Johnson, H. L. (2015) Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64-90.

TITLE

Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities

AUTHOR

Heather Lynn Johnson

ABSTRACT

Contributing to a growing body of research addressing secondary students' quantitative and covariational reasoning, the multiple case study reported in this article investigated secondary students' quantification of ratio and rate. This article reports results from a study investigating students' quantification of rate and ratio as relationships between quantities and presents the *Change in Covarying Quantities Framework*, which builds from Carlson et al.'s (2002) Covariation Framework. Each of the students in this study was consistent in terms of the quantitative operation he or she used (comparison or coordination) when quantifying both ratio and rate. Illustrating how students can engage in different quantitative operations when quantifying rate, the Change in Covarying Quantities Framework helps to explain why students classified as operating at a particular level of covariational reasoning appear to be using different mental actions. Implications of this research include recommendations for designing instructional tasks to foster students' quantitative and covariational reasoning.

The concept of change, including rate of change, permeates many areas of mathematical and scientific study (e.g., American Association for the Advancement of Science, 2008; National Research Council, 1996; Stewart, 1990). Despite the pervasiveness of rate of change, many secondary and university students have limited conceptions of rate of change (e.g., Bezuidenhout, 1998; Herbert & Pierce, 2012; Lobato, Ellis, & Muñoz, 2003; Stump, 2001; Ubuz, 2007; Zandieh & Knapp, 2006). A robust conception of rate of change involves understanding a multiplicative relationship between changing quantities, yet forming and interpreting relationships between quantities that change together is a nontrivial endeavor for secondary students (e.g., Confrey & Smith, 1995; Ellis, 2007, 2011; Johnson, 2012a, 2012b; Saldanha & Thompson, 1998). Further, the ways in which students form and interpret relationships between quantities constitutes a key distinction among different levels of sophistication of students' conceptions of ratio (Heinz, 2000; Simon, 2006; Simon & Blume, 1994; Simon & Placa, 2012). The research reported in this paper investigates the following questions: In what ways might students quantify rate of change as a relationship between quantities? What relationships might exist between students' quantification of ratio and rate[1]? In particular, this research explores how students' quantification of ratio might afford or constrain their quantification of rate.

The process of quantification involves conceiving of an attribute of an object, conceiving of a unit of measure for the attribute, and forming a relationship between the attribute's measure and the unit of measure (Thompson, 2011). For example, to quantify length, an individual would need to conceive of length as an attribute of some object that could be measured, conceive of a unit with which to measure length, and form a relationship between the unit of measure and the measure of length as an attribute. Quantifying rate is more complex than quantifying an attribute such as length, because a unit used to measure rate involves a relationship between constituent quantities composing a rate. For example, envision a bottle being filled with liquid. Now envision the rate at which the volume of liquid in the bottle is changing as the height of the liquid is increasing. To quantify this rate, one would need to conceive of a unit with which to measure the rate (the unit being a relationship between constituent quantities of volume and height). Finally, one would need to relate the unit of measure (a relationship between volume and height) to the attribute being measured (the rate of change of volume with respect to height). Importantly, a unit of measure for rate involves a relationship between quantities, while a unit of measure for length would involve only a single quantity.

Researchers have made distinctions between quantities such as length and rate, identifying length as an extensive quantity, because length can be directly measured and rate as an intensive quantity, because rate is not directly measurable (e.g., Schwartz, 1988). Although distinguishing between direct and indirect measurability is useful, it is not sufficient because it does not address an individual's conception of a measurable attribute of some object, which is central to quantification (cf. Thompson, 1994a). Recently, Simon and Placa (2012) defined intensive quantity in terms of quantification: "the relative size of the magnitudes (number of units) of measures from two different measure spaces, or of the measures of two different quantities from the same measure space, given particular units of measurement for each measure" (p. 39). In Simon and Placa's (2012) definition, the "relative size" indicates the measure of some attribute (e.g., chocolate flavor) that an individual could conceive of measuring. Therefore, to quantify an intensive quantity in the way described by Simon and Placa (2012), an individual would first need to conceive of some attribute that could be measured. Foregrounding conception and quantification, in this article I use intensive quantity to mean an individual's conception of a measurable attribute that could be quantified by forming a relationship between constituent quantities (from the same or different measure spaces). For example, an individual could conceive of chocolate flavor as a measurable attribute that could be quantified by forming a relationship between packets of chocolate and cups of water.

Decades of research have identified students' challenges with reasoning related to intensive quantity (e.g., Harel, Behr, Lesh, & Post, 1994; Howe, Nunes, & Bryant, 2010; Kaput & West, 1994; Lobato & Thanheiser, 2002; Nunes, Desli, & Bell, 2003; Simon & Placa, 2012). Rate and ratio are central concepts in school mathematics (e.g., National Governors Association Center for Best Practices Council of Chief State School Officers, 2010), and students have demonstrated persistent difficulty in forming and interpreting relationships between constituent quantities involved in ratio and rate (e.g., Herbert & Pierce, 2012; Lobato et al., 2003; Nunes et al., 2003; Simon & Placa, 2012). Assuming that an individual's conception of a quantity and quantification of that quantity are reflexively related (Thompson, 2011), investigating students' quantification of rate and ratio has the potential to explain why students' rate-related conceptions may remain so impoverished (e.g., Lobato et al., 2003; Ubuz, 2007).

As suggested by the preceding discussion, the process of quantifying rate and ratio involves forming and interpreting relationships between varying quantities. Therefore, covariational reasoning—"the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354)—is central to the quantification of rate and ratio. Carlson et al.'s (2002) Covariation Framework explicates five levels of covariational reasoning, with each level increasing in sophistication: Coordination, Direction, Quantitative Coordination, Average Rate and Instantaneous Rate. Engaging in covariational reasoning beyond the Quantitative

Coordination level is not trivial, even for successful mathematics students. After taking a university course focusing on rate and varying rate, students consistently reasoned at the Quantitative Coordination Level, but not beyond (Carlson et al., 2002). I argue that the ways in which students quantify rate as a relationship between quantities might explain, in part, students' lack of consistency in using covariational reasoning beyond the Quantitative Coordination Level. Building from Carlson et al.'s (2002) Covariation Framework, in this article I present the *Change in Covarying Quantities Framework*, which posits different quantitative operations involved in quantifying rate and makes distinctions among individuals' images of relationships between quantities involved in rate.

CONCEPTUAL AND THEORETICAL FRAMING

My use of image is rooted in the work of Piaget (Beth & Piaget, 1966; Piaget, 1970a, 1970b). In particular, I focus on Beth and Piaget's (1966) distinction between a "mental image" that stands in place of something, and imagery that is rooted in mental operations (cf. Thompson, 1994b; Thompson, 1996). By "mental image," Beth and Piaget referred to a mental picture representing "an attempt to imitate the object or the event previously perceived" (p. 216). By an individual's image of a quantity, I do not mean a static mental picture. Rather, I am referring to the "dynamics of mental operations" (Thompson, 1994b, p. 231), involving the envisioning of transformations beyond what one might have perceived in the physical world (Beth & Piaget, 1966).

To further explicate what might be involved in forming relationships between quantities involved in ratio or rate, I draw on Simon's (1996) construct of transformational reasoning, constituting "not just the ability to carry out a particular mental or physical enactment, but also the realization of the appropriateness of that process to a particular mathematical situation" (p.

203). For example, an individual might envision how different batches of hot chocolate might be prepared so that the strength, or *intensity*, of the chocolate flavor (that could be measured with a ratio) remained the same regardless of the size of the batch. Appealing to Nicole Oresme's definition of intensity, "that according to which something is said to be 'more such and such,' as 'more white' or 'more swift'" (Clagett, 1968, p. 167), by intensity I mean the degree to which an attribute is present (cf. Stroup, 2002). Relating intensity to intensive quantity, an intensive quantity would measure the intensity of some thing. Therefore, measuring intensity would involve being able to conceive of the measure of one object as multiplicatively related to the measure of another object (cf., Thompson, Carlson, Byerly, & Hatfield, 2014).

QUANTIFICATION AND OPERATION: RATIO & RATE

When engaging in quantification, an individual uses quantitative operations (Thompson, 1994a), such that operations are internalized mental actions that an individual could enact as thought processes or physical acts (Piaget, 1970b). For example, an individual could envision measuring an amount or actually measure an amount of flat surface covered by a rectangle. Thompson (1994a) distinguished quantitative operations from numerical operations, stating "A quantitative operation is nonnumerical; it has to do with the *comprehension* of a situation" (pp. 187-188). This means that a numerical operation (e.g., dividing 3 by 4) does not constitute a quantitative operation. Further, students using descriptors (e.g., greater, less) to indicate different degrees to which a quantity could be present could be reasoning in ways that are wholly quantitative (e.g., Johnson, 2012b; Monk & Nemirovsky, 1994). Although a student may determine numerical amounts (e.g., determining an area of a rectangle to be 5 square units), that student may or may not be engaging in quantitative operations. Products of quantitative operations are relationships (Thompson, 1994a), and those relationships can take different forms.

