CONCEPTION OF NUMBER AS A COMPOSITE UNIT PREDICTS STUDENTS' MULTIPLICATIVE REASONING: QUANTITATIVE CORROBORATION OF STEFFE'S MODEL

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This study[1] provides statistical analysis that corroborates a prediction implied by Les Steffe's model: the strength of children's conception of number as a composite unit predicts their ability to reason multiplicatively. In individual clinical interviews, 33 fourth graders (age ~10) correctly solved a 1-digit addition word problem (8+7). Students spontaneously used one of three strategies: counting-on, doubling, or break-apart-make-ten (BAMT). Our statistical analysis revealed that students' spontaneous use of BAMT largely predicted their ability to reason multiplicatively, counting-on predicted poor ability, and doubling fell in between. We discuss implications of these findings for research and practice.

INTRODUCTION

We examine how a central element of Steffe's (1992) model of children's mathematical thinking—children's conception of number as a composite unit—might help predict their extant ability to engage in multiplicative reasoning (MR). Like Ulrich (2015, 2016), our study addresses Lamon's (2007) call for research linking students' additive and multiplicative structures. It sheds light on a novel aspect of this link—the vital role students' conception of number may play in developing more advanced concepts, such as fraction and ratio (Hackenberg, 2013).

We argue that a child's spontaneous use of an additive strategy (counting-on, doubling, BAMT) to solve a 1-digit addition word problem indicates the strength of the child's conception of number. Our focus is not on students' *potential to learn* to reason multiplicatively. Rather, we aim to predict students' current ability to use multiplicative reasoning based on the additive strategy they spontaneously use. We follow Kilpatrick's (2001) assertion of the need for statistical corroboration of the predictive power bestowed by conceptually sound models. Drawing on constructivist theory, researchers have used qualitative methods to develop conceptual models of students' additive and multiplicative reasoning. However, little work has been done in the field to test and corroborate such qualitative models. Our study follows Norton and Wilkins' (2009) lead, by providing new quantitative analysis to corroborate a central element of Steffe's (1992) model of children's mathematical thinking.

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CONCEPTUAL FRAMEWORK

The core of the constructivist framework for this study is the depiction of children's conception of number as an abstract, symbolized composite unit (Steffe, 1992). This conception allows a child to operate on a numerical symbol as a single "thing" and to decompose it into sub-parts. When reasoning additively, a child can mentally coordinate the *same type of unit* (e.g., 8 grapes + 7 grapes = 15 grapes). In contrast, when reasoning multiplicatively, a child can simultaneously coordinate *different levels of units*: items in each composite unit (1s), number of composite units, and a total number of 1s (Ulrich, 2015). For example, consider the problem: "Sarah wants to put 8 grapes into each of 7 baskets. How many grapes does she need?" A child reasoning multiplicatively can distribute items of one composite unit *(grapes per basket)* over another composite unit (*baskets*) to find the total number of items (1s) in a collection of composite units (*total of grapes*). Such a coordination requires conceiving of number as composite unit (Steffe, 1992; Ulrich, 2016).

We sharply distinguish a child's solution to a problem from conceptions that underlie her solution (Tzur et al., 2013; Ulrich, 2016). A student may correctly solve a 1-digit addition problem (e.g., 8+7) by spontaneously using additive strategies such as: counting-all (1, 2, …, 14, 15), counting on (8; 9, …, 14, 15), doubling (7+7=14; 14+1=15), BAMT (8+2=10; 10+5=15), or fact retrieval. If we focus on the correct solution, any of these strategies would suffice. Instead, we contrast them based on the strength of a child's conception of number that we infer to underlie each strategy.

Steffe (1992) used a criterion of number as a composite unit to claim counting-all does not indicate a conception of number. We also claim that counting-on indicates a weak conception; doubling an intermediate conception; and BAMT a strong conception. In each of those latter three strategies, a child could conceive of one addend as a composite unit. Yet, a child using counting-on does not decompose numbers into sub-parts other than 1s. Rather, she accrues, one after another, units of 1 that constitute the second addend. In doubling, the child could decompose one addend to create easy fact retrieval (8 is 7+1), then add two composite units in their entirety (7+7=14), and finally 'call-back' the decomposed 1 (14+1=15). In BAMT the child could both *decompose* one addend into units larger than 1 (7 is 2+5) and *integrate* them into another unit (8+2=10) as a means to add two composite units in their entirety (10+5=15). A child's use of decomposition indicates that she can operate on a number as a unit in and of itself, without constantly reconstituting it from 1s. Because a child using BAMT uses decomposition into and integration of sub-parts that are themselves composite units, we argue that spontaneous use of BAMT indicates a stronger conception of number than does doubling.

