When a critical aspect is a conception: Using multiple theories to design dynamic computer environments and tasks to promote secondary students’ discernment of covariation

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We explicate how we used different theories of learning to design dynamic computer environments and tasks to promote secondary students’ discernment of covariation—central to students’ study of fundamental mathematical ideas such as rate and function. Using Marton’s variation theory, we designed task sequences to foster students’ discernment of the critical aspect of covariation. Using Piaget’s constructivist theory, we defined the critical aspect, covariation, in terms of students’ conceptions of a relationship between attributes whose measures vary. Using Thompson’s theory of quantitative reasoning, by quantities we mean attributes of objects that students can conceive of as being possible to measure. We provide data to demonstrate how a student’s discernment of covariation advanced during her work on a task sequence. We discuss implications for the design of dynamic computer environments and tasks focused on the mathematics of change and variation.

Keywords: Secondary school mathematics, Instructional design, Computer simulation, Learning theories.

By drawing on more than one theory of learning, researchers can combine tools and lenses to investigate complex phenomena (e.g., Cobb, 2007; Sfard, 1998; Simon, 2009). Cobb (2007) recommended that researchers “act as bricoleurs by adapting ideas from a range of theoretical sources” (p. 29). Sfard (1998) argued that researchers should not assume that theoretical “patches of coherence” somehow would combine to form a single, unifying theory of learning. Yet, using multiple theories can pose challenges, particularly if researchers view theories as competing, rather than complementary (Simon, 2009). To address challenges, it is useful for researchers to take into account the grain sizes of different theories (Kieran, Doorman, & Ohtani, 2015; Watson, 2016).

By distinguishing between the grain sizes of theories, researchers can more effectively interpret and use theory for task design purposes (Kieran et al., 2015; Watson, 2016). Broadly, grain sizes include grand theories (e.g., Piaget’s constructivist theory), intermediate theories (e.g., Marton’s variation theory), and domain specific/local theories (e.g., Thompson’s theory of quantitative reasoning). Furthermore, it is useful for researchers to acknowledge interrelationships between theories of different grain sizes. Theories of smaller grain size depend upon or address particular aspects of theories of larger grain size (Watson, 2016). For example, Thompson’s theory of quantitative reasoning depends upon Piaget’s constructivist theory to define quantities in terms of students’ conceptions. By drawing on theories of different grain sizes, researchers can adapt and interpret

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grand theories for task design and implementation (e.g., Cobb, 2007; Kieran et al., 2015; Thompson, 2002; Watson, 2016).

When researchers engage in task design, they should make explicit how their theory choice informs their task design (Watson, 2016). Using Marton's variation theory (2015), we designed task sequences to engineer opportunities for students to discern critical aspects central to fundamental mathematical ideas, such as rate and function. We posit that one such critical aspect is covariation. Using Piaget's constructivist theory (1985), we define covariation in terms of individuals' conceptions. Using Thompson's theory of quantitative reasoning (1994, 2002), we articulate the conception; by covariation we mean a conception of a relationship between attributes whose measures vary. Students' conceptions of covariation impact their understanding and use of function (Thompson & Carlson, 2017).

We build on the work of researchers who have designed dynamic computer environments and tasks to foster students' study of the mathematics of change and variation (e.g., Kaput & Roschelle, 1999; Saldanha & Thompson, 1998; Thompson, 2002). In this paper, we explicate how we used theories of different grain sizes to design dynamic computer environments and tasks to promote secondary students' discernment of covariation. To avoid remaining only in the abstract, we provide data to demonstrate the utility of our approach in fostering students' discernment of covariation.

Theoretical and conceptual framework

We use Piaget's constructivist theory to orient our research. We foreground students' conceptions to explain how researchers might design dynamic computer environments and tasks to foster students' development of difficult to learn mathematical ideas such as function and rate. We focus on students' mental operations, which refer to actions that individuals can enact in thought or in the physical world (Piaget, 1985).

We use Marton's variation theory to guide the design of our dynamic computer environments and related task sequences. Broadly, Marton (2015) argued that instructional designers should develop task sequences that provide students opportunities to discern critical aspects. The task sequence should involve patterns of variation, then invariance in the critical aspects (Marton, 2015). We draw on Piaget's constructivist theory to orient our interpretation of the critical aspect that we intend for students to discern. By critical aspect, we mean a conception. The critical aspect—covariation—refers to a conception of a relationship between attributes whose measures vary.