Researchers investigating preservice elementary teachers' conceptions of ratio have identified different conceptions of ratio that highlight distinctions among ways in which teachers form and interpret relationships between quantities involved in ratio to quantify some attribute of interest (e.g., chocolate flavor) (Heinz, 2000; Simon, 2006; Simon & Blume, 1994; Simon & Placa, 2012). A *ratio as measure* conception (Simon & Blume, 1994) involves an image of ratio as measuring the strength of an "invariant multiplicative relationship" (Simon & Placa, 2012). For example, a student with a ratio as measure conception could use 1.4 to indicate the strength of the chocolate flavor for any batch of hot chocolate such that there were 1.4 times as many chocolate packets as cups of water. In contrast, a *ratio as identical groups* conception (Heinz, 2000; Simon, 2006) and a *ratio as per-one* conception (Simon & Placa, 2012) involve an image of ratio as measuring an association of amounts of quantities. For example, a student with a ratio as identical groups conception could determine that a batch of hot chocolate with 7 chocolate packets and 5 cups of water would have the same chocolate flavor as batches with 14 chocolate packets and 10 cups of water or 1.4 chocolate packets and 1 cup of water. A student with a ratio as per-one conception could determine that any batch having 1.4 chocolate packets for every 1 cup of water would have the same chocolate flavor as a batch of hot chocolate with 7 chocolate packets and 5 cups of water.

Thompson (1994a) distinguished between two levels of conceptions of ratio, *internalized ratio* and *interiorized ratio*. At the internalized ratio level, a student could conceive of a relationship between packets of chocolate and cups of water such that there are 7 packets for every 5 cups of water (compatible with a ratio as identical groups conception). In contrast, at the interiorized ratio level, a student could conceive of varying amounts of chocolate packets and cups of water such that the relationship between the chocolate packets and cups of water maintained a constant multiplicative relationship. Students at the interiorized ratio level could have either a ratio as per-one conception (e.g., envisioning that 1.4 could represent 1.4 chocolate packets for every 1 cup of water) or a ratio as measure conception (e.g., envisioning the strength of chocolate flavor for any batch of hot chocolate to involve 1.4 times as many chocolate packets as cups of water, regardless of the amounts of chocolate packets or cups of water present). Table 1 shows students' conceptions of ratio and students' quantification of ratio associated with different levels of conceptions of ratio. Students operating at the interiorized ratio level could relate quantities involved in ratio in two distinct ways: by associating (ratio as per-one) or by coordinating (ratio as measure).

Levels of Conceptions of Ratio	***Internalized* Ratio**	***Interiorized* Ratio**	
Students' Conceptions of Ratio	Ratio as Identical Groups	Ratio as Per-One	Ratio as Measure
Students' Quantification of Ratio as a Relationship Between Quantities	An association of particular amounts of extensive quantities	An association of some amount of one extensive quantity per one unit of another extensive quantity	A coordination of quantities, such that the coordination in itself is an intensive quantity
Examples	7 packets of chocolate for every 5 cups of water	1.4, such that 1.4 represents 1.4 packets of chocolate for every 1 cup of water	1.4, such that 1.4 indicates the strength of the chocolate flavor for any batch of hot chocolate such there were 1.4 times as many packets of chocolate as cups of water

Table 1. Students' conceptions of ratio and students' quantification of ratio associated with different levels of conceptions of ratio.

Research-based definitions of rate have involved individuals' conceptions of ratio (Thompson, 1994a) and individuals' interpretation of relationships between quantities involved in rate (Kaput & West, 1994). Drawing on Piaget's construct of reflective abstraction (e.g., Piaget, 1985), Thompson (1994a) defined rate as "a reflectively abstracted constant ratio" (p. 192). In other words, conceiving of rate would entail images of a single quantity that specifies a multiplicative relationship between quantities that can vary. Kaput and West (1994) defined a *rate intensive quantity*, that "applies to a situation only when the attribute of the situation being described by that intensive quantity is possessed by that situation homogeneously" (p. 240). When an attribute is possessed homogeneously by a situation, the attribute of the situation is independent of particular amounts of quantities involved. For example, when an individual conceives of a relationship (e.g., 12 miles per hour) as homogeneous, the relationship between miles and hours does not depend on particular amounts of miles traveled or hours elapsed. Importantly, Kaput and West's (1994) definition explicates how an individual with a ratio as measure conception might conceive of rate as a homogeneous relationship between quantities.

Envisioning constant rate of change as a homogeneous relationship could support an individual's coordination of quantities involved in rate of change: "This way of thinking about constant rate of change, that corresponding changes in two quantities are homogeneous, supports thinking about continuous variation of one quantity and concomitant continuous change in the other" (Thompson, 2008, p. 39). The supported thinking Thompson described points to a *smooth* image of change (Castillo-Garsow, 2010, 2012; Castillo-Garsow, Johnson, & Moore, 2013). Smooth images of change entail the envisioning of variation as occurring through a continuing process. For example, in the filling bottle situation, an individual could envision the volume of liquid in the bottle changing concurrently with continuing change in the height of the liquid in the bottle. In contrast, *chunky* images of change (Castillo-Garsow, 2010, 2012; Castillo-Garsow et al., 2013) entail the envisioning of change as having occurred in discrete amounts[2]. For example, in the filling bottle situation, an individual could envision an amount of liquid having been poured into a bottle and an associated change in the height of the liquid that would occur. Because students using chunky thinking have demonstrated more mathematical difficulties than students using smooth thinking (Castillo-Garsow et al., 2013), if students were to have only chunky images of change, they may have more difficulty in advancing to the more sophisticated levels described by Carlson et al. (2002).

COVARIATIONAL REASONING AND QUANTITATIVE OPERATION

Because the Quantitative Coordination level (L3) of Carlson et al.'s (2002) Covariation Framework is a level at which students seem to stagnate (Carlson et al., 2002), this discussion focuses on the Quantitative Coordination level (L3), and also addresses subsequent levels, Average Rate (L4) and Instantaneous Rate (L5) (Table 2). Classifying a student as reasoning covariationally at a particular level would mean that the student is able to perform *mental actions*[3] of not only that level, but also all preceding levels (Carlson et al., 2002). The mental action associated with L3 involves relating an amount of change in one quantity with the change in another quantity (MA3) (Carlson et al., 2002). For example, a student who related amounts of change in volume to changes in height would provide evidence of MA3. A key aspect that distinguishes L3 from L4 or L5 covariational reasoning is in the object of the reasoning: amounts of change in one quantity with change in a related quantity (L3) and a rate of change (average or instantaneous) with change in a related quantity (L4 or L5). Although students' images of change can impact the products of their reasoning (Castillo-Garsow et al., 2013), MA3 lacks explication regarding how students might envision change as occurring when relating amounts of change in quantities. If students reasoning at L3 were using different quantitative operations (Thompson, 1994a), the products of those operations would involve different relationships.

Level 3 (L3). Quantitative Coordination
At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable.
Level 4 (L4). Average Rate
At the average rate level, the images of covariation can support the mental actions of coordinating the average rate of change of the function with uniform changes in the input variable. The average rate of change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable.
Level 5 (L5). Instantaneous Rate
At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate of change of the function with continuous changes in the input variable.

Table 2. Carlson et al.'s (2002, p. 358) covariational reasoning levels 3, 4, and 5

When students are quantifying rate, I argue that they could be engaging in different quantitative operations that could be inferred based on their observable actions and utterances. I use the terms comparison and coordination[4] to identify two distinct quantitative operations involved in quantifying rate. The operation of comparison involves chunky images of change, and products of the operation of comparison include associations of amounts of change in quantities (e.g., height changed more than volume in an interval). In contrast, the operation of coordination involves smooth images of change, and products of the operation of coordination include relationships between changing quantities such that change in one quantity would depend on concurrent, continuing change in another quantity (e.g., as height increases, volume continually increases).

METHOD

This article reports a multiple case study (Stake, 2005; Yin, 2006) of secondary students' forming and interpreting relationships between quantities to quantify ratio and rate. In this multiple case study, I extended an instrumental case study to multiple examples (Stake, 2005). Basic steps in case study design include: defining the case, determining whether to conduct a single or multiple case study, and deciding how/if to use theory to inform case selection, data collection, and/or data analysis (Yin, 2006). I defined a case as a secondary student's quantification of ratio and rate as relationships between quantities. I employed grounded theory methodology (Corbin & Strauss, 2008) for within-case and cross-case analysis, because of its utility for abstracting concepts from data to develop coherent explanations of complex phenomena that could be used to build theory. My decision to employ grounded theory methodology influenced my choice of research methods, including the use of the clinical interview (Clement, 2000) to elicit data from which I could develop explanations of students' forming and interpreting relationships between quantities.

Grounded theory data analysis includes aspects of open coding, comparative analysis, and conceptual saturation (Corbin & Strauss, 2008). To provide opportunity for conceptual saturation, I conducted multiple interviews with each student. Anticipating that students' forming and interpreting relationships between quantities might be dependent on their work on a particular task, I included multiple tasks incorporating different problem situations and representations. In this study, my goal was to investigate students' current forms of quantification of ratio and rate, not to support students' engaging in more advanced forms of quantification.