METHODS

This study was part of a larger project focusing on promoting and studying upper elementary teachers' shift toward a student-adaptive pedagogy (AdPed), and how such a shift impacts students' learning and outcomes. To this end, we developed and validated a written measure for assessing students' multiplicative reasoning (MR) (Hodkowski et al., 2016). The measure contains five word problems: one screener (1-digit addition) and four problems through which we intended to measure students' multiplicative reasoning. Our team includes language experts who helped design word problems appropriate for students learning English as an additional language.

In Problem #1 (screener), we intended for students to spontaneously use a strategy to add two 1-digit numbers (8+7). In Problem #2, we intended for students to iterate a composite unit (e.g., a tower of 5 cubes) to determine if it could constitute a larger composite unit (e.g., a tower of 24 cubes). In Problem #3 (MR), we intended for students to distribute items of one composite unit (3 cubes per tower) over another (6 towers) to find the total number of items in a collection of composite units (total cubes). In Problem #4, we intended for students to keep track of composite units (4 teams of 5 players each). We asked them to determine the correctness of a hypothetical student's (Joy) statement that, through 'skip-counting' by 5, she found there are 35 players in all. In Problem #5 (MR), given a total number of items (28 cookies), we intended for students to iterate one composite unit (4 cookies per bag) to determine the total number of composite units (bags) needed. In each of the MR word problems (#2-5), we included sub-questions that required students to fill in blanks with key, given information. For example, in Problem #4, students had to fill a given in the blank: "In each team there are ___ players." We included these sub-questions, in part, to assess students' comprehension of problem statements.

Our initial analysis of the interviews revealed a novel correlation that extended beyond our initial design: the spontaneous additive strategy students used to solve a 1-digit addition word problem (8+7) seemed linked with their score on the MR measure. Thus, we designed a follow-up, quantitative study (reported here) to collect and analyze data to examine this novel correlation.

Setting and Participants.

Participants were 4th graders (age ~10) at an elementary school in a large urban school district in the western USA. A total of 43 students—roughly 50% of all 4th graders in that school—completed the MR measure during an individual, clinical interview conducted by the first author. We excluded ten students from our analysis: 3 who used counting-all, 4 with no consent, and 3 who incorrectly responded to the sub-questions assessing students' comprehension of the word problems. The study sample thus consisted of 33 students (15 girls), all mainstreamed for math instruction. Most (85%) participants identified as non-white, including 17 (52%) Latino/a and 11 (33%) African-American students. Of the 33 participants, 45% were designated as English Language Learners, and three had Individual Educational Programs (IEPs).

We have established three important commonalities for this sample. First, all 33 students correctly solved Problem #1, using either counting-on, doubling, or BAMT. Second, 32 of them correctly responded to the sub-questions (fill givens in the blanks) intended to assess their comprehension of the problem statements. Third, the actions and time lapse between each student's reading and answering Problem #1 (from 4 to 30 seconds) indicated none solved 8+7 through fact retrieval.

Data Collection and Analysis.

The first author administered the MR measure to individual students, during clinical interviews lasting about 30 minutes. Each problem was first read out loud by the student or the interviewer. The student then solved the problem on her or his own, without assistance. After a student finished solving a problem, the interviewer asked follow up questions to gather further evidence of students' thinking. As each student solved Problem #1, to increase the likelihood of accurate inference of the additive strategy each student used, during the interview, he made notes of the child's actions and utterances—and the inferred additive strategy.

Students used two forms of doubling: (a) 7+7=14; 14+1=15 and (b) 8+8=16; 16-1=15. No statistically significant difference could be found between those two sub-groups. We thus combined them into a single category (doubling). We classified three ordinal levels of the independent variable: 1=incipient/weak (counting-on), 2=developing/intermediate (doubling), and 3=developed/strong (BAMT) conception of composite unit. We then conducted three tests. First, we used ANOVA to test whether means in solutions to MR problems (Problems #2-5) were significantly different for those three groups. Next, we used Kendall's Tau-b, a test of correlation that does not assume normal distribution or equal interval scaling. Finally, we used a t-test to compare between every pair of groups, supposing (based on the ANOVA) the comparison between counting-on and BAMT is the imperative one.

