mom? How?

- If the student solves correctly after prompting, move to the next problem

Figure 3.3. Prompts for Correct Responses

A second level of prompts was developed for students who do not answer correctly, seen in Figure 3.4, to determine if they have a high or low participatory level of understanding, or perhaps no understanding of this stage at all.

If response is incorrect:
Ask student to explain their thinking on this task. Produce manipulatives, and ask the student to solve with the manipulatives.

- If they does not solve correctly with manipulatives, prompt students using the following questions:
  1. How many did she have before her mom gave her more?
  2. How many does she have in the end? Can you use the number in that she had before as a starting place to see how many more she got from her mom? How?
  3. Can you use the number she had in all and count backwards to find out how many she got from her mom? How?
  - If the student solves correctly after prompting, move to the next problem
  - If the student cannot solve this problem after prompting, end testing.

Figure 3.4. Prompts for Incorrect Responses

The prompts here were designed to carefully lead a student through the activity-effect thinking process that can help them arrive at a solution. The intent of this stepwise approach is, again, to discern the specific level of the student within the participatory stage. In the next section the choice of Administration Method is discussed, with a rationale for the interview format.

Selection of Administration Method

To distinguish between the anticipatory and participatory stages in fine-grained assessment, conducting a face-to-face interview is needed to provide the assessor with the most specific feedback about what the student knows (Tzur, 2007). The format of the assessment is a clinical interview, a technique often used to delve in to the thinking of subjects, using what Zazkis and Hazzan (1998) called performance questions, followed by “Why” questions. The interviews are to be conducted with one to three students at a time, and are intended to give the students a task, observe their responses, and if needed, prompt with questions that will clarify whether they are showing evidence of the scheme or not (Ginsberg, 1981).

A clinical interview situation (Ginsberg, 1981) is best suited to this type of assessment, because it allows the assessor to control and adapt the prompts and information given to the student. The interview format provides the assessor with the unique opportunity to ask students questions as they are solving, listen to what is being verbalized, analyze and clarify figural representations created by the student in real time, and discern any movements or quiet vocalizations that the student may produce.

While a paper and pencil task is efficient from a time standpoint, it does not provide the rich data that can be gleaned from an interview setting. Furthermore, in a pencil and paper test, all of the information is present for the student to use, and it is extremely difficult to discern whether the student was able to solve a problem independently, whether they used information given in previous or subsequent problems, what kind of reasoning was used, or whether they could have solved with a little more support. Asking students to explain their answers in writing during a paper and pencil test is another option, but it often provides inaccurate results (e.g., when the student's first language is not English). Students can often explain verbally what they would not write on a test, and may know more than they show on paper.

The tasks were ordered so that students would be unlikely to be able to complete a task if they were unable to complete the previous task, a Guttman-like scale (Trochim & Donnelly, 2008). This scaling is discussed further in Chapter IV. For that reason, it is unnecessary for a student to complete the entire assessment. Rather, the assessor can simply discontinue testing when the children being assessed reach a level they are evidently unable to complete, even with prompts. As explained previously, the tasks should begin at the Composite Unit Iteration task (pairs of socks), and then either move forward to the multiplicative tasks if students reason independently, or begin the additive reasoning tasks if they do not. One reason for using the Composite Iteration Task as a starting point is that composite unit is a conception that is necessary throughout the tasks. The ability to consider a group of 1s as its own unit is something that a student would need to be able to do to reason multiplicatively or additively, but the iteration of these composite units is a more advanced concept than simply recognizing composite units. This allows a 'middle ground' for students to begin at a level that is most appropriate for them, without subjecting more advanced students to problems that are too easy for them, and possibly losing their interest, or intimidating students who are unable to reason multiplicatively. This can also save time in the administration procedure for both teachers and students. Dr. Johnson suggested, with Dr. Alan Davis' support, this change in task order at the time of thesis defense, and it was implemented in the Administration Guide.

Students from time to time may struggle with the tasks not because of the concept, but because of the numbers involved. To accommodate these students and assess their reasoning, it is possible in the course of administering the test to allow the student to respond to a task that is at the grade level below the one being assessed. For instance, if a fifth grade student is struggling to complete the Multiplicative Double Counting task, they could be administered the level 2 task, intended for second and third graders. If the student could then answer the problem, it could be surmised that they began forming the necessary multiplicative scheme, but may need some work to generalize it for larger quantities. This flexibility was included so that the assessor can be responsive to the needs of their students.

If or when a student is unable to complete the task with prompting and/or use of manipulatives, the assessor terminates the testing. Such inability indicates that the reasoning of the student has been exhausted and that they need not be tested any further. It is possible that a student who multiplies easily, and has some explanations as to what is occurring may not score as highly as expected. This result might indicate that while the student has a strong procedural knowledge or fact memorization, some of the prerequisite schemes may not be solid enough to support their future learning. As discussed in Chapter II, students will use available schemes to construct knowledge (Piaget, 1985), but if not carefully monitored, may use schemes such as additive reasoning or part-of-a-whole, which might not support them as well as multiplicative reasoning could. Next I will explain how the Administration Guide was created to walk assessors through the instrument as it is being given.

Development of Administration Guide

The administration guide is shown in Chapter IV, but this section will serve as a brief description of the thinking behind its development. The guide was created to be user-friendly, complete with scripted prompts and a description of what a person using the assessment should look for in the child's solution to each problem. The task is listed with each grade band and all prompts, in an attempt to provide for the assessor a clear picture of what will happen during the assessment, and the options they have during administration. It was also created with an eye toward some technological component being involved in the future, and considering some decisions the technology would need to make based on the student's response. The teacher or assessor can use the instrument as is to pinpoint a student's understandings and create a fine-grained assessment (Tzur, 2007). Such assessment should be helpful to determine the next steps that need to occur for the student, and help the assessor to complete the scoring guide described in the next section.

Scoring Considerations

As with the administration guide, the Scoring Guide is seen in Chapter IV, but in this section I will describe the thought process behind its development. The scoring of this assessment needed to be both detailed and simple to complete so that teachers and researchers can obtain the needed information while minimizing the time taken for the assessment. Initially, I had envisioned the assessor scoring in real time, as the administration proceeded. In testing of the instrument, however, it was noted by Mr.

Evan McClintock during this first administration that it might be simpler to have an observer score as the assessor interacts with the children. This was a very useful suggestion, and it was added as a possibility for the scoring process. It was also noted during the testing process that if students are videotaped, as they were in our first trial, scoring could happen later. I do not believe that this later scoring would be ideal, however, as the scoring process may drive further questioning of the student in real time that may not be possible after the fact. That being said, it may be used if necessary, and if care is taken to thoroughly prompt the student.

The next chapter shows the final result of the work and refinement processes described in this Methods Chapter. This instrument is still in development; but it is presented in its complete and final form at the time of submitting this thesis to the Graduate School of the University of Colorado Denver.

CHAPTER IV

RESULTS

This chapter presents the final version of the AMR, including tasks, administration guide, student problem strips, and the scoring guide. Also included is a discussion of the initial attempts made at discerning the reliability and validity of this instrument.

Tasks

The tasks, seen in Figure 4.1, are listed in the order of increasing conceptual difficulty, but not necessarily in the order that they would be administered to a student in an ordinary testing situation, because this order will vary depending on the result of each problem. These tasks are also included in the Administration Guide, and in the Student Problem Strips.