Setting

Results reported in this paper come from individual, task-based clinical interviews (Clement, 2000; Goldin, 2000) that I conducted with six secondary students (Austin, Chloe, Emily, Hannah, Jacob, and Mason) from a small rural high school in a district engaged in long-term initiatives supporting teacher development and student learning. To select students, I sent invitations to students recommended by high school mathematics faculty. I requested the faculty recommend students who would be willing to talk about their mathematical thinking, who had completed at least one year of algebra, but were not enrolled in a calculus course. In doing so, I intended to select participants familiar with Cartesian graphs included in the tasks, but not having had formal instruction related to instantaneous rate of change.

I report on three of the six students participating in the study: Austin, Jacob and Hannah. I omitted reporting on Chloe, Emily and Mason, because including their cases did not extend the scope of quantitative reasoning demonstrated by all six students. Chloe and Emily provided minimal evidence of reasoning about rate as a measurable attribute of a situation. Because this study investigated students' current quantification of ratio and rate, rather than students' development of quantification, their cases afforded less insight into the scope of the framework for reasoning about change in covarying quantities[5]. Although Mason provided considerable evidence of reasoning about rate as a measurable attribute of a situation[6], for the purposes of this article, reporting the cases of Austin and Jacob was sufficient to illustrate the scope of reasoning displayed by Austin, Jacob and Mason. At the time of the study, Austin and Jacob were in 11th grade[7] and enrolled in a PreCalculus course. Hannah was in 10th grade[8] and enrolled in a Geometry course.

Students participated in a series of five interviews, in which they completed seven mathematical tasks[9]. This paper reports students' work on two tasks, the Hot Chocolate task and the Filling Bottle task, completed during the first and last interviews, respectively. Data from these tasks were typical of students' work across all of the tasks. Students' work on the Hot Chocolate and Filling Bottle tasks provided clearest evidence of students' quantification of ratio and rate.

Data Collection and Analysis

Students were interviewed during the regular school day once per week for a period of 5 weeks[10]. Interviews occurred in a quiet room during a time when students did not have an academic class. I served as the interviewer, and another researcher operated the video camera. In addition to video recording, during some interviews, the second researcher suggested additional probing questions that I might ask of students by writing a question on a card. Depending on the flow of the interview, I decided whether asking the additional question would be useful.

Audiorecordings, videorecordings, annotated transcripts, and students' written work served as sources of data to be analyzed. Data included students' explanations, written work, and gestures. Such data included but was not limited to when students described some attribute that could be measured, formed and interpreted relationships between quantities, determined amounts of change in quantities, described intensity and/or direction of change in quantities, and/or used relationships between quantities to make predictions about some attribute that they were attempting to measure.

Data analysis incorporated ongoing and retrospective analysis. Ongoing analysis included reflective notes, compiled after each interview. Ongoing analysis informed future task-based interviews conducted with individual students as well as subsequent iterations of the same task-based interview with different students. Retrospective data analysis included multiple passes through the data. In the first pass I used open coding (Corbin & Strauss, 2008) to identify chunks of data when students were forming and interpreting relationships between change in quantities, including relationships involving amounts, direction, and/or intensity of change. The left column of Table 3 shows three different types of relationships between quantities for which I coded. In the second pass I used comparative analysis within each case (Corbin & Strauss, 2008) to develop accounts of each student's images of relationships between quantities involved in ratio and rate. For example, I worked to address how Hannah's way of relating amounts of change might explain her attending to variation in the intensity of change. In the third pass I used comparative analysis across cases (Corbin & Strauss, 2008) to make distinctions among ways in which students formed and interpreted relationships between quantities involved in ratio and rate. The terms comparison and coordination indicate two distinct quantitative operations evidenced in the students' responses. The middle and right columns of Table 3 show examples of each code that provide evidence of comparison and coordination, respectively. Hannah was the only student in the study to make distinctions between different intensities of change. Hence, her response is the only example included for the "Attending to intensity of a change" code.

Types of Relationships Between Quantities	Examples: Comparison	Examples: Coordination
Comparing or ordering quantities that can change	Austin: The one point six six (1.66) is you are going to have the greater number, meaning it is going to have more chocolate per water than the one point six one (1.61). Jacob: … the volume starts going up less than the height does	Hannah: … because that's more water per hot chocolate packet so it would be more water through it.
Relating amounts of change in quantities	Austin: Since it's only going to increase one inch in height for those four ounces, it's going to be a fatter bottle compared to… Jacob: So there's more volume than height so that means it's, like, fat at the bottom and then here it starts getting skinnier...	Hannah: … the volume increases a lot than compared to up here, from seven to eight inches, like the volume increases less.
Attending to intensity of a change (e.g., faster, slower, etc.)		Hannah: The volume looks like to me to be increasing a lot, but steadily so that is why there is that large, sort of like vase shape…

Table 3. Codes indicating types of relationships between quantities and examples providing evidence of the quantitative operations of comparison or coordination

TASKS

I designed the tasks to first address students' conceptions of an attribute of a situation that I identified (e.g., rate of change of volume with respect to height) as something that could be measured. To do so, I began by describing a situation (e.g., a bottle being filled with liquid), then asking each student how he or she made sense of the identified attribute in terms of the situation. Next I provided each student with information about constituent quantities (e.g., a graph representing volume as a function of height) that could be used to quantify the measurable attribute students had just described. In doing so, my goal was not to lead students into a particular way of quantification. Rather, my goal was to provide students with an opportunity to use constituent quantities to engage in quantifying ratio and rate.

Hot Chocolate: Quantifying Ratio

In the Hot Chocolate task, I began by asking each student to imagine he or she were making a large batch of hot chocolate for a group of people and to think about how he or she might adjust the chocolate flavor of the hot chocolate. Next I provided students with a table indicating amounts of packets of chocolate and cups of water for different batches of hot chocolate: Batch *A*, containing 5 packets of chocolate and 3 cups of water, and Batch *B*, containing 6 packets of chocolate and 4 cups of water (Figure 1). I then prompted students (a) to compare the chocolaty flavor of Batch *A* and Batch *B*, (b) to explain how to make a larger batch of hot chocolate that would taste exactly the same as Batch *A*, and (c) to make predictions about the chocolaty flavor of a third batch of hot chocolate in comparison to Batches *A* and/or *B*. Because I was not attempting to help students to develop a new way of reasoning, if students used single numerical values to compare the chocolaty flavor of Batches *A* and *B* (e.g., dividing 5 by 3 to get $1.\overline{6}$, dividing 6 by 4 to get 1.5, then using (or attempting to use) those values to compare the chocolaty flavor of Batches *A* and *B*), then I used a single number to describe the third batch of hot chocolate. Else, I provided specific amounts of chocolate packets and cups of water when asking students to make predictions about the chocolaty flavor of the third batch.

Hot Chocolate Mixtures

Batch A	Batch B
5 packets hot chocolate mix 3 cups water	6 packets hot chocolate mix 4 cups water

Figure 1. Table given to students

Filling Bottle: Quantifying Rate

In the Filling Bottle task, I began by asking each student to imagine that soda bottles are being filled at a factory, and the machine turns on and soda is dispensed into the bottle at a constant rate. For bottle B shown in Figure 2, I asked each student a series of three questions: how the height of the liquid was changing as the bottle was being filled, what volume was without telling how to find it, how the volume of the soda in the bottle was changing as the height of the soda in the bottle increases. By asking students to talk about volume without telling me how to find volume, I was attempting to determine how students were conceiving of volume as some attribute that could be measured. I began with bottle B because I thought it was the most familiar looking of the bottles. If a student demonstrated that he or she could conceive of the rate of change of volume with respect to height as something that could be measured, I then asked the student the third question for each of the remaining bottles A, C, and D. If students had difficulty with the third question, I asked follow-up questions relating specific amounts of volume and height, such as "How high do you think the liquid in the bottle would be when the bottle is half full?" If students' images of change were chunky, then providing specific amounts of volume and height might foster their forming and interpreting relationships between quantities.

Figure 2. Bottles provided to students.

The second part of the Filling Bottle task adapted the well-known bottle problem (Shell Centre for Mathematical Education (University of Nottingham), 1985), requiring students to sketch a graph such that height of the liquid in the bottle is a function of the volume of the liquid in the bottle. I adapted the well-known bottle problem in two ways: (a) by asking students to interpret rather than construct a graph relating the changing height and volume of the liquid in a bottle, and (b) by representing the height of the liquid in the bottle on the horizontal axis and the volume of the liquid in the bottle on the vertical axis. I presented students with a graph relating the changing height and liquid in a bottle (Figure 3). Then I prompted students (a) to describe how the volume of the liquid in the bottle was changing as the height of the liquid in the bottle was increasing for a given graph, (b) to use the graph shown in Figure 3 to predict what the shape of the bottle being represented by the graph might be like, and (c) to determine and/or compare how fast the volume of the liquid in the bottle was changing as the height of the liquid in the bottle was increasing in given interval(s).

