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Cite as:
Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior*, *31*(3), 313-330.

Title:

REASONING ABOUT VARIATION IN THE INTENSITY OF CHANGE IN COVARYING QUANTITIES INVOLVED IN RATE OF CHANGE

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Abstract

This paper extends work in the area of quantitative reasoning related to rate of change by investigating numerical and nonnumerical reasoning about covarying quantities involved in rate of change via tasks involving multiple representations of covarying quantities. The findings suggest that by systematically varying one quantity, an individual could simultaneously attend to variation in the intensity of change in a quantity indicating a relationship between covarying quantities. The results document how a secondary student, prior to formal instruction in calculus, reasoned numerically and nonnumerically about covarying quantities involved in rate of change in a way that was mathematically powerful and yet not ratio-based. I discuss how coordinating covariational and transformational reasoning supports attending to variation in the intensity of change in quantities involved in rate of change.

Keywords:

- Quantitative Reasoning
- Rate of change
- Quantity
- Covariational Reasoning
- Transformational Reasoning

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1. Introduction

Researchers have advocated for students' encountering conceptual underpinnings of calculus prior to a calculus course (e.g., Kaput, 1994; Kaput & Roschelle, 1999; Moreno-Armella, Hegedus, & Kaput, 2008; Roschelle, Kaput, & Stroup, 2000; Stroup, 2002, 2005). Together, this set of studies has called for students' consideration of situations involving rate of change. Stroup (2002, 2005) asserted that students should investigate situations involving both constant and varying rates of change. Further, students' investigation of situations involving constant rates of change need not be a prerequisite for their investigation of situations involving varying rates of change (Stroup, 2002).

Examining situations involving covarying quantities supports students' consideration of quantities involved in rate of change (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995). Based on observation of students' work with tables of values, Confrey and Smith (1995) noted that employing a covariation approach to the interpretation of tables of values (e.g., when values in one column increase by two the values in the other column increase by three) seemed to foster students' attention to rate of change. In a study investigating how twenty second-semester calculus students reasoned about quantities that covary, Carlson and colleagues (2002) found that students consistently coordinated amounts of change in one quantity with amounts of change in another quantity. However, only 25% of the students constructed an acceptable graph that could represent the amount of height as a function of the amount of volume for a filling bottle of varying width. Students' lackluster performance might be explained in part by differences in the ways in which students coordinated the covarying quantities. A related issue to consider is how students may reason about variation in the intensity of a change.

Secondary students have described variation in the intensity of a change when interacting with a graphing calculator simulation (Stroup, 2002), a physical device (Monk & Nemirovsky, 1994), and a Dynamic Geometry sketch (Saldanha & Thompson, 1998). Stroup (2002) articulated how eighth-grade students examining the movement of a cursor on a screen and a corresponding graph being drawn, noticed changes in the direction and intensity of the cursor's movement as "right fast" and "left slow" (p. 180). Monk and Nemirovsky (1994) detailed how a high school student working with a device that moved air through a tube, described changes in the intensity of the increase in air flow, stating: "Well – oh – It's just the amount of increase is less and less. I see-" (p. 156). Saldanha and Thompson (1998) reported how an eighth grade student, Shawn, could describe variation in the intensity of the change in the length of a line segment and coordinate that variation with the movement of a point along another segment. While all three of these studies address variation in intensity of change, Saldanha and Thompson (1998) went further by articulating how Shawn related covarying quantities.

Reasoning about rate of change is a complex, challenging activity involving understanding, recognizing, and representing different types of change, including both constant and varying rates of change (Stewart, 1990). Mathematical tasks incorporating multiple types of representations of rate of change afford students the opportunity to draw on informal, intuitive ideas of rate of change to investigate and reason about both constant and varying rates of change (e.g., Monk, 2003; Nemirovsky, 1994). There is a growing body of research investigating secondary students' reasoning related to constant and varying rate of change (Confrey, Castro-Filho, & Wilhelm, 2000; Lobato, Ellis, & Munoz, 2003; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002; Monk & Nemirovsky, 1994; Saldanha & Thompson, 1998; Stroup, 2002, 2005). This paper extends existing research by investigating both numerical and nonnumerical reasoning about covarying quantities involved in rate of change via tasks involving multiple representations of covarying quantities[1].

In this paper, I articulate how a secondary student who had not taken a calculus course reasoned about quantities involved in constant and varying rate of change when working on mathematical tasks involving multiple representations of covarying quantities. The purpose of this paper is to characterize a way of reasoning about covarying quantities involved in rate of change (see section 5.4) that could potentially serve as a cognitive root (Tall, 1989) for calculus. The objects of the reasoning include the covarying quantities involved in rate of change and quantities indicating relationships formed between the covarying quantities. A characterization refers to a viable explanation (from a researcher's perspective) of a student's way of reasoning without implying awareness of the characterization on the part of the one reasoning. Claiming that the researcher's explanation is a viable characterization implies that the explanation would hold for students' work across tasks incorporating different representations and contexts.

[1] When I use representation, it is in reference to an external product, existing in the physical world (Goldin, 2003), such that the object being represented by the product depends on the perspective of the one creating or interpreting the representation (e.g., Battista, 2008; Thompson, 1994a).

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2. Conceptual Framework

2.1 Quantity and quantitative reasoning related to rate of change

2.1.1 Quantity and rate of change

Individuals conceive of quantities, making quantities distinct from numbers (Thompson, 1994b). Quantities refer to attributes of objects that can be measured (Thompson, 1993, 1994b). I use Thompson's characterization for two reasons. First, it defines quantity as an object that can be conceptualized without actually engaging in a process of measuring. Second, it allows distinguishing between the intensity of a quantity—the degree to which a quantity is present—and the measured amount of quantity that is present. One way to distinguish between different types of quantities is to determine whether they can be measured directly or indirectly (Kaput & West, 1994). Extensive quantities (Schwartz, 1988) can be counted or measured directly. In contrast, intensive quantities (Schwartz, 1988) cannot be counted or measured directly. Because intensive quantities can only be measured indirectly, they are more difficult than extensive quantities to conceptualize and/or utilize.

When conceived of as an intensive quantity, rate of change measures the intensity of a multiplicative relationship between varying quantities. Drawing on the mental process of reflective abstraction (Piaget, 1980), Thompson (1994b) characterized rate as a "reflectively abstracted constant ratio" (p. 192) such that a ratio is the "result of comparing two quantities multiplicatively" (p. 190). The two-phased process of reflective abstraction involves an individual's reflection on and coordination of mental operations (Piaget, 1970). After reflectively abstracting a constant ratio, an individual could conceive of a single quantity as representing a constant multiplicative comparison between quantities.