Composite Unit Iteration Task

1. Blanca has 8 pairs of socks. If she counts each sock, how many will there be?

Counting on task

<u>Level 1, K-2</u>

2. Lisa had five (5) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

<u>Level 2, grades 3-4</u>

3. Lisa had seventeen (17) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

<u>Level 3, grades 5-6</u>

4. Lisa had twenty-four (24) pencils. She finds seven (7) more pencils. How many pencils does Lisa have?

<u>Level 3, grades 6-8</u>

5. Lisa had fifty-seven (57) pencils. She finds nine (9) more pencils. How many pencils does Lisa have?

Figure 4.1. Tasks

Missing second addend task

<u>Level 1 K-2</u>

6. Joseph has six (6) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has nine (9) pieces of gum. How many pieces of gum did his friend give him?

<u>Level 2, grades 3-4</u>

7. Joseph has (8) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has fifteen (15) pieces of gum. How many pieces of gum did his friend give him?

8. **<u>Level 3, grades 5-6</u>**

Joseph has fifteen (15) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has twenty-three (23) pieces of gum. How many pieces of gum did his friend give him?

9. **<u>Level 4, grades 6-8</u>**

Joseph has twenty-five (25) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has thirty-six (36) pieces of gum. How many pieces of gum did his friend give him?

Missing first addend task

<u>Level 1, K-2</u>

10. Anna has some bouncy balls. Her mom gives her three (3) more bouncy balls. She then has eight (8) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 2, grades 3-4</u>

11. Anna has some bouncy balls. Her mom gives her thirteen (13) more bouncy balls. She then has nineteen (19) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 3, grades 5-6</u>

12. Anna has some bouncy balls. Her mom gives her sixteen (16) more bouncy balls. She then has twenty-four (24) bouncy balls. How many bouncy balls did Anna have in the beginning?

<u>Level 4, grades 7-8</u>

13. Anna has some bouncy balls. Her mom gives her twenty-five (25) more bouncy balls. She then has thirty-eight (38) bouncy balls. How many bouncy balls did Anna have in the beginning?

Figure 4.1. Tasks cont'd

mDC Task 1

<u>Level 1, K-2</u>

14. Tony has his toy cars in small boxes in his room. He has four (4) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?

<u>Level 2, grades 3-4</u>

15. Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?

<u>Level 3, grades 5-6</u>

16. Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has five (5) cars in it. How many cars does Tony have in all?

<u>Level 4, grades 7-8</u>

17. Tony has his toy cars in small boxes in his room. He has seven (7) boxes, and each box has six (6) cars in it. How many cars does Tony have in all?

mDC Task 2

<u>Level 1, K-2</u>

18. Ella has five (5) baskets for her friends. She wants to put three (3) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 2, 3-4</u>

19. Ella has five (5) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 3, 5-6</u>

20. Ella has seven (7) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 4, 7-8</u>

21. Ella has eight (8) baskets for her friends. She wants to put seven (7) pieces of candy in each basket. How many pieces of candy will Ella need?

Figure 4.1. Tasks cont'd

mDC Task 3

<u>Level 1, K-2</u>

22. Aaron wants to make three (3) birdhouses, and he needs two (2) nails for each birdhouse. How many nails will he need in all?

<u>Level 2, 3-4</u>

23. Aaron wants to make six (6) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?

<u>Level 3, 5-6</u>

24. Aaron wants to make eight (8) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?

<u>Level 4, 7-8</u>

25. Aaron wants to make nine (9) birdhouses, and he needs eight (8) nails for each birdhouse. How many nails will he need in all?

Figure 4.1. Tasks cont'd

Assessment Protocol

Students are tested using the following protocol. The student(s) should be provided with pencil and paper, or some electronic alternative that allows them to write or draw only. A card containing the problem is handed to the student. These are formatted and printed with the problem number for ease of reference. Instructions for the cutting out and assembly of these problems strips are in Figure 4.2.

Student Problem Strips

- To use these strips, print out on cardstock and cut along the lines that border the tasks.
- Stack each group of problems (found on the same page) with the highest grade-band in the back.
- This should result in a stack of strips that exposes the problem at the top of the strip.
- These can be left loose or stapled on the right or left into small booklets.
- The Composite Unit Iteration Task on this page is not part of any of the booklets.

1. **Composite Unit Iteration Task**

   Blanca has 8 pairs of socks. If she counts each sock, how many will there be?

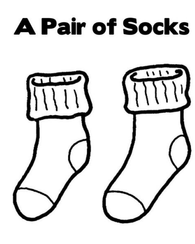

Figure 4.2. Student Problem Strips

2. Counting on task **Level 1, grades K-2**

Lisa had five (5) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

3. Counting on task **Level 2, grades 3-4**

Lisa had seventeen (17) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

4. Counting on task **Level 3, grades 5-6**

Lisa had twenty-four (24) pencils. She finds seven (7) more pencils. How many pencils does Lisa have?

5. Counting on task **Level 4, grades 6-8**

Lisa had fifty-seven (57) pencils. She finds nine (9) more pencils. How many pencils does Lisa have?

Figure 4.2. Student Problem Strips cont'd

6. Missing Second Addend Task **<u>Level 1, grades K-2</u>**

Joseph has six (6) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has nine (9) pieces of gum. How many pieces of gum did his friend give him?

7. Missing Second Addend Task **<u>Level 2, grades 3-4</u>**

Joseph has (8) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has fifteen (15) pieces of gum. How many pieces of gum did his friend give him?

8. Missing Second Addend Task **<u>Level 3, grades 5-6</u>**

Joseph has fifteen (15) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has twenty-three (23) pieces of gum. How many pieces of gum did his friend give him?

9. Missing Second Addend Task **<u>Level 4, grades 6-8</u>**

Joseph has twenty-five (25) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has thirty-six (36) pieces of gum. How many pieces of gum did his friend give him?

Figure 4.2. Student Problem Strips cont’d

10. Missing First Addend Task **Level 1, grades K-2**

Anna has some bouncy balls. Her mom gives her three (3) more bouncy balls. She then has eight (8) bouncy balls. How many bouncy balls did Anna have in the beginning?

11. Missing First Addend Task **Level 2, grades 3-4**

Anna has some bouncy balls. Her mom gives her thirteen (13) more bouncy balls. She then has nineteen (19) bouncy balls. How many bouncy balls did Anna have in the beginning?

12. Missing First Addend Task **Level 3, grades 5-6**

Anna has some bouncy balls. Her mom gives her sixteen (16) more bouncy balls. She then has twenty-four (24) bouncy balls. How many bouncy balls did Anna have in the beginning?

13. Missing First Addend Task **Level 4, grades 6-8**

Anna has some bouncy balls. Her mom gives her twenty-five (25) more bouncy balls. She then has thirty-eight (38) bouncy balls. How many bouncy balls did Anna have in the beginning?

Figure 4.2. Student Problem Strips cont'd

14. Multiplicative Double Counting Task 1	**<u>Level 1, grades K-2</u>**
Tony has his toy cars in small boxes in his room. He has four (4) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?	
15. Multiplicative Double Counting Task 1	**<u>Level 2, grades 3-4</u>**
Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?	
16. Multiplicative Double Counting Task 1	**<u>Level 3, grades 5-6</u>**
Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has five (5) cars in it. How many cars does Tony have in all?	
17. Multiplicative Double Counting Task 1	**<u>Level 4, grades 6-8</u>**
Tony has his toy cars in small boxes in his room. He has seven (7) boxes, and each box has six (6) cars in it. How many cars does Tony have in all?	

Figure 4.2. Student Problem Strips cont'd

18. Multiplicative Double Counting Task 2 **Level 1, grades K-2**

Ella has five (5) baskets for her friends. She wants to put three (3) pieces of candy in each basket. How many pieces of candy will Ella need?