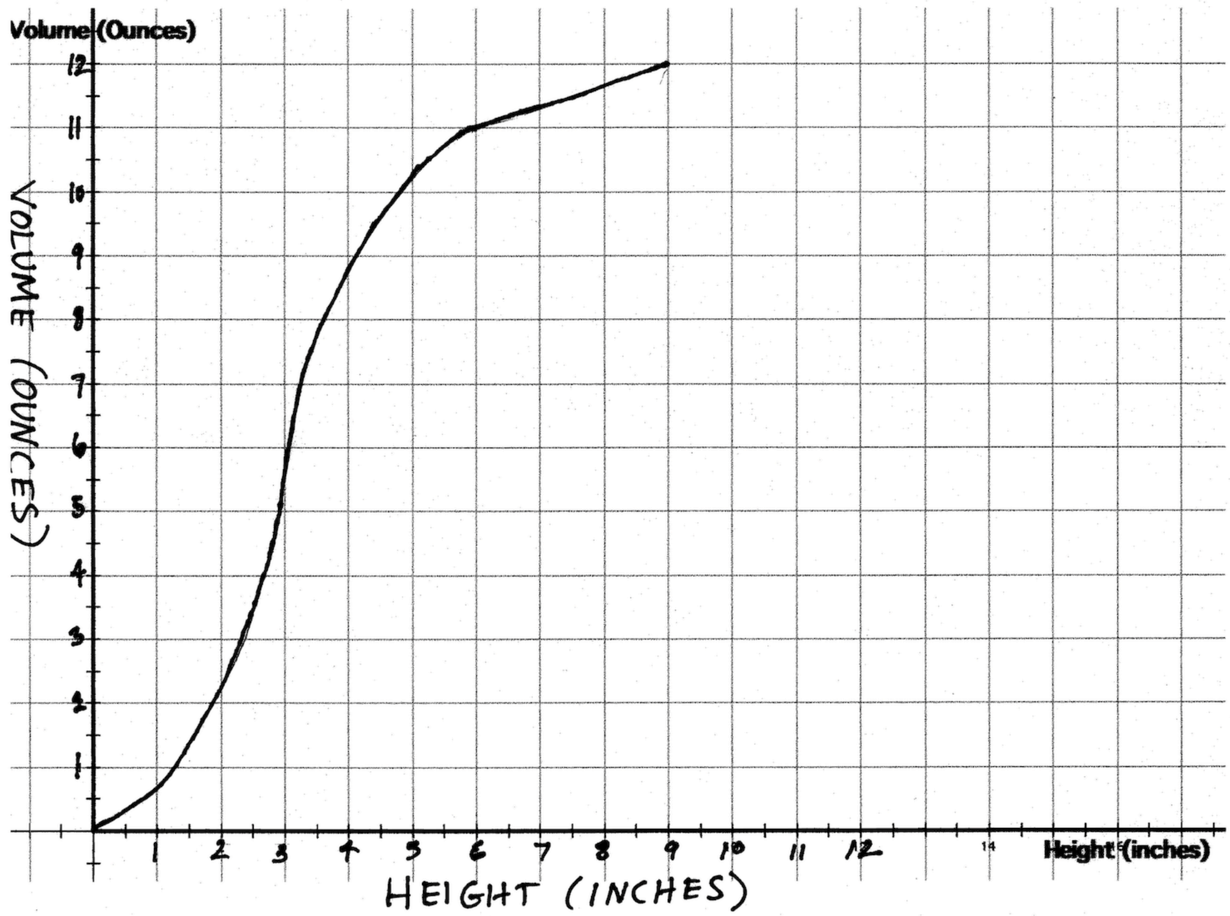

Figure 3. Graph given to students

RESULTS

The results report how Austin, Jacob, and Hannah quantified ratio and rate by forming and interpreting relationships between quantities involved in ratio and rate, respectively. In their responses to the introductory prompts to both tasks, Austin, Jacob, and Hannah all provided evidence of attending to both chocolate flavor and rate of change as measurable attributes of a situation.

Austin: Quantifying ratio by associating an amount of one quantity per one unit of another quantity

When comparing the chocolaty flavor of Batches *A* and *B*, Austin used "equal ratios" to determine that Batch *A* would be more "concentrated" than Batch *B*.

> *Austin:* You could set them up into equal ratios with the common denominators so it would be to twelve. So it [Batch A] would be twenty to twelve and this one [Batch B] would be eighteen to twelve. So this one [Batch A] would be the more concentrated one. Five to three.

Later in the interview I asked Austin if he could determine how a batch with 21 packets of chocolate and 13 cups of water would compare to the other batches without determining "equal ratios." Austin divided twenty by twelve and twenty-one by thirteen to determine two numbers, $1.\overline{66}$ and 1.61. When prompted to explain what those numbers meant in terms of chocolaty flavor, Austin used those numbers to justify why Batch *A* would be chocolatier than the batch that I invented.

> *Austin:* The one point six six (1.66) is you are going to have the greater number, meaning it is going to have more chocolate per water than the one point six one (1.61).
>
> *Researcher:* So how would that affect the chocolate flavor?
>
> *Austin:* That would make it more chocolatier.
>
> *Researcher:* So if I told you a batch of hot chocolate mix was a two point one (2.1).
>
> *Austin:* So that would be a lot more chocolatier. It's like two packets to one cups [sic] of water. And the one point six (1.6) is like one point six (1.6) packets to cups of water.

When using the numbers $1.\overline{66}$, 1.61, and 2.1 to compare the chocolaty flavor of the different batches, Austin interpreted each number in terms of chocolate packets and cups of water. As such, each number represented an association of an amount of one quantity *per-one* unit of another quantity. Although the numbers he determined could be conceived of as intensive quantities measuring the chocolaty flavor of a batch of Hot Chocolate, Austin conceived of the amounts as associations of extensive quantities, some number of packets of hot chocolate per-one cup of water.

Jacob: Quantifying ratio by associating amounts of quantities

When comparing the chocolaty flavor of Batches *A* and *B*, Jacob determined mixed numbers ($1\frac{2}{3}$ and $1\frac{1}{2}$) to represent Batches *A* and *B*, respectively. Jacob concluded that Batch *A* would be chocolatier than Batch *B,* because one and two-thirds was greater than one and one-half. When asked to explain what the one and two-thirds meant in terms of chocolaty flavor, Jacob stated it was "like percentage-wise how much of it's chocolate compared to how much is water."

As did Austin, Jacob interpreted the one and two-thirds as associating amounts of chocolate and water. Although Jacob could interpret $1\frac{2}{3}$ in terms of the chocolaty flavor of Batch *A*, he had difficulty making comparisons when given a single number, 1.75, to represent the chocolaty flavor of Batch *C*, for which particular amounts of packets of chocolate and cups of water were not given.

Researcher: Say that Batch *C* was a one point seven five (1.75). Would it be more or less chocolaty than Batch *A* or *B*, could you tell?

Jacob: Is this [Taps $1\frac{2}{3}$ written on paper.] like, is this like how much, how many milliliters or something?

Researcher: Well you wrote one and two-thirds ($1\frac{2}{3}$),

Jacob: Yeah.

Researcher: One and one-half (1.5),

Jacob: Yeah.

Researcher: One point seven five (1.75) I'm saying is like imagine a third one of those.

Jacob: So that would be like well I don't even think, I don't know what else like with these. But I'm guessing that'd be something like, seven to four or something, then I guess that would be more than these [Said hesitantly], since that'd be one point six (1.6) and one point five (1.5).

Rather than using single numerical amounts (e.g., 1.75, $1\frac{2}{3}$, and $1\frac{1}{2}$) to compare Batch *C* to Batches *A* and *B*, Jacob interpreted 1.75 as "seven to four," then made comparisons between the batches. Further, later in the interview Jacob expressed that he was not using $1\frac{2}{3}$ to make sense of the situation, but rather to "prove" what he knew immediately, that Batch *A* was stronger. Although Jacob could use single numerical amounts to compare and order the chocolaty flavor of different batches of hot chocolate, he conceived of those amounts as associations of amounts of packets of chocolate and cups of water.

Hannah: Quantifying ratio using a single, intensive quantity

When comparing the chocolaty flavor of Batches *A* and *B*, Hannah expressed that Batch *B* would be "more watery" than Batch *A*.

Hannah: Because for the five, you'll need, it has three cups but you are adding a whole new hot chocolate mix and you are only adding one more cup of water. So for Batch *B* it would be like more watery.

When asked how she might create a larger batch of hot chocolate that would have the same chocolaty flavor as Batch *A*, she began by dividing three by five "to see how many cups of water you would need per thing." Then she multiplied the result of her division (0.6) by the number of packets of hot chocolate she chose to use (8) to determine the number of cups of water (4.8) that would make a larger batch that would taste the same as Batch *A.* Next I prompted Hannah to compare the chocolaty flavor of Batches *A* and *B* to Batch *C*, given that when the number of cups of water in Batch *C* was divided by the number of packets of hot chocolate mix, the result was 0.8[11]. She determined that Batch *C* would be "more watery," providing the following explanation to support her response: "because that's more water per hot chocolate packet so it would be more water through it."