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A student who understood rate of change in a way compatible with Thompson's characterization would have an advanced conception of rate of change as an intensive quantity. Students could conceive of rate of change as a quantity indicating a relationship between varying quantities such that the relationship involves something other than a multiplicative comparison. One student in the larger study from which the results reported in this paper are drawn treated rate of change as an association of extensive quantities (Johnson, 2011). Little is known about how students' early ways of reasoning related to varying quantities involved in rate of change might support the development of more advanced conceptions of rate of change.

2.1.2 Quantitative reasoning related to rate of change

Quantitative reasoning involves mentally operating (Piaget, 1970) on quantities in a situation to create new quantities and to make relationships between quantities (Thompson, 1994b). Quantitative reasoning can be numerical or nonnumerical, because one does not actually have to know the measure of a quantity or actually measure a quantity to reason quantitatively (Thompson, 1993). For example, given a task involving an expanding square, a student could determine amounts of increase in area associated with amounts of increase in side length and use those amounts to claim that the increases in area are increasing (numerical). In contrast, a student could claim that increases in area getting larger as the side length increases without determining particular amounts of increase (nonnumerical). Although quantitative reasoning can be numerical, numerical reasoning is not by necessity quantitative reasoning (Thompson, 1994b). For example, a student could identify numerical amounts of increase in area, but not be able to interpret the meaning of the numbers in terms of area.

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The determination of numerical amounts and interpretation of those amounts in terms of quantity could provide evidence of students' attention to underlying mathematical structure (e.g., what is meant by increase). However, the essence of a quantitative mental operation is "nonnumerical; it has to do with the *comprehension* of a situation" (Thompson, 1994b, pp. 187-188). Nonnumerical quantitative reasoning related to rate of change would involve attending to variation in the intensity of a change (Stroup, 2002) and to the quantities being related by a rate of change. For example, by using the nonnumerical quantitative operation of comparison, a student could reason about the intensity of changes in varying quantities, because he or she could compare changing quantities without measuring or specifying numerical amounts of change in quantities.

2.2 Covariation

2.2.1 Static and dynamic perspectives of covariation

When quantities are covarying, they are changing simultaneously and interdependently. The covariation can be perceived as static—when amounts of one quantity are associated with amounts of another quantity—or dynamic—when changes in one quantity are associated with changes in another quantity (Clement, 1989). Confrey and Smith (1994) provided a static perspective of covariation, involving the coordination of movement between successive values in one quantity with movement between associated values in another quantity. For example, when the side length of a square moves from 2 cm to 3 cm, the area of the square moves from 4 cm^2 to 9 cm^2. A dynamic perspective of covariation can be discrete or continuous. Clement (1989) provided a discrete dynamic perspective of covariation, involving the coordination of particular amounts of change in one quantity with particular amounts of change in another quantity. For example, as the side length of a square increases by 1 cm, the increases in the area of the square increase by 2 cm^2. Saldanha and Thompson (1998) provided a continuous dynamic perspective of covariation involving the coordination of continuous change in one quantity with continuous change in another quantity. For example, as the side length of a square increases continuously, the increases in the area of the square increase at a constant rate. These three perspectives of covariation highlight distinctions between the ways in which an individual can perceive a relationship between covarying quantities involved in a given situation.

2.2.2 Covariation and function

Researchers (e.g., Confrey & Smith, 1994; 1995; Thompson, 1994a) have articulated a covariation perspective on function as a more intuitive alternative to a formal, correspondence perspective on function. Confrey and Smith (1995) provided a static perspective, characterizing functions by the linkages between the domain and range, such that the links "are relational and spatial rather that rule driven" (p. 79). In contrast, Chazan (2000) provided a dynamic perspective, characterizing functions as "*relationships between quantities* [italics added] where output variables depend unambiguously on input variables" (p. 84). Prior to formal instruction in calculus, students might reason about covarying quantities involved in rate of change in a way that would be consistent (from a researcher's perspective) with an informal covariation perspective on function. In doing so, students could investigate variation *within* the intensity of the change of a quantity that specifies a relationship between covarying quantities. For example, a student could determine that as the side length of a square increases by one-half centimeter increments, the amount by which the increases in area are increasing is constant. Students also could make comparisons *between* the intensities of change in quantities, each specifying a relationship between covarying quantities. For example, students could compare the intensity of the increases in the area and the perimeter of a square, as both area and perimeter covary with the side length of a square.

2.3. Covariational and transformational reasoning

2.3.1 Covariational reasoning

Covariational reasoning involves mentally "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). An example of covariational reasoning would be the consideration of how, given a square, the quantities of area and side length vary together. Covariational reasoning could involve static or dynamic covariation. Reasoning involving static covariation is considered to be less advanced than reasoning involving dynamic covariation (Carlson et al., 2002). In addition, covariational reasoning can be "chunky," occurring when a student envisions the variation as having occurred in discrete chunks or "smooth," when a student envisions the variation as occurring through a continuing process (Castillo-Garsow, 2010). An example of a "chunky" form of covariational reasoning would involve considering how—given a square—the quantities of area and side length vary according to incremental changes in the side length, such that the increments could be uniform or nonuniform. The object of "chunky" covariational reasoning could be static covariation or discrete dynamic covariation, depending on whether the individual's reasoning was focused on amounts of covarying quantities (static) or amounts of change in covarying quantities (discrete dynamic). An example of a "smooth" form of covariational reasoning would be the consideration of how—given a square—the quantities of area and side length vary as the side length changes continuously. The object of "smooth" covariational reasoning would involve continuous covariation because an individual engaging in "smooth" covariational reasoning could reason about a continuum of amounts of quantities that are changing simultaneously.

2.3.2 Transformational reasoning

Transformational reasoning involves the "*mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations*" (Simon, 1996, p. 201). Simon (1996) provided an example of transformational reasoning through a student using Geometric Supposer software (Schwartz & Yerushalmy, 1985) to investigate relationships between base angles and legs in isosceles triangles. The student envisioned walkers stepping out a path (operation) beginning at the endpoints of a line segment (object) to form a triangle. The student predicted that when the base angles were equal, the walkers would walk the same distance (result). As illustrated by this student's response, key components of transformational reasoning are objects, operations, and results (Simon, 1996). The operation is the component of the reasoning that involves the transformation.

In contrast to inductive and deductive forms of reasoning, the goal of an individual reasoning transformationally is to make sense of how a mathematical system works (Simon, 1996). Simon indicated that "*Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated*" (p. 201). According to Simon, the student was able to consider an isosceles triangle "not as a static figure of particular dimensions, but rather as a dynamic process that generates triangles from the two ends of a line segment" (p. 199). The dynamic process involved in transformational reasoning could include the consideration of a discrete set of states, a discrete set of states that vary by uniform increments, or a continuous set of states. For example as shown in Figure 1 (left), given a square, an individual could consider a discrete set of amounts of side length and associated area (objects), envision successive amounts of length being added to the side of a square (operation), and anticipate the corresponding changes to the area of the square (result). If the discrete set of states varied by uniform increments, then the successive amounts of length being added would be uniform (See Figure 1, middle). If the set of states were continuous, then the side length and associated area would increase along a continuum (See Figure 1, right).