19. Multiplicative Double Counting Task 2 **Level 2, grades 3-4**

Ella has five (5) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

20. Multiplicative Double Counting Task 2 **Level 3, grades 5-6**

Ella has seven (7) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

21. Multiplicative Double Counting Task 2 **Level 4, grades 6-8**

Ella has eight (8) baskets for her friends. She wants to put seven (7) pieces of candy in each basket. How many pieces of candy will Ella need?

Figure 4.2. Student Problem Strips cont'd

22. Multiplicative Double Counting Task 3	**<u>Level 1, grades K-2</u>**
Aaron wants to make three (3) birdhouses, and he needs two (2) nails for each birdhouse. How many nails will he need in all?	
23. Multiplicative Double Counting Task 3	**<u>Level 2, grades 3-4</u>**
Aaron wants to make six (6) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?	
24. Multiplicative Double Counting Task 3	**<u>Level 3, grades 5-6</u>**
Aaron wants to make eight (8) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?	
25. Multiplicative Double Counting Task 3	**<u>Level 4, grades 6-8</u>**
Aaron wants to make nine (9) birdhouses, and he needs eight (8) nails for each birdhouse. How many nails will he need in all?	

Figure 4.2. Problem Strips for Students cont'd

Once the card is given to the student, the assessor reads the problem aloud. Clarification about what the problem is asking is permissible, as long as it does not point students toward a solution. The student is asked to solve the problem by any means they wish, thinking aloud as they do so. If students do not share their thinking during the solution process, the assessor should ask the student after an answer is given what they were thinking as they solved, encouraging making marks or drawings on their paper, or using manipulatives to show their thinking.

Students should be encouraged to use writing, drawings or marks first before using manipulatives. This is because using manipulatives, such as Unifix cubes, may encourage students to count all instead of using a composite unit (Carpenter et al., 1981). When students are representing the composite units, it is preferable if they can do so without drawing in or building each of the 1s needed to construct that unit. Treating the unit as a whole, in and of itself, is important in the understanding of composite unit (Tzur, et al., 2013). For example, if the child began drawing 6 circles with 3 dots in each, the assessor should encourage him or her to first try solving the problem by just drawing the first 1-2 circles and then just draw the circles without dots. Such an intervention is done because students may count the 1s even if their reasoning supports activity on the composite units without the 1s, simply because the 1s are available to count. These instructions are seen in the Administration Guide presented later in this chapter. The manipulatives used could vary, but most commonly used are Unifix cubes and paper and pencil for drawing pictures by the student. Students can solve problems with manipulatives that they would not be able to solve without them, and so this intervention is seen as similar to a prompt (Steffe, 1970).

The assessor should do her or his best to avoid indicating to the student whether their answer or their thinking is correct or incorrect. The "burden of proof" is on the student, to be able to support their answer with reasoning, including figural or manipulative means. This approach is needed to uphold the fine-grained assessment technique (Tzur, 2007), which proceeds from asking the student to solve with the minimum amount of support necessary. In order to determine whether the student is at an anticipatory or participatory stage, it is crucial to allow the student to solve as independently as possible, anticipating as much as they are able before prompts, including those about correctness of solution, are given.

By following the guide, the assessor progresses through the questions using the prompts, either moving down a level or using questioning to determine whether a student may be able to solve a similar problem with different numbers. This will help teachers determine whether what is lacking is the ability to reason about these tasks, or the familiarity with the quantities presented. At the end of each task the assessor can find directions about the next step in the process. Generally, if the problem was solved correctly, the student moves to the next problem, if the student could not solve the problem with prompting, testing is terminated. This is due to the attempt at a Guttman-like organization (Trochim & Donnelly, 2008). This instrument is certainly not qualified as using the Guttman scale, which would be based on data from previous administrations, and not require that the student complete each item to give a reliable result. Details about this process are seen in the Administration Guide in the next section.

Administration Guide

This section provides the complete version to date of the Administration Guide, intended for use by teachers or other assessors, and showing the instructions, tasks, prompts and things to look for at each stage. This document would guide the administration of the assessment for students at any level. Using the guide, seen in Figure 4.3, should direct teachers and other assessors in how to administer the assessment, and help in the completion of the Scoring Guide seen in the following section

Administration Guide

<u>Assessment for Multiplicative Reasoning</u>

General Directions

Please read each task exactly as written. Students may also view the task and read it on their own from the problem strip as you read it aloud.

Administration begins with the Composite Unit Iteration task, and from there will move to either Counting On or Multiplicative Double Counting, depending on student response. The progression is shown in the flow chart for Administration Protocol, and is indicated in this guide.

The problem numbers are listed on the far left for easy reference.

Directions and numbers in task may be repeated indefinitely, and students may write down important information or numerals as needed. Clarification about what is meant by the problem is permitted, but students should do all solving. If possible, stick to the given prompts.

Students may use scratch paper, and all marks made by students should be kept and labeled with the problem number.

When administering these tasks, ask the student to explain their thinking, and record everything the student does on the scoring sheet.

You may use questions such as:

- "What were you thinking as you solved this problem?"
- "What strategies did you use?"
- "Did you picture something in your head to help you solve this problem?"

Begin with the level appropriate for the child's grade level. If they are not successful at this task, move down one level and test again.

Please explain to the student that you will be recording their responses and their work so that we can learn more about how they solve problems.

Do not indicate to the student whether they are correct or incorrect as they work.