Austin, Jacob, and Hannah all used single numerical amounts representing relationships between packets of chocolate and cups of water to compare the chocolaty flavor of different batches of hot chocolate (e.g., 1.66, 1.75, and 0.8, respectively). Although all three students could be classified as operating at an interiorized, rather than an internalized ratio level, only Hannah treated a single numerical amount as an intensive quantity measuring chocolaty flavor. Unlike Jacob and Austin, Hannah did not use particular amounts of packets of hot chocolate and cups of water when interpreting what a numerical amount (0.8) meant in terms of hot chocolate and water. Importantly, Hannah used a different quantitative operation (coordination) than did Austin or Jacob (comparison), and she quantified ratio as a single intensive quantity (compatible with a ratio as measure conception), than as a single extensive quantity (Austin) or as an association of extensive quantities (Jacob).

Austin: Quantifying rate by associating amounts of change in extensive quantities

When describing how the volume of the liquid would change as the height of the liquid increased for bottles A, B, C, and D (Figure 2), Austin described how the height of the liquid would increase if one were watching the bottle filling in real time. For instance, referring to bottle A, Austin said: "it's going to increase the slowest it looks like as in height, but really it's going to have the most volume and then it's going to get faster as it gets thinner, but less volume." His response provides evidence that he conceived of how the height of the liquid might change with respect to time and how that would compare to an actual amount of volume in the bottle. When he made comparisons between volume and height, he used those comparisons to measure the rate at which the height of liquid would change with respect to time. For instance, referring to bottle C, Austin said: "So volume is going to be, it's going to increase in height slower with more volume until it gets to this mid range, which it will get a little faster, less volume than height." Although I was prompting Austin to envision how the volume of liquid was changing with respect to the height of the liquid, Austin was attempting to measure how the height of the liquid was changing with respect to elapsing time.

When asked how the volume of liquid was changing as the height of liquid was increasing for a bottle represented by the graph shown in Figure 3, Austin determined amounts of increase in volume and height in three different sections of the bottle:

Austin: Okay so in this first section we'll say maybe like up to here or so. [Draws a horizontal line intersecting the graph when volume is 4 ounces.] It's increasing about two and a half inches and it has four ounces. Go maybe here [Draws a horizontal line intersecting the graph when volume is 8 ounces.] from here it's going from about two and a half to three and a half so it is increasing one inch but it is increasing four ounces. This next section [Draws a horizontal line intersecting the graph when volume is 11 ounces.] goes from about three to five and a half so two and a half inches and about three ounces in volume.

When asked what those amounts of increase meant in terms of bottle shape, Austin made comparisons (greater, less, same) between amounts of change in height and volume in different sections of the bottle to make predictions about the shape of the bottle:

Austin: Since it's only going to increase one inch in height for those four ounces, it's going to be a fatter bottle compared to—section one has four ounces as well, but it's going to cover two and a half inches so it's going to be thinner than section two. So it's going to start, it's going to start kind of thin and get fatter up through the middle here. And section three. Goes from about three and a half to six. Two point five inches, and actually I am going to change that, just so it's all the same here. Just go to twelve so they all increase four ounces. [Draws a horizontal line intersecting the graph when volume is 12 ounces.] So we go from three and a half to nine inches. Five point five (5.5) inches it's increasing four ounces. So it's going to get even more narrow here at the top because it is going to cover five inches for four ounces.

By using intervals in which the change in volume is four ounces, Austin could compare amounts of height without coordinating changes in both height and volume. Although Austin used particular amounts in his initial response, he could compare change in volume and height in different sections of the graph without determining particular amounts of change to make predictions about the shape of the bottle. When prompted to explain how he might interpret the section of the graph near where the volume was 8 ounces, Austin compared changes in volume and height without determining particular amounts of change.

Austin: The volume is changing a lot faster than the height is changing.

Researcher: Does it do that throughout?

Austin: No, when the curve is this way, [Near when the volume is 8 ounces] it's changing more in volume. Like the steeper but like in this case [Near when the volume is 1 ounce] it's a little bit the same. The more steep the line is, in this case. Basically like the more, the steeper the line is, it's going to have more volume compared to height so it's going to be like a fatter bottle. With the less steep the line, it's going to have a lot less volume compared to the height.

Although the given graph was not linear, Austin made sense of the graph as if it were linear. The curvature of the graph near when the volume is eight ounces is concave down, which would represent an increase that is increasing at a decreasing rate. Despite identifying a "curve," Austin focused only on how "steep" he perceived the curve to be, affording his making of comparisons between amounts of change in volume and height. Importantly, Austin's claim that "volume is changing a lot faster than the height is changing" referred to a section of a bottle represented by a portion of the graph that he identified to be "less steep" or "more steep."

Given Austin's work on the Hot Chocolate task, I interpret that his use of "more" when comparing volume and height extended beyond just an additive comparison of amounts of quantities to include the possibility of a multiplicative comparison between quantities. By linking steepness to an association of extensive quantities (amounts of change in volume and height), Austin went beyond just noticing an observable attribute of a graph (e.g., the slant of a line). By interpreting steepness in terms of amounts of change volume and height of liquid, Austin could determine whether the bottle would be "thin" or "fat" and thereby measure how the height of the water would change as the bottle was being filled.

When quantifying rate, Austin used the same quantitative operation (comparison) as he did when quantifying ratio. Importantly, even though Austin could make viable claims about a bottle represented by the graph, his quantification of rate involved an association of amounts of change in extensive, rather than intensive quantities.

Jacob: Quantifying rate using a single, extensive quantity that represents an association of quantities

When describing how the volume of the liquid would change as the height of the liquid increased for bottles A, B, C, and D (Figure 2), Jacob made comparisons between changes in height and volume. For instance, when referring to the upper part of bottle B, Jacob stated: "There's not as much volume put in up here as the height goes up. Like down here we said the volume was going up the same as the height but here [Points to the upper part of the bottle.] the volume starts going up less than the height does." His response provides evidence that he conceived of a quantity (an amount of volume being poured in when the height is changing) as something that could be measured.

When asked how the volume of liquid was changing as the height of liquid was increasing for a bottle represented by the graph shown in Figure 3, Jacob used a point not on the graph to illustrate how he compared amounts of volume and height to make predictions about the shape of the graph:

Jacob: Okay. Well it's, here there's more volume. [Points to the volume axis] I'll just use the point two, two. See, like it's more volume than height. [Points to the height axis] So there's more, if it was two, two [If the point (2,2) were on the graph] then it would have been constant. So there's more volume than height so that means it's, like, fat at the bottom and then here it starts getting skinnier and skinnier and skinnier.

Because he identified a greater change in volume than height in the interval [0, 2], Jacob concluded that the bottle would be "fat" at the bottom.

When asked how the volume was changing as the height increased from three inches to six inches, initially Jacob compared change in volume and height without determining particular amounts: "It's getting, it's not getting very high but it's getting filled a lot. So it's like whenever it's fat, it's getting lots in it but not very high." Importantly, Jacob's claim that the volume "is changing a lot faster than the height is changing" referred to a section of a bottle represented by the graph. When prompted to put a number to that section, Jacob divided amounts of change in volume by amounts of change in height to determine the "slope" in the interval to be 2. When prompted to explain what the 2 meant, he associated amounts of change in volume and height: "two units of volume for every one unit of height." Because Jacob was comparing the amount of units of volume to the amount of units of height, he could make comparisons between quantities from different measure spaces. Further, Jacob could identify when the same association might occur in a different interval or compare this association to other associations in different intervals. To envision when a different interval might have the same relationship (or to compare associations in intervals), he associated changes in volume and height, then compared those to his "slope" of 2.

Both Jacob and Austin made associations between amounts of change in volume and height to make accurate predictions about the width of the bottle being represented by the graph shown in Figure 3. Despite the similarity of their quantifying, they were quantifying different rates of change. Austin was quantifying a rate of change of height with respect to elapsing time, and Jacob was quantifying a rate of change of volume with respect to change in height.

As was the case for Austin, when quantifying rate, Jacob used the same quantitative operation (comparison) as he did when quantifying ratio. Unlike Austin, Jacob used single numbers to represent a rate of change (e.g., 2). Importantly, when determining and interpreting single numbers to represent relationships between amounts of change in quantities, Jacob conceived of those numbers as associations of extensive quantities. Such activity was compatible with Jacob's work on the Hot Chocolate task (e.g., interpreting 1.75 as associating 7 packets of chocolate and 4 cups of water).

Hannah: Quantifying rate as a coordination of intensive and extensive quantity

When describing how the volume of the liquid would change as the height of the liquid increased for bottles A, B, C, and D (Figure 2), Hannah described how the increase in volume with respect to change in height could vary. For instance, when referring to Bottle C, Hannah stated: "the volume would still increase as you go up, but it would slow down, the increase would slow down a little bit and then it would start to increase more again." Her response provides evidence that she conceived of a quantity (increase) as something that could be measured.