When using variation theory it is important for instructional designers to determine if the critical aspect is comprised of a single aspect or of interrelated aspects (Marton, 2015). For example, suppose a designer intends for students to discern the color blue, which students could conceive of as being comprised of a single aspect. A task sequence should begin with variation in color and invariance in some unrelated feature (e.g., blue ball, green ball, red ball), then invariance in color and variation in the unrelated feature (e.g., blue ball, blue block, blue cone), and then variation in both. In contrast, if a designer intends for students to discern the *depth of blue color*, students would

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need to conceive of interrelated aspects (depth, color) that comprise the critical aspect. In this case, the task sequence should include variation and invariance in each interrelated aspect (e.g., different colors of the same depth, then different depths of the color blue), then move to variation in both aspects. A conception of covariation necessitates a conception of a relationship between interrelated aspects (attributes whose measures vary). For example, in a situation involving the varying height and distance of a car in a turning Ferris wheel, the height and distance are the interrelated aspects, and students' conceptions of a relationship between measures of height and distance (covariation) is the critical aspect.

We found Marton's variation theory and Piaget's constructivist theory to complement each other for the purposes of our task design. From a constructivist perspective, we do not assume that a relationship between attributes whose measures vary (covariation) is something that is "out there" for students to perceive. Marton (2015) argued that researchers should not assume that students already attend to the critical aspect prior to encountering a task sequence; therefore, task sequences should include variation in critical aspects (contrast) prior to variation in noncritical aspects. To foster students' discernment of covariation, we incorporated variation in the types of interrelated aspects (height, width, distance) prior to variation in the representation of those aspects.

We use Thompson's theory of quantitative reasoning to explain what we mean by the attributes whose measures vary—the interrelated aspects comprising the critical aspect of covariation. Drawing on Piaget's constructivist theory, Thompson (1994) defined quantities in terms of individuals' *conceptions* of attributes of objects. Therefore, quantities are not "things" that exist in the physical world. Following Thompson (1994), we claim that an individual conceives of some attribute as a *quantity*, if the individual can conceive of the possibility of measuring that attribute. For example, we would claim that a student conceived of "height" as a quantity if the student provided evidence of envisioning the possibility of measuring the height of some object.

We selected Thompson's theory of quantitative reasoning because we found it to be useful for interpreting Piaget's constructivist theory. Accordingly, the mental operations on which we focus are quantitative operations, which involve actions on attributes that students can conceive of as measurable, or in other words, actions on quantities (Thompson, 1994). Specifically, we focus on students' conceptions of covariation, which entail the quantitative operations involved in forming and interpreting relationships between attributes whose measures vary. For example, a student conceiving of covariation could form and interpret relationships between the varying measures of height and distance for a Ferris wheel car traveling around one revolution of a Ferris wheel.

The diagram in Figure 1 illustrates how we used different theories to inform our task design.

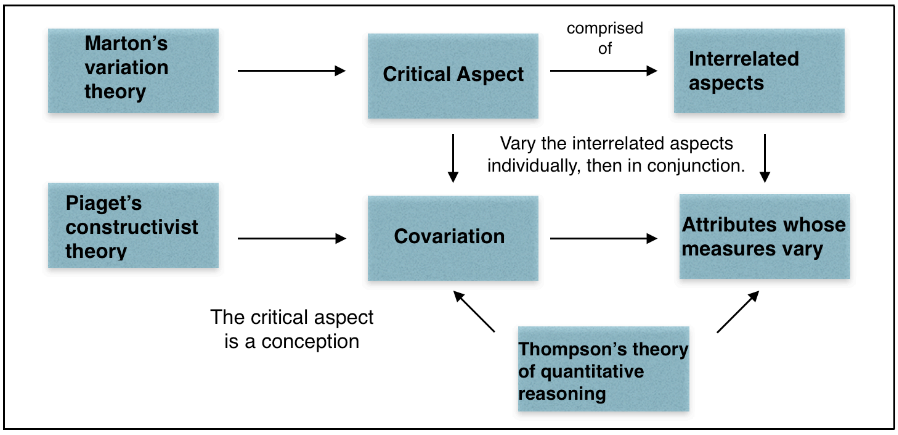

Figure 1: Relationships between the different theories we used to inform our task design