Figure 4.3. Administration Guide

Administration Sequence

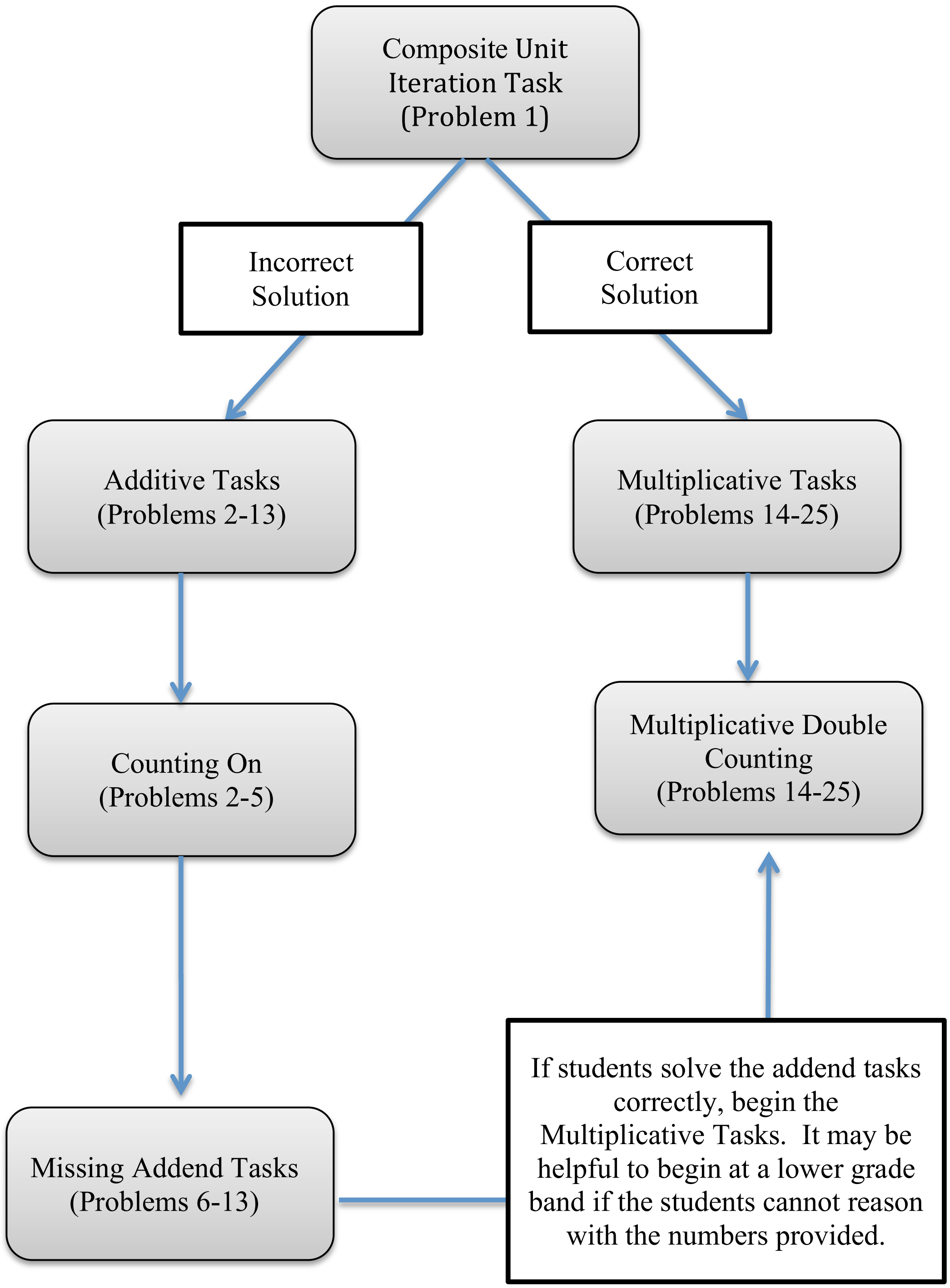

Figure 4.3. Administration Guide cont'd

Composite Unit Iteration Task

Please solve this problem, and explain as much as you can about what you're thinking as you do so.

(Clarify the meaning of "pair".)

1. Blanca has 8 pairs of socks. If she counts each sock, how many will there be?

- If the student counts one by one, and not by composite units of two, ask if they can use the way the socks are grouped into pairs.
- If the students cannot solve after this prompt, begin with the Counting On Task (problem 2).
- If the student is able to iterate the socks to find that there are 16 in all, skip to the Multiplicative Double Counting Tasks (problem 14).

Counting on task

Please solve this problem, and explain as much as you can about what you're thinking as you do so.

Level 1, K-2

2. Lisa had five (5) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

Level 2, grades 3-4

3. Lisa had seventeen (17) pencils. She finds four (4) more pencils. How many pencils does Lisa have?

Level 3, grades 5-6

4. Lisa had twenty-four (24) pencils. She finds seven (7) more pencils. How many pencils does Lisa have?

Level 3, grades 6-8

5. Lisa had fifty-seven (57) pencils. She finds nine (9) more pencils. How many pencils does Lisa have?

- o If the student has answered correctly, and shown evidence of counting on, go to the next problem.

Figure 4.3. Administration Guide cont'd.

For both correct responses without reasoning and incorrect responses:

- o Ask students to explain their thinking on this task. It is important to discourage counting all as a solution. Prompt the student to use counting on by covering the first composite unit, or asking them to make a drawing that does not include all of the ones in the unit. If the student is using marks or manipulatives, it may be necessary to cover the first composite unit and then see if the student can then proceed to count on.
- o If at any time during the administration of this task you believe that the number values are impeding the progress of the student's progress, move down to a lower grade band and test again.
- o If at any time the student answers correctly using counting on, go to the next problem.

If response is correct, but the students use an unknown strategy, cannot articulate one, use counting all or known math facts, prompt student using these questions:

1. How many did she have in the beginning?
2. How many did she find?
3. Can you use the number she had in the beginning as a starting place to see how many in all? How?

- If the student uses counting on after prompting, move to the next problem.

If response is incorrect:

Ask students to explain their thinking on this task. Produce manipulatives, and ask the student to solve with the manipulatives.

- If solved correctly with manipulatives, go to the next problem.
- If you believe that the number values are impeding the progress of the student, move down to a lower grade band and test again.
- If they do not solve correctly, prompt students using the following questions:
  - How many did she have in the beginning?
  - How many did she find?
  - Can you use the number she had in the beginning as a starting place to see how many in all? How?
- If the student uses counting on after prompting, move to the next problem
- If the student cannot solve this problem after prompting, end testing.

Additive Tasks

Missing Addend Tasks

- **Administer both of these tasks, regardless of the results, to assess the additive reasoning of the student.**

Missing second addend task

Please solve this problem, and explain as much as you can about what you're thinking as you do so.

Level 1 K-2

6. Joseph has six (6) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has nine (9) pieces of gum. How many pieces of gum did his friend give him?

Level 2, grades 3-4

7. Joseph has (8) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has fifteen (15) pieces of gum. How many pieces of gum did his friend give him?

8. **Level 3, grades 5-6**

Joseph has fifteen (15) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has twenty-three (23) pieces of gum. How many pieces of gum did his friend give him?

9. **Level 4, grades 6-8**

Joseph has twenty-five (25) pieces of gum. His friend gives him some more pieces of gum. Now Joseph has thirty-six (36) pieces of gum. How many pieces of gum did his friend give him?

- If the student uses counting on or counting back, move to the next problem.

For both correct responses without reasoning and incorrect responses:

- o Ask students to explain their thinking on this task. It is important to discourage counting all as a solution. Prompt the student to use counting on by covering the first composite unit, or asking them to make a drawing that does not include all of the ones in the unit. If the student is using marks or manipulatives, it may be necessary to cover the first composite unit and then see if the student can then proceed to count on.
- o If at any time during the administration of this task you believe that the number values are impeding the student, move down to a lower grade band and test again.
- o If at any time the student answers correctly using counting on, go to the next problem.

If response is correct, but the students use an unknown strategy, cannot articulate one, use counting all or known math facts, prompt student using these questions:

1. How many did he have in the beginning? How many does he have in the end? Can you use the number in that he had in the beginning as a starting place see how many more he got from his friend? How?
2. Can you use the number he had in all and count backwards to get to find out how many he got from his friend? How?

If the student solves correctly after prompting, move to the next problem.
If the student cannot solve this problem after prompting, end testing.

If response is incorrect:
Ask student to explain their thinking on this task. Produce manipulatives, and ask the student to solve with the manipulatives.

1. If solved correctly with manipulatives, go to the next problem.
2. If you believe that the number values are impeding the student, move down to a lower grade band and test again.
3. If they do not solve correctly, prompt students using the following questions:
4. How many did he have in the beginning? How many does he have in the end? Can you use the number in that he had in the beginning as a starting place see how many more he got from his friend? How?
5. Can you use the number he had in all and count backwards to get to find out how many he got from his friend? How?

- If the student solves correctly after prompting, move to the next problem
- If the student cannot solve this problem after prompting, end testing.

Missing first addend task
Please solve this problem, and explain as much as you can about what you're thinking as you do so.

Level 1, K-2

10. Anna has some bouncy balls. Her mom gives her three (3) more bouncy balls. She then has eight (8) bouncy balls. How many bouncy balls did Anna have in the beginning?

Level 2, grades 3-4

11. Anna has some bouncy balls. Her mom gives her thirteen (13) more bouncy balls. She then has nineteen (19) bouncy balls. How many bouncy balls did Anna have in the beginning?

Level 3, grades 5-6

12. Anna has some bouncy balls. Her mom gives her sixteen (16) more bouncy balls. She then has twenty-four (24) bouncy balls. How many bouncy balls did Anna have in the beginning?

Level 4, grades 7-8

13. Anna has some bouncy balls. Her mom gives her twenty-five (25) more bouncy balls. She then has thirty-eight (38) bouncy balls. How many bouncy balls did Anna have in the beginning?

- If the student uses counting on or counting back, end assessment.