When asked how the volume of liquid was changing as the height of liquid was increasing for a bottle represented by the graph shown in Figure 3, Hannah attended to variation in the intensity of the increase in volume:

Hannah: Towards the bottom of it like it doesn't increase as much as, but as you go along it definitely increases more, like the volume and the height, it increases more as you go, [Runs her finger along the graph, beginning near the origin and ending when the volume is approximately 6 ounces.] the volume of it increases a lot from here to here, [Using her fingers, marks off a section of the graph beginning when the volume is approximately 4 ounces and ending when the volume is approximately 8 ounces.] like compared to down here, [Points to the part of the graph near the origin.] So there would be a bigger volume for that towards the middle [Cups hands to indicate a bottle shape], and then it [the bottle] starts to get smaller as you go to the cap of the bottle.

When claiming that Hannah attended to variation in the intensity of an increase, I mean she made distinctions between the degree to which an increase was present. Hannah's statement that volume "increases more as you go" suggests that that she could envision the volume of liquid to be something that could increase to a greater or lesser degree as the height of the liquid continued to change.

When asked if she could sketch something or describe what it would be like for the volume to get smaller[12], Hannah suggested she draw the bottle. When sketching the bottle, she made distinctions between different intensities of increase in volume.

Hannah:... It would be sort of like a vase shape. So, and then it would just be sort of shaped like that because the volume down here like it says, shows on this graph, like the volume starts to increase faster as it gets in towards around here and then as you go up further and then as the height increases the volume starts to slow, like slower increasing.

Researcher: And you found a portion where you said the volume was increasing steadily

Hannah: Uh huh

Researcher: How fast is the volume increasing there at that steady portion?

Hannah: It is increasing a lot around here. The volume looks like to me to be increasing a lot, but steadily so that is why there is that large, sort of like vase shape because the volume is increasing so it would be bigger there compared to up here. *[Points to rightmost part of the graph]*

When sketching the bottle shape Hannah made distinctions between different intensities of increase in volume (e.g., increasing faster, slower increasing) occurring in conjunction with continuing change in the height of the liquid in the bottle. Importantly, Hannah's claim that volume "increases faster" refers to a relationship between change in volume and change in height rather than to a section of a bottle represented by the graph. By envisioning change in volume as occurring in coordination with continuing change in height, Hannah demonstrated a smooth image of change. Hannah could conceive of variation in the intensity of an increase in an interval that represented a section of the bottle, and she was the only student in the study to do so.

Prior to the previous excerpt, I had asked Hannah if she thought there were any parts where the volume was increasing "steadily." In her work on a previous task, Hannah introduced the term "steady increase" when interpreting a portion of a graph in which equally sized intervals appeared to have the same amounts of change in the dependent variable (Johnson, 2012b). I reintroduced this term to provide the opportunity for Hannah to quantify different intensities of increases. In her response, she distinguished increasing "a lot" from increasing steadily, suggesting that when volume "increased a lot," it may or may not increase steadily. Although Hannah typically coordinated change in volume and height without determining particular amounts of change, when prompted to explain how she knew it [volume] was "increasing a lot," she used particular amounts of change in volume and height to justify her assertion.

Researcher: You said right in here [Points to the part of the graph near when the volume is 7 inches.] it is increasing a lot.

Hannah: Yeah

Researcher: How do you know?

Hannah: Like the volume, from just these two blocks from the height of three and four inches [Puts fingers on 3 and 4 inches on the height axis], like the volume increases a lot [Marks off a section of the graph with her fingers.] than compared to up here [Marks off another section of the graph with her fingers.] from seven to eight inches [Puts fingers on 7 and 8 inches on the height axis], like the volume increases less.

Researcher: How do you know?

Hannah: Cause from three to four, it goes all the way from six to almost nine ounces of the volume but from here it doesn't even go up one ounce from the inches, like seven to eight inches.

Researcher: So how could you compare those?

Hannah: Um, you could just say, you see how much ounces it goes from inch to inch from here, it doesn't even go, like just about like half of an ounce, but from three to four it almost goes up three whole ounces, so like you can compare to see how many ounces it takes from inch to inch.

When Hannah stated that the volume increased "less" when the height increased from seven to eight inches, she spoke about the volume being in progress of going up (a smooth image of change), rather than as having already gone up by a particular amount (a chunky image of change). Although she could determine particular amounts of change in quantities when prompted to do so, she formed and interpreted relationships between volume and height by coordinating change in volume with continuing change in height. Because Hannah could envision an *increase* as a relationship between amounts of change in volume and height (rather than a result of particular amounts of change in volume and height), she could conceive of increase as an attribute possessed homogeneously by the filling bottle situation.

When quantifying rate, Hannah used the same quantitative operation (coordination) as she did when quantifying ratio. Importantly, Hannah used a different quantitative operation (coordination) than did Austin or Jacob (comparison). Further, she quantified rate in terms of intensive quantity, rather than only extensive quantity, as did Austin and Jacob. In addition, Hannah was the only student who demonstrated smooth images of change, and the only student who provided evidence of conceiving of a rate as being possessed homogeneously by a situation.

CHANGE IN COVARYING QUANTITIES FRAMEWORK

Austin, Jacob, and Hannah quantified rate in different ways. However, according to Carlson et al.'s (2002) Covariation Framework all three students would be reasoning at the same level of covariational reasoning (L3), because they were relating amounts of change with amounts of change rather than average or instantaneous rates of change with amounts of change (L4 or L5). Although making distinctions between the objects of reasoning (e.g., relationships between amounts of change [L3] vs. relationships between rates of change and amounts of change [L4 and L5]) is useful, it is not sufficient to account for students' images of change (chunky vs. smooth), students' quantitative operations (comparison vs. coordination), and students' quantification of rate (in terms of extensive or intensive quantity). Austin and Jacob, students demonstrating chunky images of change, used the quantitative operation of comparison, and quantified rate (and ratio) in terms of extensive quantity. In contrast, Hannah, a student demonstrating smooth images of change, used the quantitative operation of coordination, and quantified rate (and ratio) in terms of intensive quantity. As evidenced by Hannah, a student who quantifies ratio as an intensive quantity could quantify rate in a powerfully different way than a student who quantifies ratio in terms of extensive quantity (e.g., Jacob or Austin). Yet, Carlson et al.'s (2002) Covariation Framework does not account for such distinctions.

To account for differences in students' reasoning at L3 of Carlson et al.'s (2002) Covariation Framework, the Change in Covarying Quantities Framework distinguishes between two quantitative operations involved in quantifying rate of change: comparison (*QO-Comp*) and coordination (*QO-Coord*). For each quantitative operation, the Change in Covarying Quantities Framework explicates a possible progression through which an individual's quantification of rate might develop. A key distinction between these progressions lies in an individual's image of relationships between quantities involved in rate. For an individual using the operation of comparison, his images of relationships between quantities involved in rate of change would include associations of extensive quantities or single extensive quantities. In contrast, for an individual using the operation of coordination, her images of relationships between quantities involved in rate of change would include relationships involving both intensive and extensive quantities or single intensive quantities.

Table 4 shows the Change in Covarying Quantities Framework. The framework shows the quantitative operations of comparison (QO-Comp) and coordination (QO-Coord), with each operation containing three levels of reasoning (QO-Comp-1, 2, 3 and QO-Coord-1, 2, 3). To indicate that a student is using QO-Coord-3 (or QO-Comp-3) would entail that the student is able to use QO-Coord-1, 2 (or QO-Comp-1, 2). For each quantitative operation, the framework includes objects of reasoning and examples that could provide evidence of each level of reasoning. The quantitative operations explicate mental actions involved in quantifying rate of change. The objects of reasoning indicate images of relationships between quantities involved in rate of change as comprising intensive or extensive quantities. The examples include specific instances of students' empirical work (QO-Comp-1, 2, 3 and QO-Coord-1, 2) or an extension of students' empirical work (QO-Coord-3) that could provide evidence of the quantitative operations of comparison or coordination. A key distinction between QO-Comp-1 and QO-Comp-2 is the comparing of particular amounts of change in quantities. A key distinction between QO-Comp-2 and QO-Comp-3 is the use of a single, extensive quantity to quantify rate of change. A key distinction between QO-Coord-1 and QO-Coord-2 is the conceiving of variation in the intensity of a change (e.g., increases in volume could be slower or faster, but

Johnson, H. L. (2015) Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning, 17*(1), 64-90.