RESULTS

In this section, we present data analysis to substantiate our claim that the strength of a child's concept of number as a composite unit, inferred from her or his spontaneously used additive strategy (independent variable), can help predict the child's ability to reason multiplicatively (dependent variable). We begin with statistical analysis of all participants (N=33), followed by between-group differences.

Multiplicative Reasoning – All Participants.

In Table 1 we provide percentages of students who correctly solved each MR problem. We observe two important results. <u>First,</u> despite all 33 students solving 100% of Problem #1 correctly, they collectively solved less than 40% of each MR problem correctly. We interpret students' success on Problem #1 to indicate their ability to use additive reasoning and their difficulty with Problems #2-5 (MR) to indicate their lack of multiplicative reasoning. This contrast between additive and multiplicative reasoning lends support to researchers' theoretical predictions of a conceptual leap involved in shifting from additive to multiplicative reasoning (Hackenberg, 2013; Ulrich, 2015).

MR Problem	2	3	4	5
Percentage of correct solutions	33%	18%	36%	39%

Table 1: Percentages of students who correctly solved each MR problem.

Second, students' success rate was lowest (18%) in solving Problem #3. A paired-samples t-test comparing all students' solutions to Problem #3 and to the three other multiplicative problems shows non-significant difference with Problem #2, a statistically significant difference with Problem #4 (t=2.25, df=32, p=.032), and nearly statistically significant with Problem #5 (t=1.88, df=32, p=.07).

Between-Group Differences in Multiplicative Reasoning (MR).

In Table 2 we show percentages of students who correctly solved each MR problem, disaggregating the data by the spontaneous additive strategy students used to solve Problem #1. We found statistically significant differences among students when disaggregating by their spontaneously used additive strategy. The percentages of students who solved all MR problems correctly were highest for BAMT (56%), midway for doubling (34%), and lowest for counting-on (17%). ANOVA shows these differences are statistically significant (F=8.25, p=.001). Further t-tests on success rates for the four MR problems showed nearly statistically significant differences between counting-on and doubling (t=2.04, df=22, p=.053) and highly significant between counting-on and BAMT (t=4.29, df=23, p<.0005), but *not between doubling and BAMT*, possibly due to the smaller n of these two groups.

MR Problem	2	3	4	5	Across all 4 MR Problems
Counting on	13%	6%	19%	31%	17%
Doubling	50%	13%	38%	38%	34%
BAMT	56%	44%	67%	56%	56%

Table 2: Percentages disaggregated by students' spontaneous additive strategy.

A Kendall's Tau-b (KTb) test of correlation further demonstrates the linkage between students' additive strategy and the success rate on problems involving multiplicative reasoning (KTb= 0.5, p=.001). Data in Table 3 further highlight this: a child's spontaneous use of counting-on predicts a very low success rate on MR problems. Of students using counting-on, 94% correctly solved *at most* one problem. In contrast,

No. of MR problems Solved Correctly	0	1	2	3	4 (All)
Counting-on (N=16)	37.5%	56.3%	6.2%	-	-
Doubling (N=8)	12.5%	62.5%	-	25.0%	-
BAMT (N=9)	11.1%	11.1%	33.3%	33.3%	11.1%

Table 3: Percentages of students who correctly solved 0, 1, 2, 3, or 4 MR problems.

a child's spontaneous use of BAMT predicts a much higher success rate on MR problems: 78% of students spontaneously using BAMT correctly solved *at least two* MR problems (33.3%, 33.3%, 11.1%)

To examine the impact of spontaneous additive strategy on student success rate for each MR problem separately, we conducted ANOVA for between-group differences. Table 4 presents these results. Group contribution to this variance, calculated using S-N-K post-hoc statistics and t-tests (Table 5), showed statistically significant differences between counting-on and doubling (Problem #2), and between counting-on and BAMT (Problems #2, #3, and #4).

Problem No.	2	3	4	5
ANOVA	F=3.42 (.046)	F=3.25 (.053)	F=3.146 (.006)	-

Table 4: ANOVA of between-group differences for each MR problem.