<u>For both correct responses without reasoning and incorrect responses:</u>

- o Ask students to explain their thinking on this task. It is important to discourage counting all as a solution. Prompt the student to use counting on by covering the first composite unit, or asking them to make a drawing that does not include all of the ones in the unit. If the student is using marks or manipulatives, it may be necessary to cover the first composite unit and then see if the student can then proceed to count on.
- o If at any time during the administration of this task you believe that the number values are impeding the student, move down to a lower grade band and test again.
- o If at any time the student answers correctly using counting on or counting back, go to the next problem.

<u>If response is correct,</u> but the students use an unknown strategy, cannot articulate one, use counting all or known math facts, prompt student using these questions:

1. How many did she have before her mom gave her more?
2. How many does she have in the end? Can you use the number in that she had before as a starting place to see how many more she got from her mom? How?
3. Can you use the number she had in all and count backwards to find out how many she got from her mom? How?

- If the student solves correctly after prompting, move to the next problem

<u>If response is incorrect:</u>

Ask student to explain their thinking on this task. Produce manipulatives, and ask the student to solve with the manipulatives.

- If they does not solve correctly with manipulatives, prompt students using the following questions:
  1. How many did she have before her mom gave her more?
  2. How many does she have in the end? Can you use the number in that she had before as a starting place to see how many more she got from her mom? How?
  3. Can you use the number she had in all and count backwards to find out how many she got from her mom? How?
  - If the student solves correctly after prompting, move to the next problem
  - If the student cannot solve this problem after prompting, end testing.

Multiplicative Tasks

Multiplicative Double Counting Tasks
There are three tasks in this section. A successful solution to two out of three is considered evidence that the student has access to the Multiplicative Double Counting Scheme, and testing should be concluded when the student has successfully completed two.

mDC Task 1

Level 1, K-2

14. Tony has his toy cars in small boxes in his room. He has four (4) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?

Level 2, grades 3-4

15. Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has three (3) cars in it. How many cars does Tony have in all?

Level 3, grades 5-6

16. Tony has his toy cars in small boxes in his room. He has six (6) boxes, and each box has five (5) cars in it. How many cars does Tony have in all?

Level 4, grades 7-8

17. Tony has his toy cars in small boxes in his room. He has seven (7) boxes, and each box has six (6) cars in it. How many cars does Tony have in all?

If response is correct:
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- If the student gives an answer without units, ask what the number tells. Is it number of cars? Number of boxes?
- If correct, and using mDC strategy to find the answer, proceed to mDC task 2.
- If they use counting all, or known math facts, prompt student using these questions:
  1. How many cars in one box?
  2. Repeat original question
  3. How many cars would be in two boxes?
  4. Repeat original question

Figure 4.3. Administration Guide cont'd

<u>If response is incorrect:</u>
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- As soon as the student is able to give the answer correctly, move to mDC task 2.
- Prompt using these questions, encouraging the student to use scratch paper, fingers or manipulatives to help him/her keep track.
  1. How many cars in one box?
  2. Repeat original question
  3. How many cars would be in two boxes?
  4. Repeat original question

Regardless of whether the student answers correctly or incorrectly, go on to mDC Task 2.

mDC Task 2

<u>Level 1, K-2</u>

18. Ella has five (5) baskets for her friends. She wants to put three (3) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 2, 3-4</u>

19. Ella has five (5) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 3, 5-6</u>

20. Ella has seven (7) baskets for her friends. She wants to put four (4) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>Level 4, 7-8</u>

21. Ella has eight (8) baskets for her friends. She wants to put seven (7) pieces of candy in each basket. How many pieces of candy will Ella need?

<u>If response is correct:</u>
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- If the student gives an answer without units, ask what the number tells. Is it number of baskets? Number of candy pieces?
- If correct, and using mDC strategy to find the answer, conclude testing.
- If they use counting all, or known math facts, prompt student using these questions:
  1. How many candy pieces in one bag?
  2. Repeat original question
  3. How many candy pieces would be in two bags?
  4. Repeat original question

Figure 4.3. Administration Guide cont'd

<u>If response is incorrect:</u>
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- As soon as the student is able to give the answer correctly, move to mDC task 3.
- Prompt using these questions, encouraging the student to use scratch paper, fingers or manipulatives to help him/her keep track.
  1. How many candy pieces in one bag?
  2. Repeat original question
  3. How many candy pieces would be in two bags?
  4. Repeat original question

  - If both mDC tasks were answered correctly, conclude testing
  - If one mDC task is answered correctly, administer mDC Task 3.
  - If both mDC tasks were answered incorrectly, conclude testing.

mDC Task 3

<u>Level 1, K-2</u>

22. Aaron wants to make three (3) birdhouses, and he needs two (2) nails for each birdhouse. How many nails will he need in all?

<u>Level 2, 3-4</u>

23. Aaron wants to make six (6) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?

<u>Level 3, 5-6</u>

24. Aaron wants to make eight (8) birdhouses, and he needs four (4) nails for each birdhouse. How many nails will he need in all?

<u>Level 4, 7-8</u>

25. Aaron wants to make nine (9) birdhouses, and he needs eight (8) nails for each birdhouse. How many nails will he need in all?

<u>If response is correct:</u>
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- If the student gives an answer without units, ask what the number tells. Is it number of nails? Number of birdhouses?
- If correct, and using mDC strategy to find the answer, conclude testing.
- If the student uses counting all or known math facts, prompt student using these questions:
  1. How many nails does he need for one birdhouse?
  2. Repeat original question
  3. How many nails does he need for two birdhouses?
  4. Repeat original question

Figure 4.3. Administration Guide cont'd

<u>If response is incorrect:</u>
Ask student to explain their thinking on this task. Student may use manipulatives, fingers, or pencil and paper to prove their solution.

- As soon as the student is able to give the answer correctly, conclude testing.
- Prompt using these questions, encouraging the student to use scratch paper, fingers or manipulatives to help him/her keep track.
  1. How many nails does he need for one birdhouse?
  2. Repeat original question
  3. How many nails does he need for two birdhouses?
  4. Repeat original question.

End of Test

Figure 4.3. Administration Guide cont'd

Scoring Guide

Scoring of students' reasoning based on their solutions, including both answer and explanation, for the tasks can optimally occur during the interview process, either by the assessor or another observer. The assessment is meant to be adapted to what is seen in student responses (e.g., counting 1s or composite units, visible or figural, etc.), so that care is taken to administer appropriate tasks to appropriate students. While the time and care needed to do this type of assessment is significant, the benefits of having real-time scoring and the ability to probe and question a student as they are working gives a better understanding of the thinking and level of reasoning being employed by the student (Ginsberg, 1981; Steffe, 2002). As students complete tasks, the assessor will record the response on the scoring guide seen in Figure 4.4, including but not limited to the use of figural items, manipulatives, drawings, or computational procedures. The extra space provided in the middle of each page is intended for assessor's note taking.

Scoring Guide
Assessment for Multiplicative Reasoning

Grade Level of Student______________

Composite Unit Iteration Task (CU)
Indicate any procedures that students may use, including but not limited to the following:

☐ Counts on fingers
☐ Draws or uses marks
☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide

<u>Counting-on task</u>

Indicate any procedures that students may use, including but not limited to the following:

☐ Counts on fingers

☐ Draws or uses marks

☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)

☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide, cont'd.

Missing Second Addend (MSA)
Indicate any procedures that students may use, including but not limited to the following:
☐ Counts on fingers
☐ Draws or uses marks
☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)
☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4 Scoring Guide cont'd

Missing First Addend
Indicate any procedures that students may use, including but not limited to the following:
☐ Counts on fingers
☐ Draws or uses marks
☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)
☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide, cont'd

Multiplicative Double Counting 1 (mDC1)
Indicate any procedures that students may use, including but not limited to the following:

☐ Counts on fingers
☐ Draws or uses marks
☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)
☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide, cont'd.