Quantitative Operation: Comparison (Extensive Quantity)			
	QO-Comp-1	**QO-Comp-2**	**QO-Comp-3**
Quantitative Operations: Comparison (*QO-Comp*)	Comparing *(greater, less, same)* change in one quantity with change in a second quantity	Comparing *(greater, less, same)* particular amounts of change in one quantity with amounts of change in a second quantity	Determining a single quantity that indicates a comparison *(greater, less, same)* between change in quantities
Objects of Reasoning:	**Association of extensive quantities:** Change in one quantity with change in a second quantity	**Association of extensive quantities:** Amounts of change in one quantity with amounts of change in a second quantity	**A single extensive quantity:** An amount of change in one quantity "per" an amount of change in a second quantity is a quantity in and of itself
Examples	Forming or interpreting a relationship involving comparison of change in quantities: e.g., "So there's more volume than height so that means it's, like, fat at the bottom." *-Jacob*	Forming or interpreting a relationship involving comparison of particular amounts of change in quantities: e.g., "Since it's only going to increase one inch in height for those four ounces, it's going to be a fatter bottle" *-Austin*	Forming or interpreting a single quantity that indicates a comparison between change in quantities: e.g., "2," such that "2" represents "two units of volume for every one unit of height" in multiple intervals *-Jacob*

Quantitative Operation: Coordination (Intensive Quantity)			
	QO-Coord-1	**QO-Coord-2**	**QO-Coord-3**
Quantitative Operations: Coordination (*QO-Coord*)	Coordinating change in one quantity with continuing change in another quantity.	Coordinating variation in the intensity of change in one quantity with continuing change in another quantity	Determining a single quantity that coordinates variation in the intensity of change in one quantity with continuing change in another quantity
Objects of Reasoning	**Coordination of extensive quantities:** Change in one quantity happening in conjunction with continuing change in another quantity	**Coordination of an intensive quantity with an extensive quantity:** Variation in the intensity of change in one quantity happening in conjunction with continuing change in another quantity	**A single intensive quantity:** The variation in the intensity of change in one quantity happening in conjunction with continuing change in another quantity is a quantity in and of itself
Examples	Forming or interpreting a relationship involving continuing change in related quantities: e.g., "Cause from three to four *(inches)*, it goes all the way from six to almost nine ounces of the volume" - *Hannah*	Forming or interpreting a relationship involving variation in the intensity of a change in one quantity covarying with another quantity: e.g., "as the height increases the volume starts to slow, like slower increasing" *-Hannah*	Forming or interpreting a single quantity that indicates a relationship between change in covarying quantities: e.g., *slower increasing* such that slower increasing is a quantity in and of itself

Table 4. Change in Covarying Quantities Framework

volume would still be increasing). A key distinction between QO-Coord-2 and QO-Coord-3 is the use of a single, intensive quantity to quantify rate of change.

Students who quantified ratio in terms of extensive quantity (e.g., Austin, Jacob) used QO-Comp. In contrast, the student who quantified ratio in terms of intensive quantity (Hannah) used QO-Coord. Although the Change in Covarying Quantities Framework shows QO-Comp above QO-Coord, I am not suggesting that the quantitative operation of comparison is a prerequisite for the quantitative operation of coordination. Further, if a student using QO-Comp were to begin to use L4 reasoning (relating an average rate of change with an amount of change) as identified by Carlson et al. (2002), that student would be quantifying rate as a single extensive quantity, and therefore would seem to have an impoverished form of L4 reasoning. Importantly, if students were to use only QO-Comp and never use QO-Coord, this could explain in part students' persistent difficulty with rate.

DISCUSSION

The Change in Covarying Quantities Framework explicates two possible progressions in the quantification of rate. The progressions involve two distinct quantitative operations: comparison (QO-Comp) and coordination (QO-Coord), that students classified as operating at the Quantitative Coordination level (L3) of Carlson et al.'s (2002) Covariation Framework could be using. The operations of comparison and coordination afford different images of relationships between quantities involved in rate of change, with QO-Comp affording images of relationships comprising extensive quantity and QO-Coord affording images of relationships comprising intensive quantity. If students' images of relationships between quantities involved in rate of change were comprised only of extensive quantity, then that could provide some explanation for their difficulties with consistently demonstrating covariational reasoning beyond L3 (cf. Carlson et al., 2002).

Distinctions between QO-Comp and QO-Coord emphasize the essential nature of quantification as it relates to a central concept of mathematics, rate of change. Not only is it important that a student quantifies a rate of change, but equally important is how that student forms and interprets relationships between constituent quantities involved in rate of change. Connecting to chunky and smooth images of change, QO-Coord involves smooth images of change, while QO-Comp involves chunky images of change. QO-Coord explicates the necessity of smooth images of change (e.g., Coordinating change in one quantity with continuing change in another quantity) in quantifying rate in terms of intensive, rather than extensive quantity. In contrast, making comparisons between amounts of change across intervals (QO-Comp) is less powerful, because students using the quantitative operation of comparison quantify rate in terms of extensive, rather than intensive quantity.

Simon and Placa (2012) argued that students should have opportunities to reason about intensive quantities when developing understanding of ratio. Each student in this study was consistent in terms of the quantitative operation he or she used (either comparison or coordination) when quantifying both ratio and rate. Although the Change in Covarying Quantities Framework does not explicitly address students' quantification of ratio, the framework articulates how quantitative operations involved in quantifying ratio—comparison vs. coordination—could relate to the quantification of rate in terms of extensive or intensive quantity, respectively. The results of this study suggest that students' opportunities to reason about intensive quantities in ratio-related situations may also impact their rate-related reasoning.

Confrey and Smith (1995) posited that "the ability to recognize *variation* in a rate of change is essential for the transition to calculus" (p. 50). An important distinction between QO-Comp and QO-Coord lies in the envisioning of variation in the intensity of change. A student using QO-Comp would identify different intervals in which quantities are changing, form and interpret relationships between changing quantities in each of those intervals, then compare the relationships between the changing quantities across those intervals. In contrast, a student using QO-Coord would coordinate change in related quantities and envision that change progressing in an interval. For a student using QO-Comp, an interval is central to the comparison. However, for a student using QO-Coord, the relationship between the changing quantities is central to the coordination. Going further, distinctions between QO-Coord and QO-Comp also might help to explain students' difficulty in performing mental actions associated with L5 covariational reasoning, specifically the coordination of an instantaneous rate of change with continuing change in a related quantity (Carlson et al., 2002, p. 357). Given that smooth images of change are central to the mental actions associated with L5, the use of smooth thinking in prior levels (e.g., L3) might afford students' consistent application of L4 or L5 reasoning. Because a student using QO-Coord-2 could envision change in one quantity as being dependent on continuing change in another quantity, she already would be operating with aspects of L5 reasoning, unlike a student using QO-Comp-3. Therefore, it would seem that a student using QO-Coord-2 (e.g., Hannah) would be better equipped to consistently apply L5 reasoning than would a student using QO-Comp-2 (e.g., Jacob).

Thompson (2008) asserted that conceiving of constant rate of change as a homogeneous relationship between quantities would support envisioning change in one quantity occurring in conjunction with continuing change in another quantity (e.g., a smooth image of change).

Hannah's rate-related reasoning suggests that smooth images of change involved in quantifying rate may be reflexively related to a conception of a rate as a homogeneous relationship. All of the students in this study used the same quantitative operation (comparison or coordination) when quantifying ratio that they used when quantifying rate. Importantly, even when using the same quantitative operation, a student's quantification of ratio might be more sophisticated than her quantification of rate, as evidenced in Hannah's case. Her case provides an example of a student who quantified ratio using a single intensive quantity, but who did not yet demonstrate evidence of having quantified rate using a single intensive quantity.

Thompson (2011) argued that an individual's conception of a quantity and an individual's quantification of that quantity are inseparable. When investigating students' quantification of rate and ratio in this study, that quantification was intertwined with students' conceptions of rate and ratio. Although all of the students quantified a rate by forming and interpreting relationships between constituent quantities involved in rate, claiming that all of the students conceived of rate as an intensive quantity does not satisfactorily communicate what appear to be tangible differences in students' conceptions of rate. I argue that differences in students' quantification of rate could help to explicate differences in students' conceptions of rate. Although this study did not investigate how students' conceptions of rate progressed, it would seem that an individual's conception of rate and an individual's quantification of rate are reflexively related.

IMPLICATIONS

The Change in Covarying Quantities Framework could be used to inform the design of instructional tasks related to covariation and rate of change. These findings suggest that an important aspect of instructional tasks focusing on covariation and rate of change is the opportunity for students to envision running through calculations to form and interpret relationships between quantities involved in rate of change (smooth images of change). Although numerical calculations can be an entry point for prompts related to covariation and rate of change, tasks that emphasize generalization from numerical calculations may not predictably foster an advance in covariational reasoning, because those tasks focus students' attention on amounts of change that have already occurred (chunky images of change). For example, tasks requiring students to use the graph shown in Figure 3 to determine amounts of volume for given amounts of height may not provide students with opportunities to envision how the volume is changing as the height increases. Prompts requiring students to predict how change might continue can provide opportunities for students to quantify rate of change by envisioning change in one quantity as occurring in conjunction with change in a continuing quantity. Further, students' responses to such prompts could provide evidence to distinguish students' quantifying rate of change in terms of extensive, rather than intensive quantity.

Although the student in this study who used QO-Coord also quantified ratio in terms of intensive quantity, it is not known if a ratio as measure conception would be necessary for students to reason using QO-Coord. Future research examining students' conceptions of ratio and rate may investigate further interrelationships between students' conceptions of ratio and rate. Although it might seem logical that a student would first quantify ratio and rate in terms of extensive quantity, then in terms of intensive quantity, I argue that it may not necessarily be the case. Students' consistency in using a single quantitative operation (either comparison or coordination) when quantifying both ratio and rate provides further evidence to suggest the importance of students having opportunities to reason about ratio as an intensive quantity (cf. Simon & Placa, 2012). Perhaps given instructional opportunities that foster the quantification of ratio (or rate) in terms of intensive quantity, students may be less likely to quantify ratio (or rate) in terms of extensive quantity and thereby be less likely to develop more impoverished conceptions of ratio and rate compatible to ratio as identical groups or ratio as per-one. The Change in Covarying Quantities Framework explicates a quantitative operation, coordination, important for students to develop through instruction.