Problem No.	2	3	4	5
Count-on vs. Doubling	t=2.1 (.048)	-	-	-
Count-on vs. BAMT	t=2.49 (.021)	t=2.47 (.021)	t=2.61 (.015)	-

Table 5: Independent samples t-test values of between group-pairs differences on each problem (equal variance **not** assumed; p-values in parentheses).

Results presented in Tables 4 and 5 indicate two main points that, combined, support our claim that the strength of a child's conception of number as composite unit holds predictive power for her current ability to reason multiplicatively. First, we focus on responses to Problem #4, on which students who used counting-on were most successful. To solve Problem #4 correctly, students needed to (a) determine that Joy's response is wrong (in 4 teams of 5 players each, there are not 35 players), (b) select an appropriate reason for Joy's mistake, and (c) figure out the correct number of *teams* that Joy counted (35 players would make 7 teams of 5 players each). Only three students (19%) who used counting-on could solve this problem correctly, seemingly by their ability to skip-count by 5s to arrive at 20. The other thirteen (81%) students were unsuccessful. Among those thirteen, ten students (63%) incorrectly selected "35" as the number of teams that Joy counted. We interpret the students' error to provide empirical evidence to support Ulrich's (2015) claim that such students rely on operating on 1s—a reliance that may *often be masked* by their successful performance when iterating familiar numbers, such as 5.

Students' performance on Problems #2, #3, #4 provides further support of Steffe's (1992) model. In each of these problems, a child would have to carry out the simultaneous, coordinated monitoring of the accrual of *both* 1s and composite units. Among the 16 students who spontaneously used counting-on to solve Problem #1, only two (13%) could solve Problem #2, only one (6%) could solve Problem #3, and only three (19%) could solve Problem #4. These data corroborate the prediction that students with weak composite unit—solving addition tasks by adding 1s—are unlikely to reason multiplicatively. In contrast, among the nine students who spontaneously used BAMT, five (56%) could solve Problem #2, four (44%) could solve Problem #3, and six (67%) could solve Problem #4. These data corroborate the prediction that students with strong composite unit—solving addition tasks by decomposing the second addend—are more likely to reason multiplicatively.

DISCUSSION

We examined how an element of Steffe's (1992) model—children's conception of number—might help predict their ability to reason multiplicatively. We provided analysis of the conceptual foundations of students' spontaneous use of three additive strategies—counting-on, doubling, and BAMT. Importantly, students' use of any of these strategies provides evidence that they have constructed a conception of number. Our study corroborated Steffe's model: the strength of a child's conception of number, as evidenced by their spontaneous use of an additive strategy, can help to predict their extant ability to engage in multiplicative reasoning. Specifically, a child who spontaneously uses counting-on is highly unlikely to engage in multiplicative reasoning. In contrast, a child who spontaneously uses BAMT is likely to do so. Keeping with Kilpatrick's (2001) assertion, our study thus contributes to the field's knowledge base by testing a long-known and well-articulated conceptual model that links, developmentally, students' additive and multiplicative reasoning.

Implications for Research.

We note three implications of this study. First, it opened the way for identifying 1-2 tasks that can indicate a child's likelihood for advanced ways of reasoning based on observable, lower-level solutions. A future, larger N study may confirm the predictive power of a child's additive strategy. Second, a related measure to the one we used could be developed to examine the linkage between a child's comprehension of a realistic word problem and her or his conception of number and/or multiplicative reasoning. Third, this study implies the need to carefully examine the design and findings of studies intended to determine the impact of an instructional intervention on student learning and outcomes. Lack of impact of such interventions may be rooted not in the intervention per se (Woodward & Tzur, in press), but in students' lack of a cognitive prerequisite that affords the intended learning (e.g., lack of strong enough composite unit; see Tzur, Xin, Si, Kenney, & Guebert, 2010).

Implications for Practice.

For practice, our study implies the possibility to use a quick measure (screener Problem #1) to assess the strength of each student's conception of number. In our current project, *teachers* are learning to use it so they: (a) link between a child's additive strategy and her MR, (b) can conduct short, task-based interviews to elicit students' strategies, (c) document the results of their assessments, and (d) *adapt their subsequent instruction to meet the needs of students in each group*. Teachers with whom we work seem to deeply appreciate the main goal for each student who uses counting-on and doubling: strengthen her or his conception of number as composite unit by learning to decompose addends into sub-composite units.

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