Multiplicative Double Counting 2 (mDC2)
Indicate any procedures that students may use, including but not limited to the following:
☐ Counts on fingers
☐ Draws or uses marks
☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)
☐ Other behaviors (please describe)

Student Response	**Stage Indicated by this Result**
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide, cont'd.

Multiplicative Double Counting 3 (mDC3)
Indicate any procedures that students may use, including but not limited to the following:

☐ Counts on fingers
☐ Draws or uses marks
☐ Writes algorithm (Orient the student to avoid using an algorithm before other methods of computing mentally have been exhausted.)
☐ Other behaviors (please describe)

Student Response	Stage Indicated by this Result
☐ Correct Response to grade level problem **without** prompts.	Anticipatory at grade level for this scheme
☐Correct Response to grade level problem **with** prompts.	Participatory at grade level for this scheme
☐Correct Response to below grade level problem **without** prompts.	Anticipatory, but below grade level for this scheme
☐Correct Response to below grade level problem **with** prompts.	Participatory and below grade level for this scheme
☐Incorrect Response with prompts.	Pre-Participatory for this scheme

Figure 4.4. Scoring Guide.cont'd.

Using the Scoring guide should allow assessors to determine at which stage, participatory or anticipatory, students seem to operate in each scheme tested. It may also help to guide prompts during the interview process, as it becomes clear the evidence needed to prove competence at each level.

Reliability and Validity

The AMR is work-in-progress in its preliminary stages. It has not yet been fully tested for reliability or validity; however, attempts made to increase these two measures of the instrument are discussed here. The validity measures taken in the development of the AMR were based upon content validity according to expert opinion, both classroom teachers and researchers. The items were developed under the advisement of Dr. Ron Tzur of the University of Colorado, Denver. His guidance focused on ensuring that the tasks were created to reflect the theoretical stances described in Chapter 2. Using information about the numbers that would be most appropriate at each grade band, and tying the tasks closely to the six Schemes of Multiplicative Reasoning discussed earlier, the items are believed to be good indicators of the reasoning at each level. As explained in Chapter III, these items were refined both by feedback from teachers and through informal testing with fourth-grade students. I was able to observe students as they participated in the assessment, and also received feedback from the reports of Dr. Tzur and Mr. McClintock on later administrations of the tasks I was unable to witness. Additionally, I solicited critique by way of a Feedback Form from classroom teachers, as discussed in context in Chapter IV. The Form used to collect this data is found in the Appendix to this thesis.

Another reliability measure is the two-fold problem delivery of verbal directions and printed student problem strips shown earlier in this section. These seemingly redundant steps are to remove as many barriers to the students' interpretation of the problem as possible. Students who have difficulty interpreting verbal questions have access to the printed problem, while those who have difficulty with reading have the verbal instructions read by the assessor. The ability to refer to the card and clarify numbers or any other aspect of the problem, as well as review any information, was deliberately included to ensure that incorrect solutions were not due to misunderstandings or difficulty remembering the information given in the tasks.

The choice of numbers for each problem type and each grade level was purposeful, as supported by the work of Sherin and Fuson (2005). The strategy used by students in that study varied depending on the numbers and context used. It showed that the choice of each number used in a task must be made mindfully to elicit the strategy for which it is designed. If the numbers and situations for these problems were not chosen carefully, students could use some learned strategies, or known facts, that do not necessarily reflect their understanding of composite unit, such as how to count on. Sherin and Fuson also note that the only route to this clear, complex, flexible knowledge is the passage through earlier stages, through which they develop strategies rooted in understanding. Clearly, an understanding of the problem is an important facet, and was treated as such using these methods, to provide a reliable assessment.

Because the focus of this thesis was on multiplicative reasoning, there are three tasks to add validity to this portion of the assessment. If a student is able to correctly and independently solve two tasks in a row, testing is terminated, and the student is considered to have access to the scheme for Multiplicative Double Counting. If, however, the student solves only one out of the first two tasks independently and correctly, the third task is administered to determine whether the student can be considered to have this scheme, or at which stage. A score of two out of three correct would indicate the student's readiness to move to the next level of multiplicative reasoning.

These steps were taken to further increase reliability within the given time and resources available, and could certainly help to improve the AMR further in future endeavors. The following section includes a discussion of the implications and benefits of the development of the AMR, as well as a further discussion of the limitations indicated here.

CHAPTER V

DISCUSSION

Contributions to Research and Practice

This section will focus on the ways that the work seen in Chapter IV can benefit the field of mathematics education in terms of research and classroom teaching. The ability to test for and determine not only the multiplicative schemes available for students but also the anticipatory or participatory stages within each scheme could be a great benefit to teachers and researchers alike. In both the classroom and the field of research, there is a need to find out what students understand about multiplicative reasoning, and whether they have constructed a scheme at the participatory or anticipatory stage.

A classroom teacher could use this instrument to determine what students need to be able to move ahead, into multiplicative reasoning, or whether they are solid in their initial stage of multiplicative reasoning, and are ready to move on to more advanced topics. For example, a student could be presented with the Missing Second Addend task for the 3-4 grade-band. It reads: "Joseph brings eight (8) pieces of gum to school. His friend gives him some more pieces of gum. When Joseph counts them all, he finds that he has fifteen (15) pieces of gum. How many pieces of gum did his friend give him?" The student's response allows a teacher to distinguish whether a students is at the anticipatory or participatory stage of using composite units within a given, encompassing composite unit. If at the participatory, (i.e., can only solve the problem with prompting), teaching would then focus on moving the student toward an anticipatory stage, using her current knowledge and making prompting explicit so that she will begin to self-prompt. However, if the student is at the anticipatory stage (can solve independent of any prompting while using strong reasoning), teaching can move forward to missing second addend tasks, by using the missing first addend concepts as a bridge to these missing second addend concepts. It may also indicate moving ahead to teaching mDC while keeping in mind to focus on the student's way of reasoning with 1s vs. with composite units.

In addition, the assessment should be flexible enough to include a wide range of learners. An advanced first grader could be assessed for compaction of curriculum or advancement to a more challenging math group based on the AMR results. A struggling seventh grader might be able to receive needed intervention based on results that show he or she is not able to perform Multiplicative Double Counting. A fifth grade student who does have the availability of the appropriate scheme, but does not compute fluently with whole numbers, could test at a lower grade band and show this reasoning in spite of their computational difficulties.

I see this awareness as essential for teachers, because the connection between multiplicative reasoning and topics such as fractions is not always a focus of instruction in those topics, though it might be a way to improve it. Fractions are multiplicative quantities because they require a reference to the unit. When fractions are an extension of multiplication, the idea that there are multiple ratios to consider in multiplication (Confrey and Smith, 1995) can help create a bridge from the whole number to the fractional concepts. Some of the ratios indicated in 5 x 7 are each composite unit of five to seven ones, as the relation of the seven ones in one group and the five groups to the thirty-five ones in total. This understanding of multiplication as multiple ratios builds the foundation for students to understand fractional quantities in a way that repeated addition does not.