FIGURE 1

Figure 1. Discrete set of states (left), a discrete set of states that vary by a uniform increment (middle), and a continuous set of states (right)

2.4 A cognitive orientation

I employ a cognitive perspective by considering mathematical reasoning to be a purposeful mental activity on the part of the one reasoning. Purposeful activity involved in mathematical reasoning includes association (Thompson, 1996), sense-making (National Council of Teachers of Mathematics, 2009; Simon, 1996), and operation (Piaget, 1970). Association refers to the making of relationships between objects (Thompson, 1996). Sense-making refers to apprehending how a mathematical situation holds together (Simon, 1996). Operation refers to "an action that can be internalized; that is, it can be carried out in thought as well as executed materially" (Piaget, 1970, p. 21). From this perspective mathematical thinking—and hence, mathematical reasoning—is not directly observable. Students' explanations and justifications, as well as their observable actions (e.g., hand gestures), are key pieces of observable evidence from which students' mathematical reasoning can be inferred (Yackel & Hanna, 2003). For example, a student could explain that as the length of a side of a square increases, the amount of square units of area increase faster than the amount of units of perimeter. The explanation would provide evidence of the student's association of area, perimeter, and side length, sense-making of a situation involving varying area, perimeter, and side length, and carrying out an operation such that changes in area and perimeter are dependent upon changes in side length. Although I am foregrounding the cognitive perspective by attending to reasoning of individual students, I consider an individual student's reasoning to involve the context in which the individual is interacting. Therefore, explanations of individual students' reasoning reflect my consideration of the mathematical tasks and the types of representations of covarying quantities with which students are interacting.

3. Research Questions

Carlson and colleagues (2002) asserted that covariational and transformational reasoning shared mental operations and called for research addressing the transformational aspect of the reasoning. Further, Carlson and colleagues (2002) were able to offer "no information about the process of coming to generate a particular transformational approach" (p. 375). For the study reported in this paper, I examined how combining covariational and transformational reasoning might support numerical and/or nonnumerical quantitative reasoning about covarying quantities involved in rate of change. The research questions I investigated were: (1) How, prior to taking a calculus course, do students reason about quantities involved in constant and varying rate of change when working on tasks involving multiple representations of covarying quantities, and (2) How might students combine covariational and transformational reasoning when considering quantities involved in rate of change?

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4. Methods

4.1 Setting

The case (Stake, 1994) reported in this paper was part of a larger study that consisted of six high school students from a small rural school district that has been involved in long-term initiatives to support staff development and student learning. The six students volunteered to participate in the study and were paid $8 per hour for their participation. Student participants were drawn from a list of recommended students given to me by the district mathematics teachers and the mathematics coach. I requested to work with students who would be willing and able to talk about their mathematical thinking in an interview setting, had completed at least one year of algebra, and were not currently enrolled in a calculus course. I chose students who had completed at least one year of algebra, because I expected them to be familiar with Cartesian graphs that were included in the tasks. I chose students not currently in a calculus course, because my goal was to learn more about how students reason about changing quantities prior to formal instruction in rate of change that typically occurs in a calculus course.

The results presented in this paper focus on the reasoning of one of the six students in the larger study—Hannah (pseudonym), a tenth grader who had completed one year of algebra at the time of the study. Hannah was the only student who combined covariational and transformational reasoning in a way that supported attention to variation in the intensity of change in quantities involved in rate of change. My investigation of Hannah's reasoning explicates complexities involved in rate-related reasoning.

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4.2 Data collection

The larger study consisted of a series of five individual task-based clinical interviews (Clement, 2000; Goldin, 2000) with each student. Consistent with clinical interview methodology, no attempt was made to teach these students. By utilizing a series of interviews, I was able to present a variety of tasks incorporating different contexts and multiple representations of covarying quantities involved in different kinds of rate of change. The task-based interviews with each student occurred once per week[2] in the fall of 2009, with each interview lasting approximately 40 minutes. Another researcher video and audio recorded the interviews. I collected students' written materials and took digital photographs of their work. A transcriber produced verbatim written transcripts from the audio-recordings. After receiving the written transcripts, I checked them for accuracy, incorporating figures and annotations. This series of five interviews provided time for me to develop rapport with students, allowed for the use of a variety of tasks, and facilitated further investigation of students' work from prior interviews.

4.3 Tasks

The set of five interviews contained a total of seven tasks that each student completed. I designed each task with an overarching goal of better understanding students' reasoning about covarying quantities involved in rate of change. Each task afforded students the opportunity to utilize verbal and geometric representations to represent covarying quantities and to quantitatively reason numerically and/or nonnumerically. I designed each task to support

[2] Due to time constraints, I interviewed one student twice during one week.

combinations of covariational and transformational reasoning, anticipating that students' covariational reasoning could be "chunky" or "smooth" and students' transformational reasoning could be discrete or continuous.

In this paper, I focus on three of the tasks used in the larger study—the changing square, typical high temperature, and filling bottles. This set of three tasks incorporates dynamic and static representations of covarying quantities, with quantities varying at constant and varying rates. The dynamic representations consist of researcher-developed dynamic sketches created using Geometer's Sketchpad software (Jackiw, 2001). The descriptions of the tasks appear in the order in which students encountered them. The first task required students to compare variation between two quantities, each covarying with respect to a third quantity. All three tasks required students to investigate variation within a quantity covarying with another quantity. The first and third tasks included quantities involved in both constant and varying rates of change. The first task included quantities involved in rate of change that varied at a constant rate; the second and third tasks included quantities involved in rate of change that varied at varying rates. Interviewer questions investigated how students might relate covarying quantities, make distinctions between constant and varying rates of change, and predict how change might continue. For a more detailed description of all of the tasks used in the larger study see Johnson (2010).

4.3.1 Changing square

The changing square task required students to investigate and compare the change in the area and perimeter of a square as each covaried with the side length of the square. In developing the changing square task, I adapted Lamon's (2005, 2007) tasks by including a dynamic sketch and asking interview questions that focused on relationships between the covarying quantities.

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Students first were given a dynamic sketch (See Figure 2) and then were given tables of values relating side lengths and amounts of area and perimeter (See Figure 3). When interacting with the dynamic sketch, students could change the size of the square by dragging point B, and the measures of the length of a side, the area, and the perimeter changed accordingly. Figure 2 shows change that occurred after dragging point B to two new locations. Interviewer questions included prompts such as: As the length of the side of the square changes, how do the area and perimeter change? If you were to increase the side length of another square by the same amount, would the perimeter still increase by the same amount? My overarching goal for the changing square task was to better understand students' comparison of variation between quantities involved in a constant rate of change and quantities involved in a constantly varying rate of change.