We placed Marton's variation theory at the top to foreground our intention to design a task sequence to foster students' discernment of a critical aspect comprised of interrelated aspects. We placed Piaget's constructivist theory in the center to communicate how we used this grand theory to define the critical aspect, covariation, in terms of individuals' conceptions. We placed Thompson's theory of quantitative reasoning at the base to show how we used this local theory to explain what we mean by covariation—a conception of a relationship between attributes whose measures vary.

Ferris wheel dynamic computer environments

Using Geometer's Sketchpad software, Johnson developed two dynamic computer environments for use with the task sequence. The environments consisted of a Ferris wheel animation and linked graph, each of which students could control separately or in conjunction. The environments related either the *height* of a Ferris wheel car from the ground or the *width* of the car from the center to the *distance* traveled around one revolution of the wheel (Figure 2 shows height and distance). See Johnson (2015) for more details about the environments.

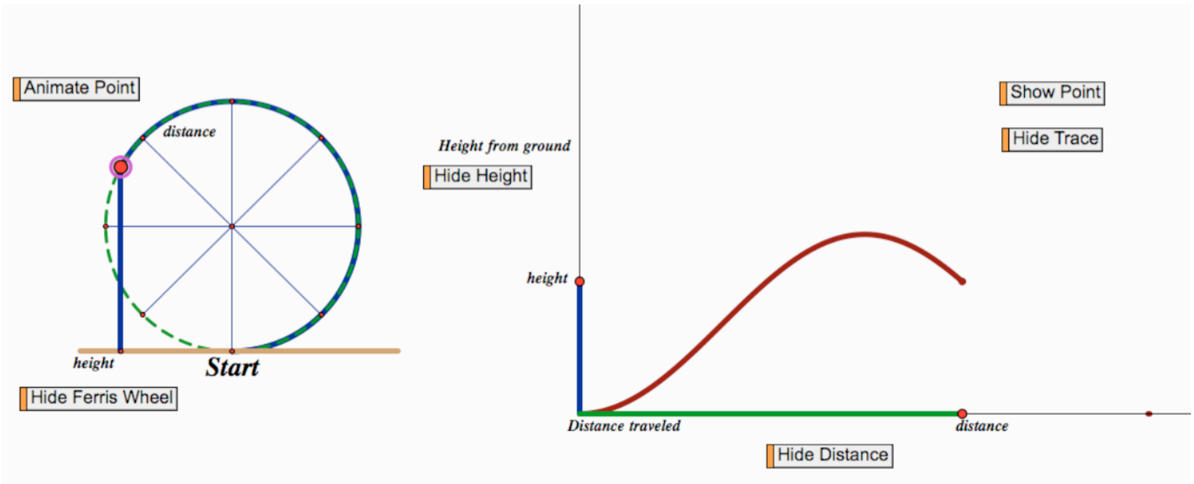

Figure 2: Ferris wheel dynamic computer environment, distance and height

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The Ferris wheel environments contained three affordances particularly relevant to our use of variation theory. First, students could vary each of the interrelated aspects (e.g., height and distance) individually by dragging or animating the dynamic segments on the vertical and horizontal axes. Second, the environments included different interrelated aspects—height and distance (Figure 2), and width and distance (not shown). Third, the environments included variation in the axes used to represent the interrelated aspects on the Cartesian plane (e.g., height and distance represented on the vertical and horizontal axes [Figure 2], then horizontal and vertical axes, respectively [not shown]).

Our use of Piaget's constructivist learning theory and Thompson's theory of quantitative reasoning informed our choices about the types of quantities to include on each of the axes. Specifically, we included quantities measurable with linear units, because it is less difficult for students to conceive of using linear units to measure quantities (see also Piaget, 1970). Furthermore, Thompson (2002) recommended students use their fingers as tools to represent change in individual quantities. In the Ferris wheel environments, students could use either their fingers or the dynamic segments on the vertical and horizontal axes to represent change in individual quantities.