When the meaning of five groups of seven is conceptualized by the student, the understanding that 2/3 means that there are some units, of which the whole is three times as much, and that we are considering two of these units, as 2 * 1/3 or 2 'groups' of 1/3, can equip a student to compare relative values of these fractions. This thinking also allows a student to understand why in a problem such as 2/3 + 4/5, we must use a common measure for operating on both thirds and fifths. When students recognize thirds and fifths as different units, and see this as a way to build upon their knowledge of multiplication to iterate those units, the search for a common measure ('denominator') is no longer some mysterious invention of teachers of mathematics, but a necessary operation of multiplicative unit coordination needed to find common units. Lamon (1996) saw a similar conception being built in students who use equal sharing tasks to develop their fraction concepts. When students began to see certain quantities as equivalent, they were using multiplicative rather than additive concepts to understand the size of the portion. Being able to see and assess this reasoning could go a long way for classroom teachers seeking to improve the teaching and learning of fractions and/or multiplicative reasoning in their classrooms.

In research the AMR may be used for a similar purpose, or to find correlations between multiplicative reasoning and other types of mathematical conceptions. For example, it could be investigated whether a student who has a strong knowledge of area and perimeter was more equipped with multiplicative reasoning than one who does not. The AMR could be used to quickly determine the level of a student and then use it to correlate with other factors. There are some limitations to this instrument, however, and they are discussed next.

Limitations

While the AMR provides benefits for teachers and researchers, it is important to be aware of its limitations. It is understood that the time involved in completing an assessment of this type is limiting for most classroom situations. One way to decrease the amount of time needed for testing is to administer, as a written test, the portion that the teacher believes the student has already constructed as a solid conception. In this way the AMR's tasks function as a preliminary screener. For example, if a third grade teacher believes students are proficient in missing second addend problems, the counting on and missing second addend section could be given to students simply as a written test. The students who show clearly that they are anticipatory for these tasks can begin with the missing first addend for the interview. Those who did not show that they are anticipatory could then be tested further, by interview, to determine if they may be participatory for that stage or do not yet have the conception at all. As explained in Chapter II, the distinction between anticipatory and participatory stages requires the task to be first given without prompts, to determine whether the students can anticipate the entire process, and if not, with prompts to assess for a participatory level of understanding.

The AMR is a prototype, and only addresses the first scheme in the multiplicative reasoning continuum delineated by Tzur et al. (2013), namely, Multiplicative Double Counting. This was done for simplicity, and to allow testing and feedback to determine the foci of later instruments. The remaining stages of multiplicative reasoning could be created in similar ways, and would show where more advanced students fall on the continuum of multiplicative reasoning.

Instead of progressing from mDC to more advanced multiplicative schemes, the AMR instrument includes schemes that are prerequisite to multiplicative reasoning, such as counting on. These were included to determine the extent to which students have constructed composite units needed for their journey from additive to multiplicative reasoning, or are still developing the understandings on which additive reasoning should be built (Steffe, 1994). In a study conducted with 5th grade students designated by their school system as learning disabled, who did not reason multiplicatively (Tzur, 2010), it was found that one of the major impediments to their progress was the lack of a concept of number as an abstract, symbolic, composite unit.

The AMR is meant to begin with these prerequisite skills to save teachers time when doing assessments. Clearly the additive portion of the test is not comprehensive enough to account for all aspects of additive reasoning and all of the different types of problem structures and what they may reveal about student thinking. Rather, the additive portion was designed to give a glimpse into the schemes available to students, and whether they have the conceptual prerequisites needed to engage in and learn to reason multiplicatively. With this in mind, the assessment only covers counting on, missing second addend problems with the missing portion being smaller, and a missing first addend with the missing potion being smaller. It is understood that this spans a limited range in the additive abilities of students, but does so while providing essential information about the students' use and coordination of units. An assessor who finds that students are in this general stage of reasoning and would like to further pinpoint their development, would need to consult a more comprehensive additive reasoning assessment.

It is also important to note that this instrument has never been tested in its entirety. Significant changes were made following the initial testing, up to and including the defense phase. The following section suggests possibilities for future research, some of which address the aforementioned limitations.

Implications for Future Research

A continuation of the assessment to include the remaining items in the framework for multiplicative reasoning, outlined by Tzur et al. (2013), would be helpful to teachers who need to be able to assess all schemes, and stages within them, up to the point in which they may be ready to learn fractions. In particular, items could be created for the remaining schemes in this framework: Same Unit Coordination (SUC), Unit Differentiation and Selection (UDS), Mixed Unit Coordination (MUC), Quotitive Division (QD), and Partitive Division (PD), so the assessment is expanded for the entire scope of these schemes. When a student has progressed through all of these schemes, they may be considered proficient in multiplicative reasoning.

Additionally, the prototype of the AMR was written such that a computer-based assessment could be built to either complete the preliminary screening questions, used to decide where a student might need to begin testing, or as a tool to help the assessor. A software program that could record responses of students, coded by the assessor, and indicate the next task and level that should be administered, could be a valuable tool. For example, if the student answers a question correctly with strong reasoning, the technological component could advance the student to the next appropriate question. It seems unlikely that the assessment could be productively administered without the human interview component, due to the importance of the interaction between the assessor and student, and the careful attention to student thinking gleaned from bodily gestures and whispers, which may not be detected, let alone understood, by existing technology.

Additionally, teachers may want to know how to meet student needs once their reasoning, both scheme and stage, has been assessed. This is an excellent opportunity for future work, and would greatly assist teachers in using the data obtained to foster the understandings that they now know students need to prepare them for future work.

Concluding Remarks

This project was born out of my own frustration, as a middle school teacher, with the low level of knowledge that my students often have about fractional quantities. Too often I saw that though students gained procedural knowledge with repetition and practice, their conceptual understanding remained weak, despite activities that I designed to try to build these conceptions. I truly believe that for teachers and other educators to first and foremost become aware of the gaps in reasoning that cause students in middle school, high school and even higher education to not only dread, but show a limited understanding of the meaning of fraction, is key. A teacher's possession of both this awareness and the knowledge about schemes used for reasoning multiplicatively, and of anticipatory and participatory stages within each scheme, could go a long way towards improving the teaching and learning of fraction and ratio concepts. Further, I am glad as an educator to have an instrument that allows me to begin assessing the underlying multiplicative reasoning that my students may or may not have. I foresee more productive lessons when students are equipped with the necessary conceptual prerequisites with which to construct fraction knowledge, as opposed to asking them to operate on conceptions that they are not able to access due to incomplete multiplicative schemes.

REFERENCES

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. *Acquisition of mathematics concepts and processes*, (pp. 91-126). New York: Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. A. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Bonato, M., Fabbri, S., Umilta, C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology-Human Perception and Performance*, *33*(6), 1410-1419.

Bright, G. W., Joyner, J. M., & Wallis, C. (2003). Assessing proportional thinking. *Mathematics Teaching in the Middle School*, *9*(3), 166-172.

Bryant, B. R., & Maddox, T. (1996). Using alternative assessment techniques to plan and evaluate mathematics instruction." *Ariel*, 129, 6-1.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, *12* (1), 27-39.

Clark, F. B., & Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997a). "Rational Number Project: Fraction Lessons for the Middle Grades: Level 1." Dubuque, IA: Kendall/Hunt Publishing.

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997b). "Rational Number Project: Fraction Lessons for the Middle Grades: Level 2." Dubuque, IA: Kendall/Hunt Publishing.

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education*, *18* (5), 363-381.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. NY: Oxford University.

Dewey, J. (1938). *Experience and education*. New York: Collie

Erlwanger, S. H. (1973). Benny's Conception of Rules and Answers in IPI Mathematics. *Journal of Children's Mathematical Behavior, 1*(2), 7-26.

Fan, L., & Zhou, L. (2006). How Have Chinese Students Performed in Mathematics? A Perspective from Large-Scale International Mathematics Comparisons. In L. Fan, N.-Y. Wong, J. Cai and S. Li (Eds.), *How Chinese Learn Mathematics: Perspectives from Insiders* (pp. 3-25).