FIGURE 2

Figure 2. Original square (left) and two new squares resulting from dragging point B

FIGURE 3

Figure 3. Tables of values relating amounts of side length and amounts of perimeter and area

4.3.2 Typical high temperature

The typical high temperature task required students to investigate the way in which a city's typical high temperature increased and decreased over the course of a year. In the typical high temperature task students were given a dynamic Cartesian graph (See Figure 4). The graph contained an active point that students could drag by clicking and holding or animate by pressing an action button. Although I recognize that a city's typical high temperature will be constant for any given day, I chose to use a sine function rather than a discrete or step function to model the situation. Interviewer questions included prompts such as: How would you determine by how much the temperature is increasing between day 40 and day 140? How would you compare that increase to the increase between day 40 and day 90? How do the decreases in temperature compare to the increases? My overarching goal for the typical high temperature task was to better understand students' reasoning about quantities involved in rates of change that varied at a varying rate.

FIGURE 4

Figure 4. Dynamic Cartesian graph representing the typical high temperature as a function of the day of the year

4.3.3 Filling bottles

The filling bottles task required students to investigate how the volume of liquid in a bottle would change as the height of the liquid in the bottle increased, given that liquid was being dispensed into the bottle at a constant rate. Students were given a static, Cartesian graph representing the volume of liquid in a bottle as a function of the height of liquid in the bottle, as shown in Figure 5. I adapted the filling bottles task from previous work (Carlson et al., 2002; Heid, Lunt, Portnoy, & Zembat, 2006) by representing the volume of liquid as a function of the height of the liquid in the bottle. I chose to represent volume as a function of height in part because early college students working on a task such that height was a function of volume operated with the independent variable, volume, as if it were time (Carlson, Larsen, & Lesh, 2003). By focusing on how the volume of liquid in a bottle would be changing as the height of the liquid in the bottle increased, I anticipated that it would reduce the likelihood of students reasoning about the independent variable as if it were time, which was not one of the covarying quantities explicitly represented by the graph. Interviewer questions included prompts such as:

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How is the volume of the soda in the bottle changing as the height of the soda in the bottle increases? Sketch a bottle that the graph (shown in Figure 5) could represent. My overarching goal for the filling bottles task was to better understand students' reasoning about quantities involved in constant rates of increase and rates of increase that varied at varying rates.

FIGURE 5

Figure 5. Static Cartesian graph representing the volume of liquid as a function of the height of the liquid in the bottle

4.4 Data analysis

Data analysis involved three phases: choosing and describing the data, using open coding (Corbin & Strauss, 2008) to develop characterizations of students' reasoning, and articulating how students might use different forms of reasoning. Making multiple passes through the data, I built from students' explanations, written work, and gestures to characterize students' reasoning. As I developed characterizations of each student's reasoning, I used a constant comparative method (Corbin & Strauss, 2008), vetting the viability of evolving characterizations with two other researchers. These characterizations, grounded in the data, captured a researcher's perspective of the essence of each student's reasoning. Working from the characterizations, I analyzed how students' reasoning might be quantitative, covariational and/or transformational. Consistent with the methodology of grounded theory (Glaser & Strauss, 1967), my analysis did not assume that students would engage in particular forms of reasoning or combine different forms of reasoning.

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5. Results

The results section is organized by the tasks that students encountered. In this section, I provide evidence of Hannah's use of numerical and nonnumerical quantitative reasoning and her coordination of covariational and transformational reasoning. When I indicate that Hannah is engaging in a particular form of reasoning, I am providing a researcher's perspective of her reasoning. I am not claiming that Hannah was aware she was engaging in that form of reasoning.

5.1 Changing square

In Hannah's work on the changing square task, she reasoned quantitatively about increases in area and perimeter, because for her area and perimeter were measurable attributes of the square. Sketching a square unit, she indicated that area would be the amount of square units needed to cover "everything that's yellow[3]." Running her finger along the edges of the square shown in the dynamic sketch, she indicated that perimeter would be "the measure of all these lines, like the outside of the square added together."

5.1.1 Making comparisons between increases

To investigate how Hannah might distinguish between different amounts of change occurring in the same direction, I prompted her to describe how the area and perimeter changed as the side length increased.

Interviewer: So as the side of the square changes, how do the area and perimeter change?

Hannah: They both increase as you make the—you increase the side of the

[3] The interior of the square shown on the dynamic sketch (See Figure 2) was shaded yellow.

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square, like if you drag point B, both area and perimeter increase.

Interviewer: And if you had to describe—to compare, the ways in which area and perimeter increase?

Hannah: It seems that area increased a lot faster as you—or decreased a lot faster as you increase or decrease the side of the square. Perimeter—increases slower.[4]

Interviewer: How do you determine that?

Hannah: Um, as you increase it, as it is now at the area at one hundred eight point eight three (108.83) *[sic]* and the perimeter at forty-one point seven one (41.71). If you increase it a little bit, the area increased a lot more than the perimeter did, cause now it's like—I increase it just a little way and it's one hundred sixteen point two one (116.21) and the perimeter only went up to forty-three point one two (43.12). *[While responding, Hannah dragged point B, changing the side length of the square from 10.43 to 10.78 cm. See Figure 6.]*

FIGURE 6

Figure 6. Hannah dragged the active point B (lower right) to increase the side length of the square from 10.43 cm to 10.78 cm.

[4] As do many students, Hannah extensively used the word "like" in her responses. For example, in this response she said "Perimeter—**like** increases slower." To ease reading of the transcript, I removed superfluous uses of the word "like."

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Cite as:
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When Hannah described an increase, she was reasoning about an extensive quantity measuring amounts of increase in area and/or perimeter associated with amounts of increase in side length. As she dragged point B she increased and decreased the side length in nonuniform increments, pausing briefly at different values of the side length. When Hannah used the descriptor "faster," she was comparing the amounts of increase in area to the amounts of increase in perimeter as each covaried with side length. Using particular numerical amounts of increase in side length to explore associated amounts of increase in area and perimeter was not necessary for her. She predicted that area would increase faster than perimeter, and when prompted, she used numerical amounts to illustrate that her prediction was viable. As long as the amount by which she increased the side length was small, it suited her purpose of demonstrating that the amount of increase in area would exceed the amount of increase in perimeter.

5.1.2 Investigating variation within increases

After Hannah's work with the dynamic sketch, I provided her with tables of values relating amounts of side length and amounts of perimeter and area (See Figure 3). Hannah determined that when the side length of the square increased from 0.5 cm to 1.0 cm the amount of increase in area was less than the amount of increase in perimeter: "for the first one to the second one, when you increase the area, it only goes up by point seven five (.75), and the perimeter goes up by a whole two." When prompted to consider how the perimeter would increase for side lengths not included on the table, Hannah appealed to relationships between side length and perimeter to explain why a 2 cm increase in perimeter always would be associated with a 0.5 cm increase in side length:

Interviewer: If you were to start with a larger line segment AB, um, say we have this ten point seven eight (10.78) centimeters that's on the screen here. If you were to increase this square by point five (.5) centimeters, do you think the perimeter would still increase by two centimeters?