A potential limitation of the Change in Covarying Quantities Framework is that it focuses primarily on one level of Carlson et al.'s (2002) Covariation Framework (L3-Quantitative Coordination). Although the focus is specific, this study targeted a key level of Carlson et al.'s (2002) Covariation Framework, beyond which even successful university students do not reason consistently (cf. Carlson et al., 2002). Future research could use the Change in Covarying Quantities Framework to investigate a broader sample of students at different ages and different levels of mathematical knowledge.

Using this framework, it would seem that a student using QO-Comp would not be quantifying rate of change in terms of intensive quantity. However, if students were able to use both QO-Coord and QO-Comp, their use of QO-Comp likely would be much richer than what is reported in this framework. Further, unanswered questions remain regarding how students might develop QO-Coord or QO-Comp. Might students begin to reason about rate-related situations by using QO-Comp, QO-Coord, or some combination of both? How might students using only QO-Comp shift to using QO-Coord? Future research investigating such questions could provide insight into students' development of understanding of the difficult to learn concepts of ratio and rate.

ACKNOWLEDGEMENT

This research was completed to fulfill the dissertation requirement for a doctoral degree at The Pennsylvania State University under the advisement of Rose Mary Zbiek. The framework reported in this paper resulted from subsequent analysis of dissertation data. I am grateful to Patrick Thompson and Evan McClintock for their many thoughtful comments on prior versions of this paper and for the many insights that resulted from our conversations related to this paper. This paper is supported in part by the National Science Foundation under Grant ESI-0426253 for the Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL). Any opinions, findings, or conclusions expressed in this document are those of the author and do not necessarily reflect the views of the National Science Foundation.

An earlier version of this article appeared in:

Johnson, H. L. (2014). Images of intensive and extensive quantity: A framework for reasoning about change in covarying quantities. In L. P. Steffe, K. Moore & R. Mayes (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 267-280). Laramie, WY: University of Wyoming College of Education.

REFERENCES

American Association for the Advancement of Science. (2008). *Benchmarks for science literacy*. New York: Oxford University Press.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology*. Dordrecht-Holland: D. Reidel Publishing Company.

Bezuidenhout, J. (1998). First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology, 29*(3), 389-399.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Castillo-Garsow, C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth.* (Ph. D. doctoral dissertation), Arizona State University, Phoenix, AZ.

Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 55-73). Laramie, WY: University of Wyoming College of Education

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31-37.

Clagett, M. (1968). *Nicole oresme and the medieval geometry of qualities and motions*. Madison, WI: University of Wisconsin Press.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handook of research design in mathematicss and science education* (pp. 547-589). Mahwah, NJ: Lawrence Erlbaum Associates

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). London: Sage Publications.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439-478.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. J. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215-238). New York: Springer

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, NJ: Lawrence Erlbaum Associates

Harel, G., Behr, M., Lesh, R. A., & Post, T. R. (1994). Invariance of ratio: The case of children's anticipatory scheme for constancy of tast. *Journal for Research in Mathematics Education, 25*(4), 324-345.

Heinz, K. (2000). *Conceptions of ratio in a class of preservice and practicing teachers.* (doctoral dissertation), The Pennsylvania State University, University Park, PA.

Herbert, S., & Pierce, R. (2012). Revealing educationally critical aspects of rate. *Educational Studies in Mathematics, 81*(1), 85-101.

Howe, C., Nunes, T., & Bryant, P. (2010). Intensive quantities: Why they matter to developmental research. *British Journal of Developmental Psychology, 28*, 307-329.

Johnson, H. L. (2010). *Making sense of rate of change: Secondary students' reasoning about changing quantities.* (doctoral dissertation), The Pennsylvania State University, University Park, PA.

Johnson, H. L. (2012a). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In R. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 39-53). Laramie, WY: University of Wyoming College of Education

Johnson, H. L. (2012b). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *Journal of Mathematical Behavior, 31*(3), 313-330.

Johnson, H. L. (2013). Reasoning about quantities that change together. *Mathematics Teacher, 106*(9), 704-708.

Kaput, J. J., & West, M. M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 237-287). Albany, NY: State University of New York Press

Lobato, J., Ellis, A. B., & Muñoz, R. (2003). How "focusing phenomena" in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning, 5*(1), 1-36.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 162-175). Reston, VA: National Council of Teachers of Mathematics

Monk, S., & Nemirovsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education, I* (pp. 139-168)

National Governors Association Center for Best Practices Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.

National Research Council. (1996). *National science education standards*. Washington, DC: National Academy Press.

Nunes, T., Desli, D., & Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research, 39*, 651-675.

Piaget, J. (1970a). *The child's conception of movement and speed*. New York: Basic Books, Inc.

Piaget, J. (1970b). *Genetic epistemology*. New York: Columbia University Press.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development*. Chicago: University of Chicago Press.

Saldanha, L., & Thompson, P. W. (1998). Re-thinking covariation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson, K. R. Dawkins, M. Blanton, W. N. Coloumbe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the 20th annual meeting of the Psychology of Mathematics Education North American Chapter* (Vol. 1, pp. 298-303). Raleigh, NC: North Carolina State University

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2, pp. 41-52). Reston, VA: National Council of Teachers of Mathematics

Shell Centre for Mathematical Education (University of Nottingham). (1985). *The language of functions and graphs: An examination module for secondary schools*: Shell Centre.

Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. *Educational Studies in Mathematics, 30*(2), 197-209.

Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359-371.

Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 13*(2), 183-197.

Simon, M. A., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics, 32*(2), 35-41.

Stake, R. E. (2005). Qualitative case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *The SAGE handbook of qualitative research* (pp. 443-466). Thousand Oaks, CA: Sage Publications, Inc.

Stewart, I. (1990). Change. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 183-224). Washington, DC: National Academy Press

Stroup, W. (2002). Understanding qualitative calculus: A structural synthesis of learning research. *International Journal of Computers for Mathematical Learning, 7*, 167-215.

Stump, S. L. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics, 101*(2), 81-89.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press

Thompson, P. W. (1994b). Images of rate and operation understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*, 229-274.

Thompson, P. W. (1996). Imagery and the development of mathematical reasoning. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds.), *Theories of mathematical learning* (pp. 267-284). Mahwah, NJ: Lawrence Erlbaum Associates

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31-49). Morelia, Mexico

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. A. Chamberlain & L. L. Hatfield (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for wisdom^e* (Vol. 1, pp. 33-56). Laramie, WY: University of Wyoming College of Education

Thompson, P. W., Carlson, M. P., Byerly, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore & L. L. Hatfield (Eds.), *Epistemic algebraic students: Emerging* *models of students' algebraic knowing* (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International journal of mathematical education in science and technology, 38*(5), 609-637.

Yin, R. K. (2006). Case study methods. In J. L. Green, G. Camilli & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 111-122). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc.

Zandieh, M., & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *Journal of Mathematical Behavior, 25*, 1-17.

---

[1] In this article I use both rate and rate of change. When using rate of change rather than rate, my intent is to emphasize the change involved in rate. When speaking about both ratio and rate, I use rate rather than rate of change to maintain parallel structure. I interpret both rate and rate of change to be single quantities that are capable of varying.

[2] Students envisioning change as occurring in discrete amounts could occur in situations perceived to be continuous by an outside observer (Castillo-Garsow et al., 2013). The focus here is on the reasoning of the individual, not on the classification of a situation (e.g., as discrete or continuous).

[3] Piaget (1970b) defined operations as internalized mental actions that an individual could enact as thought processes or physical acts. I interpret Carlson et al.'s use of mental actions to refer to operations, or internalized mental actions as defined by Piaget.

[4] Piaget distinguished coordinations from observable actions, such that coordinations involve "constructing new relationships that go beyond what can be observed" (Piaget, 1985, p. 38). In distinguishing between quantitative operations of comparison and coordination, I do not intend to communicate that the former is an observable action while the latter is a coordination; both operations are coordinations as defined by Piaget. My intent is to draw distinctions between different types of quantitative operations involving different conceptions of relationships that go beyond what is observable.

[5] Both Chloe and Emily demonstrated evidence of using QO-Comp-1, see Table 4.

[6] Mason demonstrated evidence of using QO-Comp-2, see Table 4. For additional discussion related to Mason, see Johnson (2012a, 2013).

[7] In the US, 11th grade students are typically 16-17 years old.

[8] In the US, 10th grade students are typically 15-16 years old.

[9] See Johnson (2010) for a detailed description of all of the tasks.

[10] Due to scheduling constraints, Jacob was interviewed twice during one week.

[11] By specifying that 0.8 was a result of a division of an amount of cups of water and packets of chocolate, as the interviewer I may have had a more guiding role with Hannah than I did with either Jacob or Austin. I added this clarification during the interview because Hannah had been explicit about her use of division to determine numerical amounts to represent different batches of hot chocolate.

[12] The researcher misspoke. She meant to say when the bottle would get smaller.