The Ferris wheel task sequence

Purpose and setting

We view tasks as problems designed for particular audiences and settings (see Sierpinska, 2004) Johnson designed the Ferris wheel task sequence to provide students opportunities to discern covariation. In a small neighborhood school in an industrial region of a large U.S. city, Johnson conducted a series of small group interviews with five ninth grade students (~15 years old), enrolled in an introductory algebra course. Interviews occurred approximately once per week. During the interviews, students completed the Ferris wheel task sequence (see Table 1). Johnson designed the task sequence for a small group interview setting; however, teacher/researchers could adapt the tasks for use in different settings (see Johnson, Hornbein, & Azeem, 2016).

Variation and invariance in the Ferris wheel task sequence

To foster students' discernment of critical aspects comprised of interrelated aspects, Marton (2015) recommended that instructional designers begin with task sequences containing variation in individual interrelated aspects, then variation in the both interrelated aspects, against a background of invariance. In the Ferris wheel task sequence, we intended for the situation of a turning Ferris wheel to provide a background of invariance. Furthermore, Marton (2015) recommended variation in features (or dimensions) of those interrelated aspects. We provided two types of variation in features: the type of interrelated aspects (width, height, or distance), and the representation of each aspect on the Cartesian plane (horizontal or vertical axis). Table 1 shows the Ferris wheel task sequence, including variation in interrelated aspects and representations on the Cartesian plane.

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Task	Interrelated aspects	Representation on axes on Cartesian Plane
1	Height, Distance	Distance – horizontal, Height – vertical
2	Width, Distance	Distance – horizontal, Width– vertical
3	Height, Distance	Height – horizontal, Distance – vertical
4	Width, Distance	Width – horizontal, Distance – vertical

Table 1: Ferris wheel task sequence: Variation in interrelated aspects and representation

Covariation and quantity in the Ferris wheel task sequence

When the critical aspect is a mental operation, instructional designers should provide students opportunities to engage in activities in thought as well as action. Each task in the Ferris wheel task sequence contained five parts: (1) Explain what the interrelated aspects measure in the Ferris wheel situation; (2) Sketch a graph relating both aspects; (3) Use dynamic segments to represent change in individual aspects (e.g., height or distance); (4) Predict a car's location on the Ferris wheel given only dynamic segments representing the changing individual aspects; and (5) Compare the computer generated graph to the sketch in (2). Through each of the five part tasks, Johnson provided students multiple opportunities to discuss and represent their thinking about how the interrelated aspects (height and distance or width and distance) were changing individually and together. For example, students sketched a graph relating the interrelated aspects prior to viewing any facets of the dynamic graph. Furthermore, Johnson provided students opportunities to discuss and show the possibility of measuring interrelated aspects of the Ferris wheel situation (e.g., "height" represents the vertical distance from the car to the base of the Ferris wheel, see Figure 2).

A case of a student's discernment of covariation

We use the work of one student—Ana—to demonstrate the promise of this design approach for fostering students' discernment of covariation. Ana's work demonstrates the range of reasoning of all five students who completed the Ferris wheel task sequence. Building from Ana's work, we present a case of a student's discernment of covariation.

We share data from Ana's work for Part 2 of Tasks 1 and 3: *Prior to viewing the dynamic graph, sketch a graph relating both aspects*. We selected data from Part 2 to illustrate how Ana's sketches changed prior to viewing aspects of the dynamic graph. We attribute the changes in her sketches to changes in her conceptions of a relationship between varying measures of height and distance.

Figure 3 shows Ana's written work for Part 2 of Tasks 1 and 3. For Part 2 of Task 1 (left), Ana drew the curved graph, labeling it "height," and then drew the line graph, labeling it "distance" (Figure 3, left). When asked what her labels meant, Ana stated: "This (points to the curved graph) would be the graph shape if we were dealing with the height, and this (points to the line graph) would be the shape if we were dealing with the distance." For Part 2 of Task 3, using one continual motion, Ana sketched a single graph (Figure 3, right). When asked to explain her thinking, she stated that the "distance keeps on going," but the height will reach "a certain amount," and then "it goes back down." To illustrate, she drew an arrow along the left of the distance axis. Next, she drew a small, darkened segment on the graph, and two arrows extending along the graph.