Hannah: Um—yeah because they're both being measured by two centimeters, so they're in centimeters so if you increase it, it should still be the same increase for both. If I increase it up point five (.5) it should be two centimeters.

Interviewer: And what are you using to make that claim?

Hannah: Um, no matter how many centimeters it's always going to be the same length. No matter how much you measure it, so if you increase the centimeters by point five (.5), the perimeter will go up because each line, each side of the square, there's four squares *[sic]*, and if you increase both sides by point five (.5), because all the sides have to be equal to each other, it will equal two centimeters that you're adding to it.

Interviewer: And where will that two centimeters come from?

Hannah: Uh, it'll just add a longer line to it.

Interviewer: Could you draw it?

Hannah: *[Drawing]* If you want to add, increase this one by say two centimeters of each, you just make this line longer (See Figure 7). You'd have to make every line longer of the square by the point

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five (.5) that you want it to make the line segment.

FIGURE 7

Figure 7. Hannah represents an increase in perimeter resulting from an increase in side length.

Hannah predicted that the increase in the perimeter of a square would be constant (2 cm) when the side length increased uniformly (0.5 cm increments). Even though Hannah's covariational reasoning involved discrete amounts of change in the side length and perimeter, the particular amounts of change seemed secondary to the dynamic covarying relationship between the increase in side length and the increase in perimeter. As she drew the representation shown in Figure 7, Hannah enlarged a square (object), claiming that the new square that must have perimeter 2 cm larger than the original square (result) by envisioning length being added to all of the sides of the square (operation). Although the use of a table of values potentially could foster discrete transformational reasoning, Hannah's representation indicating from where the two-centimeter increase in perimeter would come suggests a square whose side lengths can continuously increase.

Hannah was also able to reason about the constantly varying increase in area resulting from uniform changes in side length. When prompted to determine the amounts of change in area when the side length increased from 1 cm to 1.5 cm and 1.5 cm to 2 cm, Hannah recorded 1.25 and 1.75 respectively (See Figure 8). When prompted to explain what the numerical amounts meant in terms of the quantities of area and side length, Hannah identified a constant difference between the consecutive differences in amounts of area. She used that constant difference to predict how the area would change for side lengths not included on the table.

Interviewer: And those three numbers that you have, if you had to say what the

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point seven five (.75) and the one point two five (1.25) and the one point seven five (1.75) mean in terms of the square and the area, what would you say?

Hannah: I'd say each time they're increasing it by zero point five (0.5) the um, the difference between them goes up point five (.5) between the area of the squares.

Interviewer: How did you determine the difference between them went up by point five (.5)?

Hannah: Because each time, the difference of the difference that they're going up by is point five (.5) between each of them.

Interviewer: And you can write that down as well.

Hannah: All right. *[Pause 10 sec. Writing.]* (See Figure 8, right)

Interviewer: And you used a phrase: "the difference of the difference." Could you tell me what that means in terms of the square?

Hannah: The difference of the area of the square that we're taking off of, the zero point seven five (0.75). It's the difference between the zero point seven five (0.75), between the one point two five (1.25), the next difference of the area of the square.

Interviewer: And say you had to predict kind of going out, um, how much the area would increase by as the length of the side increased from say nine centimeters to nine point five (9.5) centimeters. And you can go ahead and write that down at the bottom of the table. Like as the length of the side went from say nine centimeters to nine point five (9.5) centimeters.

Hannah: Mmm Hmm. *[Pause 15 sec, records 9 cm and 9.5 cm]* (See Figure 8, bottom)

Interviewer: Do you have a sense of how much the area would increase by?

Hannah: It's um, you would, the difference between the five point five (5.5) and the six would be um, five point seven five (5.75) it increased. So you would just keep adding on zero point five (0.5) to the difference of, as it goes, as you went out until you got to the nine centimeters to nine point five (9.5) centimeters.

FIGURE 8

Figure 8. Hannah records the "difference of the difference," then records the lengths 9 cm and 9.5 cm in response to the prompt.

Hannah used the term "difference of the difference" to refer to a constant numerical amount by which the increase in the area of the square would increase. While working with the table she did not specify that the numerical amounts would be measured in square centimeters. However, near the end of the interview, when prompted, she was able to describe those amounts in terms of "yellow space" [referring to the interior of the square shown on the dynamic sketch, see Figure 2] and square centimeters.

Interviewer: And way back at the beginning you talked about perimeter and you told me that perimeter was the distance around-

Hannah: Yeah.

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Interviewer: And area was the yellow space inside-

Hannah: Mmm Hmm.

Interviewer: Um, I guess, kind of relating these numbers to the distance around and the yellow space inside, um, can you make sense of these point seven five (.75) and one point two five (1.25) and one point seven five (1.75) *[Referring to the numbers Hannah recorded to the right of the table shown in Figure 7.]* in terms of the yellow space?

Hannah: Um that's, you'd have, point seven five (.75) of a square centimeter that you're increasing into a yellow space of the square and with the next time you'd have one point two five (1.25) of a square centimeter that you're increasing into the yellow square.

Interviewer: And what about those point fives (.5) and the yellow space?

Hannah: It'd just be, point five (.5) of the square centimeter.

Hannah's responses indicate that she can quantify the relationship between the increase in area and increase in side length as an extensive quantity—an increasing amount of area—associated with a particular increase in side length (0.5 cm). When referring to the increasing "yellow space," she seemed to be envisioning a square being transformed as it continued to increase in area by successively larger amounts. Although she related the constant difference between the successive increases in area (.5) to square centimeters, she did not provide any sense of how that amount related to an enlarging square. This suggests that the "difference of a difference" may not be a quantity for Hannah. Further, in her work with the tabular representations, her assertion "you would just keep adding on zero point five (0.5)" may indicate

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that she was identifying and continuing a numerical pattern rather than reasoning transformationally.

When prompted to explain how her work with the tables of values could inform her comparisons of the increases in the area and perimeter, Hannah provided further clarification regarding her comparison of the increases: “For the area, it gets bigger as you make the line segments bigger, so the difference them between like the bigger it gets, the area will increase faster. But with the perimeter, it’s a steady increase, just like it goes up by two each time.” In her work across the tasks, Hannah used the descriptor “steady” to indicate variation that typically would be called constant. Her use of “faster” in this response had a different meaning than her earlier use of “a lot faster” to compare the increase in area to the increase in perimeter. In this instance, when she claims that area will increase “faster,” she is referring to the successively larger amounts of area being added to a square as the length of the side is increased by 0.5 cm.