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for research in mathematics education*, 3-17.

Fuson, K. C., & Willis, G. B. (1988). Subtracting by counting up: More evidence. *Journal for Research in Mathematics Education*, 19(5), 402-420.

Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the learning of mathematics*, *1*(3), 4-11.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). Highlights from TIMSS 2007. *US Department of Education, Institute of Education Sciences, National Center for Education Statistics, http://nces. ed. gov/pubs2009/2009001. pdf.*

Hartnett, P., & Gelman, R. (1998). Early understandings of numbers: paths or barriers to the construction of new understandings? *Learning and instruction*, *8*(4), 341-374.

Jacob, L. & Nieder, A. (2009) Notation-Independent Representation of Fractions in the Human Parietal Cortex. *The Journal of Neuroscience,* 29(14), 4652-4657

Jacob, L., Willis, S.G., 2003, The development of multiplicative thinking in young children, In *Mathematics Education Research: Innovation, Networking, Opportunity: Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australia*, 06/07/2003 to 10/07/2003, Deakin University, Melbourne Vic Australia, pp. 460-467.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193.

Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology*, *53*(1), 72-103.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26* (5), 422-441.

McClintock, E. (Personal Communication, May 13, 2012)

Medina, J. (2008). *Brain rules: 12 principles for surviving and thriving at work, home, and school*. Seattle, WA: Pear.

Mulligan, J. T., & Mitchelmore, M. C. (1997). Young children's intuitive models of multiplication and division. *Journal for Research in Mathematics Education, 28*(3), 309-330.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics* Reston, VA: NCTM.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. US Department of Education.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40*(1), 27-52.

Norton, A., & Hackenberg, A. J. (2010). Continuing research on students' fraction schemes. In L.P. Steffe and J. Olive (Eds.), *Children's Fractional Knowledge* (pp. 341-352). New York, NT:Springer US.

Olive, J. (2001). Children's number sequences: An explanation of Steffe's constructs and an extrapolation to rational numbers of arithmetic. *The Mathematics Educator*, *11*(1), 4-9.

Piaget, J. (1985). The equilibration of cognitive structures: The central problem of intellectual development (T. Brown & K.J. Thampy, Trans.). Chicago: The University of Chicago.

Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26*(2-3), 165-190.

Raykov, T.,Marcoulides, G. A. (2011). *Introduction to Psychometric Theory*. Taylor & Francis. Retrieved 11 June 2013, from <http://lib.myilibrary.com?ID=303716>

Sherin, B., & Fuson, K. (2005). Multiplication strategies and the appropriation of computational resources. *Journal for research in mathematics education*, 36(4), 347-395.

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2012). Fractions: the new frontier for theories of numerical development. *Trends in cognitive sciences, 17 (*1), 13-19.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive psychology*, *62*(4), 273-296.

Siemon, D., & Breed, M. (2010, May). Assessing multiplicative thinking using rich tasks. In *Annual Conference of the Australian Association for Research in Education.*(Vol. 13). Downloaded from http://publications.aare.edu.au/06pap/sie06375.pdf.

Siemon, D., Breed, M., & Virgona, J. (2005). From additive to multiplicative thinking–the big challenge of the middle years. In *Proceedings of the 42nd Conference of the Mathematical Association of Victoria*.

Siemon, D., Izard, J., Breed, M., & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking.

Sousa, D. A. (2008). *How the brain learns mathematics*. Thousand Oaks, CA: Corwin (Sage).

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5), 503-518.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*, 1-41.

Steffe, L. P., & Cobb, P. (1998). Multiplicative and divisional schemes. *Focus on Learning Problems in Mathematics, 20*(1), 45-62.

Steffe, L. P. (1970). Differential performance of first-grade children when solving arithmetic addition problems. *Journal for Research in Mathematics Education*, *1* (3), 144-161.

Steffe, Leslie P. (2002). The constructivist teaching experiment: Illustrations and implications. In *Radical constructivism in mathematics education*. (pp. 177-194). Netherlands: Springer.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research*. Dordrecht, The Netherlands: Kluwer.

Streefland, L. (1993). Fractions: A realistic approach. In Carpenter, Thomas P., Fennema, Elizabeth, and Romberg, Thomas A. (Eds.), *Rational numbers: An integration of research*, 289-325. New York, NY :Routledge.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *A research companion to Principles and standards for school mathematics* (pp. 95-113). Reston, VA:

Torbeyns, J., Ghesquière, P., & Verschaffel, L. (2009). Efficiency and flexibility of indirect addition in the domain of multi-digit subtraction. *Learning and Instruction*, *19*(1), 1-12.

Trochim, W. M., & Donnelly, J. P. (2008). *Research methods knowledge base*. Mason, OH: Atomic Dog/Cengage Learning.

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *Journal of Mathematical Behavior, 23*, 93-114.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics, 66*(3), 273-291.

Tzur, R., Johnson, H. L., McClintock, E., Kenney, R. H., Xin, Y. P., Si, L., ... Jin, X. (2013). Distinguishing schemes and tasks in children's development of multiplicative reasoning. *PNA, 7(3),* 85-101.

Tzur, R., Johnson, H., McClintock, E., Xin, Y. P., Si, L., Kenney, R., et al. (2013). Children's development of multiplicative reasoning: A schemes and tasks framework. In T.-Y. Tso (Ed.), *Proceedings of the 36th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 155-162). Taipei, Taiwan: National Taiwan Normal University.

Tzur, R., & Lambert, M. A. (2011). Intermediate participatory stages as Zone of Proximal Development correlate in constructing counting-on: A plausible conceptual source for children's transitory 'regress' to counting-all. Journal for Research in Mathematics Education, 42(5), 418-450.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. *International Journal of Science and Mathematics Education, 2*, 287-304.

Tzur, R., Xin, Y. P., Si, L., Kenney, R., & Guebert, A. (2010). *Students with learning disability in math are left behind in multiplicative reasoning? Number as abstract composite unit is a likely "culprit"*. Paper presented at the American Educational Research Association. Retrieved from http://www.eric.ed.gov/

Tzur, R., Xin, Y. P., Si, L., Woodward, J., & Jin, X. (2009). Promoting transition from participatory to anticipatory stage: Chad's case of multiplicative mixed-unit coordination (MMUC). In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 249-256). Thessaloniki, Greece: PME.

Tzur, R. (Personal Communication, November 15, 2012)

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D.C.: Falmer.

Wilkins, J. L., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education*, *42*(4), 386-416.

Zazkis, R., & Hazzan, O. (1998). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, *17*(4), 429-439.

Zhou, Z., Peverly, S. T., & Lin, J. (2005). Understanding early mathematical competencies in American and Chinese children. *School Psychology International*, *26*(4), 413-427.

APPENDIX A

Assessment for Multiplicative Reasoning Feedback

Dear Expert/Teacher:

This assessment will be taken by students to determine their level of multiplicative reasoning.

Please provide me with your feedback regarding the tasks.

Validity - Will this task allow me to tell if a student can reason in this way?
Grade level appropriateness -Are the problem settings and numbers used appropriate for the grade level indicated?
Wording (Comments) – Please suggest changes that will make the problem clearer for students.
Keep/change/omit: Indicate if item can/should be kept in the assessment as is (1), maintained with some changes (2 – add changes in the Comments column), or omitted (3 – provide reason).

Task	Validity? Y/N	Grade Level Appropriate? Y/N	Comments, Wording Suggestions	Keep/Change/Omit
Counting On				
Missing Second Addend				
Missing First Addend				
Composite Unit 1				
Composite Unit 2				
mDC 1				
mDC 2				
mDC 3				