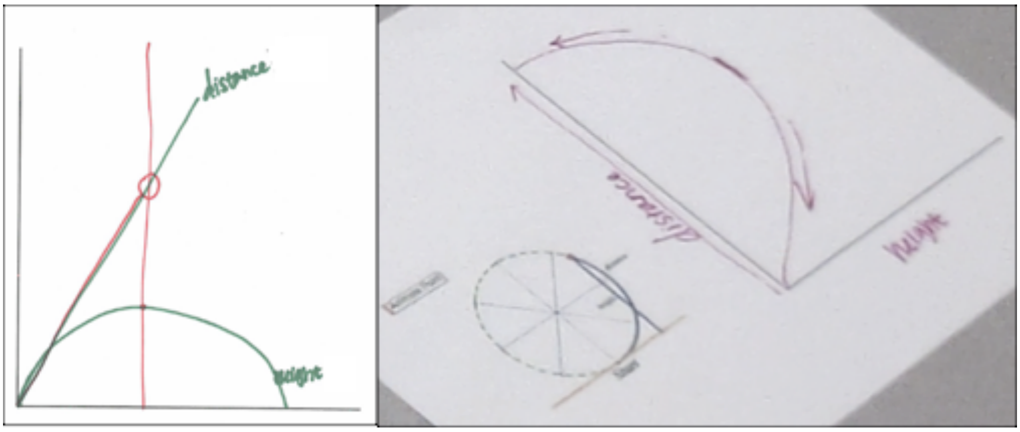

Figure 3: Ana's graphs in Part 2 of Task 1 (left) and Part 2 of Task 3 (right)

For Part 2 of Task 1, we interpret that Ana represented individual variation occurring in the measures of height and distance. Not only did she sketch two graphs, she labeled the actual sketches, rather than the axes. We use Ana's work for Part 2 of Task 1 to demonstrate that Ana did not enter the Ferris wheel task sequence already conceiving of a relationship between the varying measures of height and distance, or in other words, conceiving of covariation. Moving forward to Part 2 of Task 3, Ana used a single graph to represent a relationship between the varying measures of height and distance. Not only did she sketch a single graph, she annotated the graph to show how the single graph represented variation in the measures of both height and distance. Therefore, we claim that Ana demonstrated discernment of covariation during her work for Part 2 of Task 3 (conceived of a relationship between attributes whose measures vary). Furthermore, Ana's discernment of covariation was not limited to the aspects of height and distance. She also demonstrated discernment of covariation when working with width and distance in Tasks 2 and 4.

Discussion/Implications

When critical aspects involve interrelated aspects, Marton (2015) recommended that instructional designers develop task sequences that include different backgrounds. In our task sequence, we used only a Ferris wheel situation, and we recommend that researchers designing task sequences to foster students' discernment of covariation also include different situations. However, we provide our recommendation with a caveat—the different situations should include interrelated aspects measurable with linear units. For example, if we were to design a task sequence for a filling bottle situation, we might ask students to relate the height of the liquid in the bottle to the diameter of the liquid in the bottle. Our caveat stems from our use of Piaget's constructivist theoretical perspective.

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It is less difficult for students to conceive of the possibility of using linear units to measure attributes (e.g., Piaget, 1970). Therefore, we recommend that task sequences designed for students to discern covariation (a critical aspect involving interrelated aspects) should include interrelated aspects measurable with linear units. Researchers have shown that even successful university students have difficulty using graphs to represent relationships between height and volume in a filling bottle situation (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). If students have difficulty conceiving of the possibility of using a three-dimensional unit to measure volume, it may impact their discernment of covariation for situations involving such attributes.

By using theories of different grain sizes, we were able to employ multiple, compatible lenses to engage in task design that looked both *across* and *within* the sequence of tasks. By guiding our variance and invariance of interrelated aspects, Marton's variation theory informed design *across* the task sequence. By fostering our choices about the kinds of aspects to vary, Thompson's theory of quantitative reasoning informed our design *within* tasks in the sequence. The ability to view a task sequence from different perspectives—in our research, looking both across and within—is a productive result emerging from the use of multiple theories to do compatible explanatory work to augment the design of task sequences intended to foster students' discernment of critical aspects.

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