When Hannah investigated variation *between* increases in quantities each covarying with a third quantity, she could make distinctions between magnitudes of the amounts of increase. She could then use qualitative descriptors to make comparisons between the increases (e.g., perimeter increased “slower” than area). In contrast, when Hannah investigated variation *within* the increase in a single quantity covarying with a third quantity, she used “steady” and “faster” to indicate different intensities of increases. For example, she was certain that when the length of the side of a square increased by 0.5 cm, the perimeter would always increase by 2 cm. Further, she appealed to an enlarging square to claim that the increases in the area would keep getting larger as the side length increased. By coordinating transformational and covariational reasoning, Hannah could envision how area and perimeter of a square would increase as the side length of the square increased by a uniform amount.

By describing the increases in area and perimeter occurring as a result of changing the side length, Hannah treated the increases in area and perimeter as if they were a function of side length. Throughout her work, Hannah never explicitly used the word function, and I suspect she might not have been able to provide a mathematical definition for function. She appeared to be reasoning about the increases in area and perimeter such that each depended unambiguously on variation in the side length. Her informal way of reasoning seems consistent (from a researcher's point of view) with a dynamic covariation perspective on function.

5.2 Typical high temperature

5.2.1 Using numerical amounts to illustrate a varying increase

When prompted to discuss how the temperature was changing throughout the year, Hannah had nonnumerically described variation in the increase and decrease in temperature. To further investigate her reasoning, I prompted her to explain how she would determine an amount by which the typical high temperature was increasing.

Interviewer: So you talked to me about the temperature increasing and you said it increased more quickly and then it increased a little bit less. If you had to determine how much it would be increasing, how would you go about doing that?

Hannah: You could take the temperature of like one of the higher days and then subtract maybe like the temperature from the previous day, you could find out how much it increased or decreased. Like you can keep doing that and see if there's a steady increase or decrease or close to that.

Similar to what she had done on the changing square task, Hannah described how she would engage in calculations without actually performing the calculations. When further prompted to "go ahead and mark some things on the graph and do some of those calculations like you described," Hannah wrote that there was a ".1 increase from day 54 to day 55". In her work on this task, when Hannah described an increase (or decrease), she was reasoning about an extensive quantity measuring amounts of increase (or decrease) in temperature associated with a set of consecutive days. Again, Hannah used the descriptor "steady" to refer to increases that would typically be referred to as constant, and I prompted her to elaborate on what she meant by a "steady" increase.

Interviewer: So if the increase was steady, what would that mean for you?

Hannah: Like, it would increase by point one degree Fahrenheit every day and that would be like a steady increase or decrease.

Interviewer: Do you think it will be steady?

Hannah: Um, it doesn't look like it would be steady, like right along here, but if you got up here it looks like it would be pretty much at a steady increase.

Interviewer: What about the way the graph looks makes you say that?

Hannah: Because it's not really like a straight line. Like around here it's more straight, *[Runs her finger along the graph, moving left from approximately day 140 to day 100. Graph is shown in Figure 4.]* so that, that means it's

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going up at the same amount of degrees around the same amount each day, but it's curving, so it's changing.

Interviewer: So if you were to gather information to help you figure out whether there would be a steady increase or not, what would you do next?

Hannah: Um, you could, you'd just take like if you wanted to find out if there was a steady increase here, you'd go to day fifty-six and you'd see if the same increase from day fifty-five to fifty-six was the same as for fifty-four, fifty five.

Hannah associated a "steady" increase with a physical attribute of the graph—straightness that is not necessarily linearity—using numerical amounts to illustrate what she meant by a "steady" increase. Specifically, Hannah showed that the amount of increase between days 55 and 56 was not the same as the amount of increase in temperature between days 54 and 55, indicating that the increase was "not really steady." By indicating that "it" would be changing when a graph is curving, Hannah was accounting for variation in the amount of increase in temperature associated with a uniform increment (a set of consecutive days).

5.2.2 Making distinctions between variation in the intensity of a varying increase

After Hannah used numerical amounts to illustrate that the increases were not "steady" near day fifty-four, I prompted her to make a prediction about how the increases "might go."

Interviewer: So, say you were to do the next couple of days, could you predict for me how the decrea-, the increases might go?

Hannah: I'd say the increase would increase, the increase would get bigger as you went along, until you got to that steady and then after the steady one it looks like there's a decrease of how much it increases *[Points to the portion of the graph just to the left of the maximum. Graph is shown in Figure 4.]* until it reaches the peak and then it turns around and decreases.

Interviewer: So you talked about an increase in increases,

Hannah: Yeah

Interviewer: And then a decrease in increases,

Hannah: Yeah

Interviewer: Could you tell me a little bit about what that means in terms of temperature?

Hannah: Like the, how much the increase of temperature between the days. It looks like from the beginning there's a smaller increase from day to day and as it goes on the increase from day to day gets bigger or larger and then it looks like it reached a steady increase from day to day. Then eventually, after it gets after that steady increase, it looks like the increase from day to day starts to go down, like smaller increases.

In this episode, Hannah predicted that a "steady" increase would occur in some subsection of the graph shown in Figure 4. Although this is not actually the case, if one were to consider a continuum of intensities of increases, a "steady" increase would occur between an increasing increase and a decreasing increase. Hannah's attention to a continuum of intensities of increases represented by a continuous graph suggests she was mentally running through amounts of increase in temperature associated with the sets of consecutive days. Because Hannah was running through sets of consecutive days, her covariational reasoning was “chunky” and her transformational reasoning was discrete. Despite the lack of smoothness in her reasoning, Hannah did not need to calculate numerical amounts of change in degrees associated with each set of consecutive days to make predictions about the variation in the intensity of the increases. By allowing the increase in temperature to depend on her incremental movement “from day to day,” Hannah seemed to reason as if she were using a dynamic covariation perspective on function.

Throughout her work on the typical high temperature task, Hannah made distinctions between the intensities of increases that varied at a varying rate. Coordinating transformational and covariational reasoning, Hannah was able to mentally run through variation in the intensities of increase without relying on numerical amounts. By appealing to the shape of the graph, she could predict how the quantity—amounts of increase in temperature associated with sets of consecutive days—would vary. This is not to say that Hannah was unable to use numerical amounts to support her nonnumerical predictions. As she had done on the changing square task, Hannah drew on her nonnumerical work to make predictions about variation in the intensity of change. In her work on this task, she used numerical amounts to demonstrate variation in the intensity of the increase.

5.3 Filling bottles

In Hannah’s work on the filling bottles task, she reasoned quantitatively about the increases in volume, because for her the volume of the liquid measured how much space was taken up by the liquid. Referring to a physical bottle of soda, she described the volume of the liquid in the bottle as measuring: “how much, I guess, drink you could get inside of it.”

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5.3.1 Drawing on variation in the intensity of change to predict the shape of a bottle

Given a graph relating the volume of liquid in a filling bottle with the height of the liquid in the bottle (See Figure 5), I prompted Hannah to describe the way in which the volume of liquid increased as the height of the liquid increased. In her response, she included a prediction about the shape of a bottle that would be represented by the graph. Running her finger along the graph, Hannah asserted: "as you go along it definitely increases more." After running her finger along the graph, Hannah described how a bottle being represented by the graph shown in Figure 5 would be shaped, saying that it "gets smaller as you go to the cap of the bottle." When prompted to justify how she knew that the bottle would get smaller, Hannah focused on the increasing quantities of volume and height, moving her hand along the volume axis and then along height axis.

Interviewer: How do you know that the bottle starts to get smaller?

Hannah: Because the volume starts to not increase as much as the height goes on.

Noteworthy was Hannah's movement of her hands in conjunction with coordinating the intensity of the increase in the extensive quantity—volume—with an increase in the extensive quantity—height. When specifying that the volume would "not increase as much as the height goes on" Hannah moved her hand along the volume axis and then along the height axis. Hannah's hand motions indicate that she attended to the covarying quantities of volume and height, suggesting that for her both volume and height are dynamically increasing, and potentially "smoothly" covarying, despite the fact that she was working with a static graph containing specific measurements. By specifying that the "not as much" increase in the volume occurred as the height "goes on," Hannah was reasoning about the variation in the change in volume as being dependent upon variation in the height.

5.3.2 Drawing on variation in intensity of change to sketch a bottle represented by a graph

Immediately after Hannah said that the "volume starts to not increase as much as the height goes on," I prompted her to "sketch something or to describe to me what that would be like." In response to my prompt, Hannah suggested that she just draw the bottle. As Hannah sketched the shape of a bottle, she related variation in the intensity of the increases in volume to the shape of the bottle.

Interviewer: Could you maybe sketch something or describe to me what that would be like? I have a blank sheet if that helps.

Hannah: Do you want me to just draw the bottle?

Interviewer: Sure.

Hannah: Like how the shape would be? It would start off smaller because the increase of volume starts off slower and then it would start to get larger, so it would have this kind of curve to it. And then, as like this, and then as soon as you get up here into this, here the bottle would get smaller again, as it went to this cap, so it would be sort of like a vase shape. So, and then it would just be sort of shaped like that because the volume down here like it says, shows on this graph, like the volume starts to increase faster as it gets in towards around here and then as you go up further and then as the height increases the volume starts to slow, like slower increasing.

[Hannah sketches the bottle while talking. Sketch is shown in Figure 9.]

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FIGURE 9

Figure 9. Hannah's sketch of the shape of a bottle that the graph would represent.

When drawing a viable bottle shape associated with the graph, Hannah simultaneously coordinated variation in the intensity of change in volume with "smooth" chunks of increase in the height. She chose the chunks based on when the variation in the intensity of increase in volume would change from one type to another (e.g., from faster increasing to slower increasing). For Hannah, *slower increasing* referred to variation in a quantity that related the changing volume with the changing height, measuring a decreasing trend in a set of increases in the volume of liquid in the bottle per nonspecific amounts of change in height. By running through the increase in the height of the liquid in the bottle in "smooth" chunks while allowing the volume of the liquid in the bottle to increase simultaneously, Hannah coordinated transformational and covariational reasoning. The transformational aspect involved the smooth run-through of variation in the intensity of the increase in volume throughout each chunk. The covariational aspect involved the coordination of the covarying quantities of volume and height such that volume was dependent on height. By systematically increasing the height, Hannah was able to attend to variation in the intensity of the increase in a quantity relating the changing volume and the changing height.

It is noteworthy that Hannah's systematic variation of the height—in "smooth" chunks—is different from what one would actually see if one were watching a bottle of varying width being filled with liquid being dispensed at a constant rate. For a bottle of varying width, the height of the liquid would not increase at a constant rate. By increasing the height in "smooth" chunks, Hannah worked on the task as if the height of the liquid in the bottle were increasing at a constant rate. By allowing the height of the liquid in the bottle to increase at a constant rate and treating the volume of the liquid in the bottle as being dependent on the height of the liquid in the bottle, she could then attend to the variation in the increase of the volume of the liquid in the bottle.

5.4 A characterization of reasoning about variation in the intensity of the change in covarying quantities involved in rate of change

Across the set of tasks, Hannah investigated variation *within* the intensity of the change of a quantity that specifies a relationship between covarying quantities involved in rate of change. When investigating such variation, Hannah's way of reasoning remained consistent even when the tasks involved different contexts, quantities, and representations. The way in which Hannah reasoned about quantities involved in rate of change seemed compatible (from a researcher's perspective) with a dynamic covariation perspective on function. By engaging in such a way of reasoning, Hannah was able to consider variation in the intensity of change of a quantity indicating a relationship between covarying quantities.

Based on Hannah's work across tasks, a way of reasoning about covarying quantities involved in rate of change can be characterized as follows: *systematically varying one quantity and simultaneously attending to variation in the intensity of change in a quantity indicating a relationship between covarying quantities.* The systematic variation could be discrete (involving uniform or nonuniform increments) or continuous. The quantity being systematically varied serves as the independent variable. The quantity indicating a relationship serves as the dependent variable. The quantity indicating the relationship could be an intensive, ratio-based quantity or an extensive quantity coordinating the two covarying quantities. For example, in the filling bottle task, Hannah envisioned systematic variation in the height and made distinctions between the intensity of the increase (e.g., "slower increasing") in the volume as it covaried with the height. A dynamic, systematic way of reasoning about relationships between covarying quantities involved in rate of change is mathematically powerful and useful to consider for a variety of reasons, three of which I articulate next.

First, this way of reasoning supports attention to variation in the intensity of the change in a quantity indicating a relationship between one quantity covarying with another quantity. It is known that attending to the intensity of change is more advanced than just attending to the direction of change (Carlson et al., 2002). By systematically varying one quantity, a student utilizing this way of reasoning would be able to attend to variation in the intensity of change in a quantity indicating a relationship between covarying quantities without needing to determine numerical amounts of change. Further, attending to variation in the intensity of change is essential to reasoning about a second derivative, a key concept of calculus.

Second, this way of reasoning provides an empirically based model of how covariational and transformational reasoning could be coordinated into a coherent process. The covariational aspect of the reasoning involves the coordination of two varying quantities, one serving as the independent variable and the other serving as the dependent variable. Systematically varying the quantity serving as the independent variable allows a student to mentally run through variation in the intensity of the change in the quantity indicating a relationship between the covarying quantities, which is the transformational aspect of the reasoning. The transformational aspect is intertwined with the covariational aspect because of the systematic nature of the variation of the quantity serving as the independent variable. In the context of reasoning about covarying quantities involved in rate of change, the need to systematically vary one of the quantities might foster transformational operations.

Third, this way of reasoning provides an empirically based model of how a student could coordinate "chunky" and "smooth" covariational reasoning about quantities related to rate of change prior to engaging in ratio-based reasoning. By utilizing "smooth" chunks, a student could coordinate change in two covarying quantities such that the quantities are changing simultaneously as each quantity passes through all of the intermediate values in the chunk. Further, utilizing "smooth" chunks could be useful for reasoning about the Mean Value Theorem, which indicates that for a continuous, differentiable function, the average and instantaneous rate of change must be equal at some point in a closed interval. By envisioning a tangent line passing through all of the intermediate slopes in an interval (chunk), a student could predict when the tangent line would be parallel to the secant line.

6. Discussion and Implications

This study contributes to a growing body of knowledge on how students can attend to variation in the intensity of change without formal instruction in calculus (e.g., Monk & Nemirovsky, 1994; Stroup, 2002, 2005). Stroup (2002) asserted that non ratio-based reasoning about rate of change is mathematically powerful and more than just transitional to ratio-based reasoning about rate of change. The characterization in section 5.4 provides a way of reasoning that coordinates covarying quantities involved in rate of change in a way that is mathematically powerful and yet not ratio-based. At the heart of the reasoning is the coordination of covarying quantities in a way that supports the creation of a new quantity specifying a relationship between those covarying quantities. Saldanha and Thompson (1998) suggested that once an individual coordinates covarying quantities such that the quantities are changing simultaneously, a multiplicative object (ratio) would be formed. This research highlights mathematical reasoning that could underlie the construction of a multiplicative object (e.g., ratio) or nonmultiplicative object (e.g., association of quantities).

When creating a new quantity by coordinating covarying quantities, a ratio becomes one of many possible objects constructed as a result of the reasoning rather than the essence of the reasoning itself. Therefore, this way of reasoning is not just transitional to ratio-based reasoning but a distinct way of thinking about quantities covarying simultaneously and interdependently. Further, by employing a way of reasoning involving the systematic variation of one quantity and the simultaneous attention to variation in the intensity of change in a related quantity, an individual could make sense of variation in the intensity of change numerically or nonnumerically and with or without forming a ratio. By positing such a way of reasoning, I am not advocating that ratio-based reasoning related to rate of change be deemphasized. Given the complexity of this domain, it seems important to consider various viable ways of reasoning that support attention to variation in intensity of change.

When an individual is coordinating covariational and transformational reasoning, that individual can attend to variation in the intensity of change in a quantity that indicates a relationship between covarying quantities. By using coordinating, I mean that the forms of reasoning are integrated into a coherent process. A student's coordination of covariational and transformational reasoning when reasoning about rate of change could be conducive to more formal reasoning about rate of change. For example, in her work on the typical high temperature task, Hannah's way of reasoning afforded her attention to an *increase in increases* and a *decrease in increases*, both of which could be productive ways of verbally describing a second derivative as a rate of change of a rate of change.

Nonnumerical reasoning can be useful for students coming to understand the concept of derivative (Zandieh & Knapp, 2006). However, students engaging in nonnumerical derivative-related reasoning may not be attending to underlying process or structure, including concepts of ratio, limit and function (Zandieh, 2000; Zandieh & Knapp, 2006). For example, the description *decrease in increases* provides no evidence of consideration of underlying concepts of ratio, limit, or function as related to the second derivative. In Hannah's case, the object of her reasoning when she made the statements about the intensity of the increase was the quantity she indicated—the amount of change in temperature associated with a set of consecutive days.

Hannah was able to consider the underlying structure of the objects of her reasoning, as evidenced by her numerical justification for her nonnumerical claims. When students use numerical justification to support their nonnumerical descriptions of rate of change, it could provide evidence of their attention to multiplicative relationships specified by rates of change. To support students' consideration of underlying mathematical structure, educators could utilize mathematical tasks that foster students' integration of numerical and nonnumerical reasoning related to rate of change.

This study has implications for future research in the area of covariational reasoning. This paper addressed reasoning involved when a student considered variation in the intensity of extensive quantities (e.g., a *decrease in increases*, such that "decrease" denotes the variation in the intensity of the extensive quantity, an "increase") indicating relationships between covarying quantities (e.g, temperature and time). However, in each of these task settings, a student could have considered variation in the intensity of an intensive quantity (e.g., rate of change conceived of in a way compatible with Thompson's characterization in section 2.1.1). It seems productive to investigate students' reasoning about intensive quantities indicating relationships between covarying quantities (which may be intensive or extensive).

Systematically varying the quantity that serves as the independent variable and simultaneously considering variation of the intensity of change in a related quantity shares characteristics with what Carlson and colleagues (2002) referred to as "coordinating the average rate of change of the function with uniform changes in the input variable" (p. 358). Results of this study suggest that the systematic variation could be uniform, and the variation of the intensity of the change could be represented by an average rate of change. Further examination of the operations involved when students combine covariational and transformational reasoning could inform the expansion of the covariational reasoning framework articulated by Carlson and colleagues (2002).

7. Limitations

In closing, I make a few comments related to the nature of the tasks used to investigate students' use of covariational and transformational reasoning. It seems possible that dynamic sketches could provide structure to support students' continuous transformational and "smooth" covariational reasoning. The dragging feature of Dynamic Geometry Environments, such as Geometer's Sketchpad, affords the seemingly continuous stretching of one object to create an infinite amount of new objects sharing some properties of the first object (Goldenberg, Scher, & Feurzeig, 2008). By design, only squares could be created through clicking and dragging in the dynamic sketch for the changing square task. The design could have imposed constraints that afforded particular transformational operations. Further, the dynamic sketch for the typical high temperature task imposed a continuous structure on a discrete situation. Such structure might support continuous ways of reasoning by students who might otherwise have engaged in discrete ways of reasoning. While this study suggests how a student could coordinate covariational and transformational reasoning, it does not provide insight into how students might develop such forms of reasoning.

This is a post-peer-review, pre-copyedit version of an article published in The Journal of Mathematical Behavior. The final authenticated version is available online at: doi: 10.1016/j.jmathb.2012.01.001
Cite as:
Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior*, *31*(3), 313-330.

Acknowledgements

This research was completed to fulfill the dissertation requirement for a doctoral degree at The Pennsylvania State University under the advisement of Rose Mary Zbiek. The author would like to thank Evan McClintock, Ron Tzur, Rose Mary Zbiek, and anonymous reviewers for their feedback on prior drafts of this paper. This paper is supported in part by the National Science Foundation under Grant ESI-0426253 for the Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL). Any opinions, findings, or conclusions expressed in this document are those of the author and do not necessarily reflect the views of the National Science Foundation.

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